香港中文大學
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THE CHINESE UNIVERSITY OF HONG KONG

DEPARTMENT OF MECHANICAL & AUTOMATION ENGINEERING

MAEG5080 Smart Materials & Structures

Assignment #1

by

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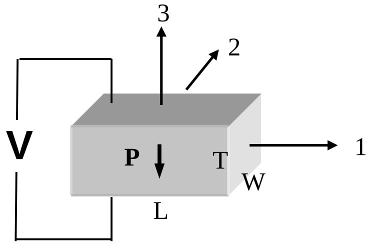
Problem 1

The piezoelectric constant matrix $\mathbf{d}$ of PZT is described as

$$\begin{bmatrix} 0 & 0 & 0 & 0 & d_{15} & 0 \\ 0 & 0 & 0 & d_{15} & 0 & 0 \\ d_{31} & d_{31} & d_{33} & 0 & 0 & 0 \end{bmatrix} \tag{1}$$

Consider a PZT element above used as a micro positioning device, in which in $L = 30$ mm, $T = 5$ mm, and $W = 12$ mm. With 110 volts applied, compute the changes in L, T, and W for a PSI-5A-S4 piezoceramic ($d_{31} = -190 \times 10^{-12}$ Meters/Volt; $d_{33} = 390 \times 10^{-12}$ Meters/Volt; $d_{15} = 550 \times 10^{-12}$ Meters/Volt).



Solution:

The electric fields after 110 volts are applied are given by:

$$E = \begin{bmatrix} 0 \\ 0 \\ \frac{110\text{ V}}{T} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \frac{110\text{ V}}{5\text{ mm}} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2.2 \times 10^4 \end{bmatrix} \text{V/m} \tag{2}$$

The mechanical strains are calculated as:

$$S = d^t E = \begin{bmatrix} 0 & 0 & 0 & 0 & d_{15} & 0 \\ 0 & 0 & 0 & d_{15} & 0 & 0 \\ d_{31} & d_{31} & d_{33} & 0 & 0 & 0 \end{bmatrix}^t \cdot \begin{bmatrix} 0 \\ 0 \\ 2.2 \times 10^4 \end{bmatrix} = \begin{bmatrix} -4.18 \times 10^{-6} \\ -4.18 \times 10^{-6} \\ 8.58 \times 10^{-6} \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{3}$$

Therefore, the changes in L, T, and W for a PSI-5A-S4 piezoceramic can be computed as

$$\Delta L = S_1 L = -4.18 \times 10^{-6} \times 30\text{ mm} = -1.254 \times 10^{-7}\text{ m} \tag{4}$$

$$\Delta W = S_2 W = -4.18 \times 10^{-6} \times 12\text{ mm} = -5.02 \times 10^{-8}\text{ m} \tag{5}$$

$$\Delta T = S_3 T = 8.58 \times 10^{-6} \times 5\text{ mm} = 4.29 \times 10^{-8}\text{ m} \tag{6}$$

Problem 2

Given the following differential equation

$$m\ddot{x} + c\dot{x} + kx = 0 \tag{7}$$

or

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0 \tag{8}$$

where $\omega_n = \sqrt{\frac{k}{m}}$ and $\zeta = \frac{c}{2m\omega_n}$. For initial conditions: $x(0) = x_0$, $\dot{x}(0) = v_0$,

(a) Show the solutions for the following cases in details:

(i) $\zeta = 0$ (undamped):

$$x(t) = A\cos(\omega_n t - \phi) \tag{9}$$

where $A = \sqrt{x_0^2 + \left(\frac{v_0}{\omega_n}\right)}$ and $\phi = \tan^{-1}\frac{v_0}{x_0\omega_n}$.

(ii) $0 < \zeta < 1$ (underdamped):

$$x(t) = Ae^{-\zeta\omega_n t}\cos(\omega_d t - \phi) \tag{10}$$

where $\omega_d = \sqrt{1-\zeta^2}\omega_n$, $A = \sqrt{x_0^2 + \left(\frac{\zeta\omega_n x_0 + v_0}{\omega_d}\right)^2}$, and $\phi = \tan^{-1}\frac{\zeta\omega_n x_0 + v_0}{x_0\omega_d}$.

