香港中文大學
The Chinese University of Hong Kong

THE CHINESE UNIVERSITY OF HONG KONG

DEPARTMENT OF MECHANICAL & AUTOMATION ENGINEERING

MAEG5080 Smart Materials & Structures

Assignment #2

by

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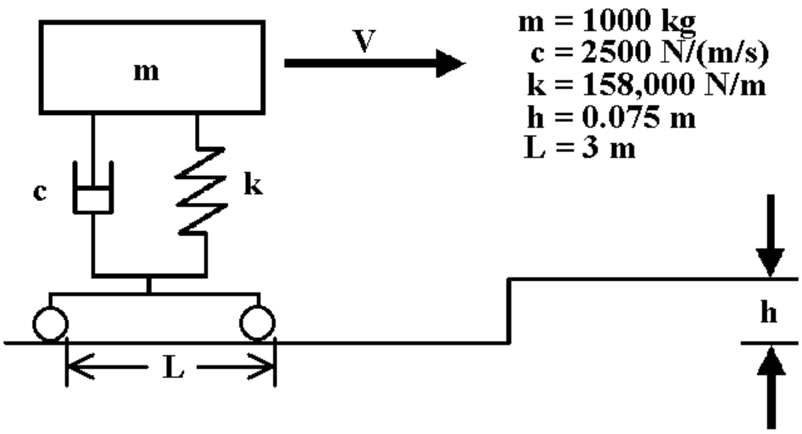
2022-23 Term 1

Problem 1

The vertical dynamics of a two-axle vehicle with a suspension system are modeled as a SDOF system, as shown in the following figure. The vehicle travels to the right at a speed V and encounters a step change in the road height.



(a) Develop the equation of motion for this "base-driven" system.

(b) Develop an expression for the vertical motion of the vehicle following its encounter with the road height change. Use nominal speed V = 24 m/s, plot this motion as a function of time. Discuss your findings.

(c) Possibly computer-assisted: What is the maximum height attainable by the vehicle at any speed? At what speed(s) is this achieved? At what speed(s) will the residual suspension motion (when both wheels are on the higher part of the road) be minimized?

Solution:

(a) The free-body diagram of m is shown on the right side of Figure 1. The force exerted by the spring k on the mass m is downward as it tends to restore to the undeformed position. Note that the gravitational force, mg, is not included in the free-body diagrams.

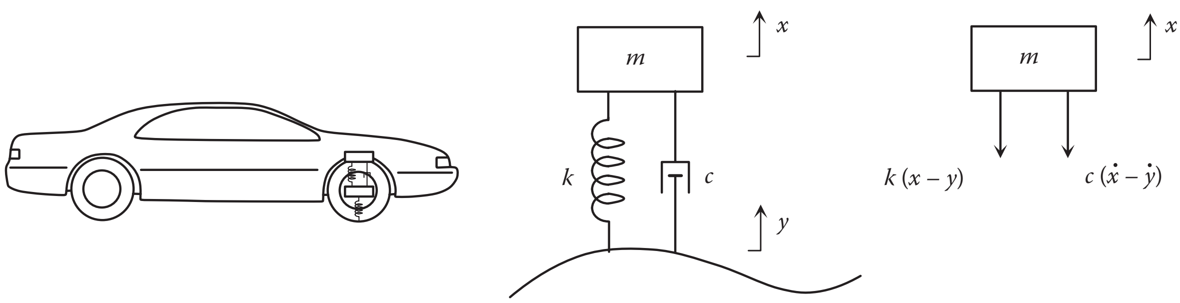


Figure 1: Simplified Suspension System of Car Model.

Applying Newton's second law to the mass m gives

$$+\uparrow x: \sum F_x = ma_x \tag{1}$$

$$k(x-y)+c(\dot{x}-\dot{y}) = m_2\ddot{x}_2 \tag{2}$$

Rearranging the equations into the standard input–output form,

$$m\ddot{x}+c\dot{x}+kx = c\dot{y}+ky \tag{3}$$

(b) When the vehicle encounters a step change in the road height, the equation of motion for this "base-driven" system can be expressed as

$$m\ddot{x}+c\dot{x}+kx = \frac{ch}{2}\delta\left(t\right)+\frac{ch}{2}\delta\left(t-\frac{L}{V}\right)+\frac{kh}{2}u\left(t\right)+\frac{kh}{2}u\left(t-\frac{L}{V}\right) \tag{4}$$

where $\delta\left(t-a\right)$ is the Dirac Delta Function, and $u\left(t-a\right)$ is the Heaviside Function. Taking the Laplace transform to Equation (4) with the initial condition $x\left(0\right)=0$ and $\dot{x}\left(0\right)=0$ yields that

$$ms^2X\left(s\right)+csX\left(s\right)+kX\left(s\right) = \frac{ch}{2}+\frac{ch}{2}e^{-\frac{L}{V}s}+\frac{kh}{2}\frac{1}{s}+\frac{kh}{2}\frac{e^{-\frac{L}{V}s}}{s} \tag{5}$$

Solving the Laplace transform in Equation (5) gets that

$$\begin{aligned} x\left(t\right) =& \frac{3u\left(t-\frac{1}{8}\right)}{80} \\ &+ \frac{3\sqrt{2503}u\left(t-\frac{1}{8}\right)e^{-\frac{5}{4}t+\frac{5}{32}}\sin\left(\frac{\sqrt{2503}t}{4}-\frac{\sqrt{2503}}{32}\right)}{40048} \\ &- \frac{3u\left(t-\frac{1}{8}\right)e^{-\frac{5}{4}t+\frac{5}{32}}\cos\left(\frac{\sqrt{2503}t}{4}-\frac{\sqrt{2503}}{32}\right)}{40048} \\ &+ \frac{3\sqrt{2503}e^{-\frac{5}{4}t}\sin\frac{\sqrt{2503}t}{4}}{40048} - \frac{3e^{-\frac{5}{4}t}\cos\frac{\sqrt{2503}t}{4}}{40048} + \frac{3}{80} \end{aligned} \tag{6}$$

The plot of this motion as a function of time is shown in Figure 2. From the system response shown in Figure 2, we can see that when the vehicle encounters a step change in the road height, the system is underdamped and the steady time is about 2 seconds.

(c) I use the following MATLAB codes to find the answer to this question:

```
clc; clf; clear all;
syms s t V
m = 1000; c = 2500; k = 158000; h = 0.075; L = 3;
f1 = ilaplace(c*h/2*1/(m*s^2+c*s+k),t);
f2 = ilaplace(c*h/2*exp(-L/V*s)/(m*s^2+c*s+k),t);
f3 = ilaplace(k*h/2*1/s*1/(m*s^2+c*s+k),t);
f4 = ilaplace(k*h/2*1/s*exp(-L/V*s)/(m*s^2+c*s+k),t);
```