(iii) $\zeta = 1$ (critically damped):

$$x(t) = [x_0(v_0 + \omega_n x_0)t]e^{-\omega_n t} \tag{11}$$

(iv) $\zeta > 1$ (overdamped):

$$x(t) = C_1 e^{\left(-\zeta+\sqrt{\zeta^2-1}\right)\omega_n t} + C_2 e^{\left(-\zeta-\sqrt{\zeta^2-1}\right)\omega_n t} \tag{12}$$

where

$$C_1 = \frac{x_0\omega_n\left(\zeta + \sqrt{\zeta^2-1}\right) + v_0}{2\omega_n\sqrt{\zeta^2-1}} \tag{13}$$

and

$$C_2 = \frac{-x_0\omega_n\left(\zeta - \sqrt{\zeta^2-1}\right) - v_0}{2\omega_n\sqrt{\zeta^2-1}} \tag{14}$$

(b) Consider the following values of damping ratio:

(1) $\zeta = 0$; (2) $\zeta = 0.1$; (3) $\zeta = 1$; (4) $\zeta = 5$

where $\omega_n = 1.2\pi$ rad/sec, $x_0 = 1.5$ mm, and $v_0 = 2$ mm/sec.

Plot the following three figures (MATLAB is recommended):

(i) $x(t)$ versus t (0~8 sec)

(ii) $\dot{x}(t)$ versus t (0~8 sec)

(iii) $\dot{x}(t)$ versus $x(t)$ (called phase plane)

(c) Discuss the results in part (b)

Solution:

(a)

(i)

Assume that the solution $x(t)$ is of the form (Inman & Singh, 1994)

$$x(t) = ae^{\lambda t} \tag{15}$$

where a and λ are nonzero constants to be determined. Upon successive differentiation, Equation (15) becomes $\dot{x}(t) = \lambda a e^{\lambda t}$ and $\ddot{x}(t) = \lambda^2 a e^{\lambda t}$. Substitution of the assumed exponential form into Equation (8) yields

$$m\lambda^2 a e^{\lambda t} + kae^{\lambda t} = 0 \tag{16}$$

Since the term $ae^{\lambda t}$ is never zero, Equation (16) can be divided by $ae^{\lambda t}$ to yield

$$m\lambda^2 + k = 0 \tag{17}$$

Solving this algebraically results in

$$\lambda = \pm\sqrt{-\frac{k}{m}} = \pm\sqrt{\frac{k}{m}}j = \pm\omega_n j \tag{18}$$

where $j = \sqrt{-1}$ is the imaginary number and $\omega_n = \sqrt{k/m}$ is the natural frequency as before. Note that there are two values for λ, $\lambda = +\omega_n j$ and $\lambda = -\omega_n j$, because the equation for λ is of second order. This implies that there must be two solutions of Equation (8) as well. Substitution of Equation (18) into Equation (15) yields that the two solutions for $x(t)$ are

$$x(t) = a_1 e^{+j\omega_n t} \tag{19}$$

and

$$x(t) = a_2 e^{-j\omega_n t} \tag{20}$$

where a_1 and a_2 are complex-valued constants of integration. The Euler relations for trigonometric functions state that $2\sin\theta = \left(e^{\theta j} - e^{-\theta j}\right)$ and $2\cos\theta = \left(e^{\theta j} + e^{-\theta j}\right)$, where $j = \sqrt{-1}$. Using the Euler relations, Equation (20) can be written as

$$x(t) = A\cos(\omega_n t - \phi) \tag{21}$$

where A and ϕ are real-valued constants of integration. Each set of two constants is determined by the initial conditions, x_0 and v_0:

$$x_0 = x(0) = A\cos(\omega_n 0 - \phi) = A\cos\phi \tag{22}$$

and

$$v_0 = \dot{x}(0) = -\omega_n A\sin(\omega_n 0 - \phi) = \omega_n A\sin\phi \tag{23}$$

Solving these two simultaneous equations for the two unknowns A and ϕ yields

$$A = \sqrt{x_0^2 + \left(\frac{v_0}{\omega_n}\right)} \tag{24}$$

and

$$\phi = \tan^{-1}\frac{v_0}{x_0\omega_n} \tag{25}$$

(ii)

Let $x(t)$ have the form given in Equation (15), $x(t) = ae^{\lambda t}$. Substitution of this form into Equation (8) yields

$$\left(m\lambda^2 + c\lambda + k\right)ae^{\lambda t} = 0 \tag{26}$$

Again, $ae^{\lambda t} \neq 0$, so that this reduces to a quadratic equation in λ of the form

$$m\lambda^2 + c\lambda + k = 0 \tag{27}$$

called the characteristic equation. This is solved using the quadratic formula to yield the two solutions

$$\lambda_{1,2} = -\frac{c}{2m} \pm \frac{1}{2m}\sqrt{c^2 - 4km} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} \tag{28}$$

In this case, the damping ratio ζ is less than 1 ($0 < \zeta < 1$) and the discriminant of Equation (28) is negative, resulting in a complex conjugate pair of roots. Factoring out (-1) from the discriminant in order to clearly distinguish that the second term is imaginary yields

$$\sqrt{\zeta^2 - 1} = \sqrt{(1-\zeta^2)(-1)} = \sqrt{1-\zeta^2}\,j \tag{29}$$

Thus the two roots become

$$\lambda_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{1-\zeta^2}\,j \tag{30}$$

Following the same argument as that made for the undamped response of Equation (20), the solution is then of the form

$$x(t) = e^{-\zeta\omega_n t}\left(a_1 e^{+j\sqrt{1-\zeta^2}\omega_n t} + a_2 e^{-j\sqrt{1-\zeta^2}\omega_n t}\right) \tag{31}$$

where a_1 and a_2 are arbitrary complex-valued constants of integration to be determined by the initial conditions. Using the Euler relations, this can be written as

$$x(t) = Ae^{-\zeta\omega_n t}\cos(\omega_d t - \phi) \tag{32}$$

where A and ϕ are constants of integration and ω_d, called the damped natural frequency, is given by

$$\omega_d = \sqrt{1-\zeta^2}\,\omega_n \tag{33}$$

in units of $\mathrm{rad/s}$. Each set of A and ϕ is determined by the initial conditions, x_0 and v_0:

$$x_0 = x(0) = Ae^{-\zeta\omega_n 0}\cos(\omega_d 0 - \phi) = A\cos\phi \tag{34}$$

Differentiating Equation (32) yields

$$\dot{x}(t) = -\zeta\omega_n A e^{-\zeta\omega_n t}\cos(\omega_d t - \phi) - \omega_d A e^{-\zeta\omega_n t}\sin(\omega_d t - \phi) \tag{35}$$

Let $t = 0$ and $A = x_0/\cos\phi$ in this last expression to get

$$v_0 = \dot{x}(0) = -\zeta\omega_n x_0 + x_0\omega_d \tan\phi \tag{36}$$

Solving this last expression for ϕ yields

$$\tan\phi = \frac{\zeta\omega_n x_0 + v_0}{x_0\omega_d} \tag{37}$$

With this value of ϕ, the cosine becomes

$$\cos\phi = \frac{x_0\omega_d}{\sqrt{(\zeta\omega_n x_0 + v_0)^2 + (x_0\omega_d)^2}} \tag{38}$$

Thus the value of A and ϕ are determined to be

$$A = \sqrt{x_0^2 + \left(\frac{\zeta\omega_n x_0 + v_0}{\omega_d}\right)^2} \tag{39}$$

and

$$\phi = \tan^{-1}\frac{\zeta\omega_n x_0 + v_0}{x_0\omega_d} \tag{40}$$

(iii)

In this last case, the damping ratio is exactly one ($\zeta = 1$) and the discriminant of Equation (28) is equal to zero. This corresponds to the value of ζ that separates oscillatory motion from nonoscillatory motion. Since the roots are repeated, they have the value

$$\lambda_1 = \lambda_2 = -\omega_n \tag{41}$$

The solution takes the form

$$x(t) = (a_1 + a_2 t)\, e^{-\omega_n t} \tag{42}$$

where, again, the constants a_1 and a_2 are determined by the initial conditions. Substituting the initial displacement into Equation (42) and the initial velocity into the derivative of Equation (42) yields

$$a_1 = x_0, \quad a_2 = v_0 + \omega_n x_0 \tag{43}$$

(iv)

In this case, the damping ratio is greater than 1 ($\zeta > 1$). The discriminant of Equation (28) is positive, resulting in a pair of distinct real roots. These are

$$\lambda_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} \tag{44}$$

The solution of Equation (8) then becomes

$$x(t) = C_1 e^{\left(-\zeta+\sqrt{\zeta^2-1}\right)\omega_n t} + C_2 e^{\left(-\zeta-\sqrt{\zeta^2-1}\right)\omega_n t} \tag{45}$$

which represents a nonoscillatory response. Again, the constants of integration C_1 and C_2 are determined by the initial conditions:

$$x_0 = x(0) = C_1 e^{\left(-\zeta+\sqrt{\zeta^2-1}\right)\omega_n 0} + C_2 e^{\left(-\zeta-\sqrt{\zeta^2-1}\right)\omega_n 0} = C_1 + C_2 \tag{46}$$

Differentiating Equation (32) yields

$$\dot{x}(t) = \left(-\zeta + \sqrt{\zeta^2-1}\right)\omega_n C_1 e^{\left(-\zeta+\sqrt{\zeta^2-1}\right)\omega_n t} + \left(-\zeta - \sqrt{\zeta^2-1}\right)\omega_n C_2 e^{\left(-\zeta-\sqrt{\zeta^2-1}\right)\omega_n t} \tag{47}$$

Let $t = 0$ in this last expression to get

$$v_0 = \dot{x}(0) = \left(-\zeta + \sqrt{\zeta^2-1}\right)\omega_n C_1 + \left(-\zeta - \sqrt{\zeta^2-1}\right)\omega_n C_2 \tag{48}$$

Solving Equation (46) and (48) for C_1 and C_2 yields

$$C_1 = \frac{x_0\omega_n\left(\zeta + \sqrt{\zeta^2-1}\right) + v_0}{2\omega_n\sqrt{\zeta^2-1}} \tag{49}$$

and

$$C_2 = \frac{-x_0\omega_n\left(\zeta - \sqrt{\zeta^2-1}\right) - v_0}{2\omega_n\sqrt{\zeta^2-1}} \tag{50}$$

(b) The MATLAB code in the main file is shown below:

```
clc; clf; clear all;
hold on;
t0 = 0; tf = 8;
tspan = [t0 tf];
x0v0 = [2; 1];
[t,x1] = ode45('Q2MotionFunction1', tspan , x0v0);
figure(1);
hold on;
plot(t, x1(:,1),'color',[238, 64, 53]/256,'LineWidth',2.5);
figure(2);
hold on;
plot(t, x1(:,2),'color',[238, 64, 53]/256,'LineWidth',2.5);
[t,x2] = ode45('Q2MotionFunction2', tspan , x0v0);
figure(1);
plot(t, x2(:,1),'color',[128, 0, 128]/256,'LineWidth',2.5);
figure(2);
plot(t, x2(:,2),'color',[128, 0, 128]/256,'LineWidth',2.5);
[t,x3] = ode45('Q2MotionFunction3', tspan , x0v0);
figure(1);
```

```
plot(t, x3(:,1),'color',[123, 192, 67]/256,'LineWidth',2.5);
figure(2);
plot(t, x3(:,2),'color',[123, 192, 67]/256,'LineWidth',2.5);
[t,x4] = ode45('Q2MotionFunction4', tspan , x0v0);
figure(1);
plot(t, x4(:,1),'color',[3, 146, 207]/256,'LineWidth',2.5);
figure(2);
plot(t, x4(:,2),'color',[3, 146, 207]/256,'LineWidth',2.5);
figure(3);
hold on;
plot(x1(:,1), x1(:,2),'color',[238, 64, 53]/256,'LineWidth',2.5);
plot(x2(:,1), x2(:,2),'color',[128, 0, 128]/256,'LineWidth',2.5);
plot(x3(:,1), x3(:,2),'color',[123, 192, 67]/256,'LineWidth',2.5);
plot(x4(:,1), x4(:,2),'color',[3, 146, 207]/256,'LineWidth',2.5);
figure(1);
grid on;
xlabel('$t, \mathrm{\ \left(s\right)}$','interpreter','latex');
ylabel('$x\left(t\right), \mathrm{\ \left(mm\right)}$', ...
    'interpreter','latex');
legend('$\zeta=0$', '$\zeta=0.1$', '$\zeta=1$', '$\zeta=5$', ...
    'interpreter','latex');
a = get(gca,'XTickLabel');
set(gca,'XTickLabel',a,'FontName','Times','fontsize',12);
set(gcf,'renderer','painters');
hold off;
filename = "x_vs_t"+".pdf";
saveas(gcf,filename);
figure(2);
grid on;
xlabel('$t, \mathrm{\ \left(s\right)}$','interpreter','latex');
ylabel('$\dot{x}\left(t\right), \mathrm{\ \left(mm/s\right)}$', ...
    'interpreter','latex');
legend('$\zeta=0$', '$\zeta=0.1$', '$\zeta=1$', '$\zeta=5$', ...
    'interpreter','latex');
a = get(gca,'XTickLabel');
set(gca,'XTickLabel',a,'FontName','Times','fontsize',12);
set(gcf,'renderer','painters');
hold off;
filename = "xdot_vs_t"+".pdf";
saveas(gcf,filename);
figure(3);
grid on;
xlabel('$x\left(t\right), \mathrm{\ \left(mm\right)}$', ...
    'interpreter','latex');
ylabel('$\dot{x}\left(t\right), \mathrm{\ \left(mm/s\right)}$', ...
    'interpreter','latex');
```

```
legend('$\zeta=0$', '$\zeta=0.1$', '$\zeta=1$', '$\zeta=5$', ...
    'interpreter','latex');
a = get(gca,'XTickLabel');
set(gca,'XTickLabel',a,'FontName','Times','fontsize',12);
set(gcf,'renderer','painters');
hold off;
filename = "xdot_vs_x"+".pdf";
saveas(gcf,filename);
```

The MATLAB code in the function that defines the motion dynamics of the spring-damper system is shown below:

Case (1)

```
function xdot = Q2MotionFunction1(t,x)
omega_n = 1.5*pi;
zeta = 0;
xdot(1) = x(2);
xdot(2) = -omega_n^2*x(1)-2*zeta*omega_n*x(2);
xdot = xdot(:);
```

Case (2)

```
function xdot = Q2MotionFunction2(t,x)
omega_n = 1.5*pi;
zeta = 0.1;
xdot(1) = x(2);
xdot(2) = -omega_n^2*x(1)-2*zeta*omega_n*x(2);
xdot = xdot(:);
```

Case (3)

```
function xdot = Q2MotionFunction3(t,x)
omega_n = 1.5*pi;
zeta = 1;
xdot(1) = x(2);
xdot(2) = -omega_n^2*x(1)-2*zeta*omega_n*x(2);
xdot = xdot(:);
```

Case (4)

```
function xdot = Q2MotionFunction4(t,x)
omega_n = 1.5*pi;
zeta = 5;
xdot(1) = x(2);
xdot(2) = -omega_n^2*x(1)-2*zeta*omega_n*x(2);
xdot = xdot(:);
```

(i) The results for $x(t)$ are plotted in Figure 1.

(ii) The results for $\dot{x}(t)$ are plotted in Figure 2.

(iii) The results for phase portraits are plotted in Figure 3.

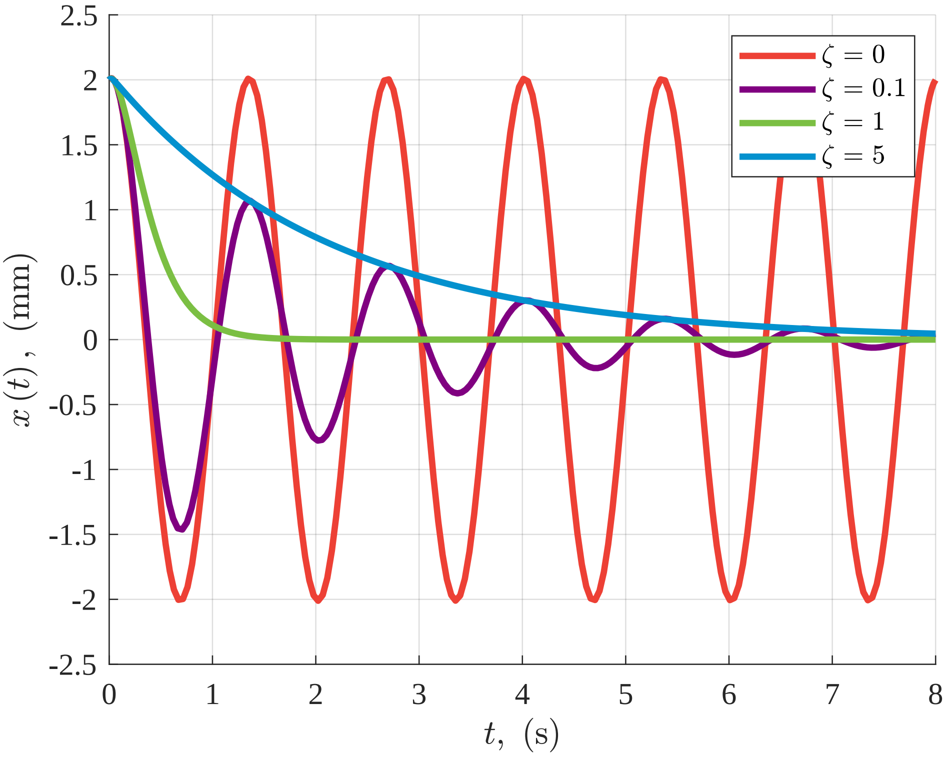


Figure 1: Results for $x(t)$ versus t.

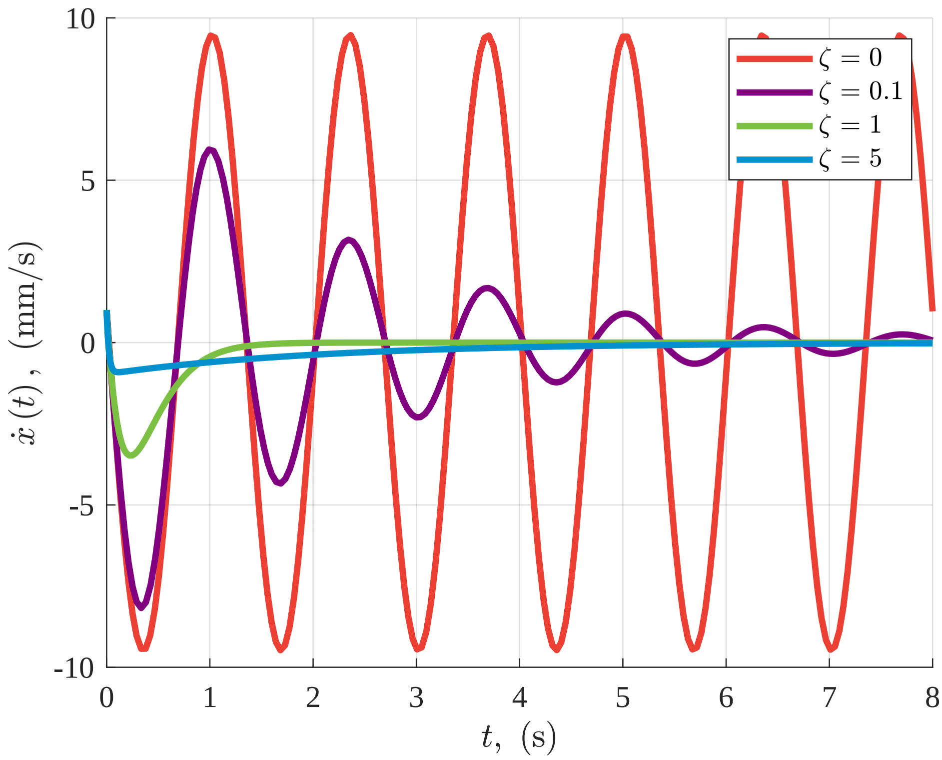


Figure 2: Results for $\dot{x}(t)$ versus t.

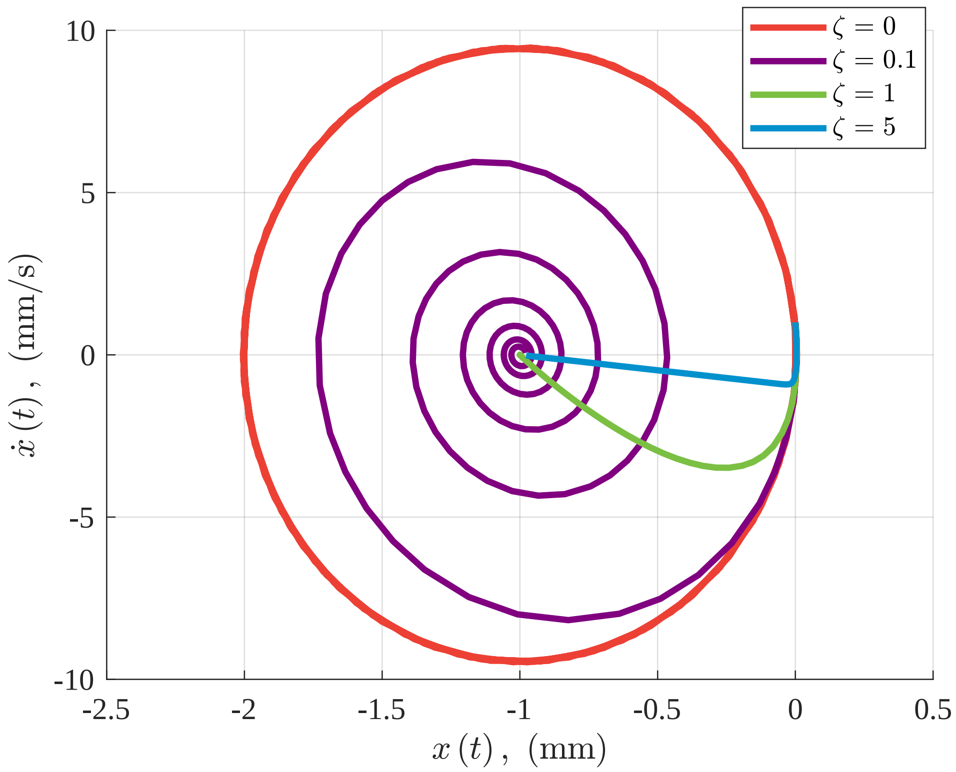


Figure 3: Results for $\dot{x}(t)$ versus $x(t)$ (called phase plane).

(c) The discussion is listed below:

- For undamped case, the system is in the harmonic motion and will never end, which is marginal stable.
- Critical damping returns the system to equilibrium as fast as possible without overshooting.
- An underdamped system will oscillate through the equilibrium position.
- An overdamped system moves more slowly toward equilibrium than one that is critically damped.

Problem 3

For a single degree of freedom damped system under harmonic force, the magnification factor M is found as

$$M = \frac{1}{\sqrt{\left(1 - r^2\right)^2 + (2\zeta r)^2}} \tag{51}$$

where $r = \frac{\omega}{\omega_n}$.
Show the maximum value of M for $0 < \zeta < \frac{1}{\sqrt{2}}$,

$$M_{\max} = \frac{1}{2\zeta\sqrt{1 - \zeta^2}} \tag{52}$$

where $r = \sqrt{1 - 2\zeta^2}$.
Solution:

To get the maximum value of M, we need to minimize $\left(1 - r^2\right)^2 + (2\zeta r)^2$. We can regard $\left(1 - r^2\right)^2 + (2\zeta r)^2$ as a function with respect to r:

$$f(r) = \left(1 - r^2\right)^2 + (2\zeta r)^2 = r^4 + \left(4\zeta^2 - 2\right) r^2 + 1 \tag{53}$$

Because $0 < \zeta < \frac{1}{\sqrt{2}}$, $f(r)$ is a quadratic function with respect to r^2. This function reaches the minimum point at

$$r^2 = -\frac{4\zeta^2 - 2}{2} \tag{54}$$

that is

$$r = \sqrt{1 - 2\zeta^2} \tag{55}$$

In this case, the magnification factor M reaches maximum, which equals

$$M_{\max} = \frac{1}{2\zeta\sqrt{1 - \zeta^2}} \tag{56}$$

References

Inman, D. J. & Singh, R. C. (1994). *Engineering vibration*, volume 3. Prentice Hall Englewood Cliffs, NJ.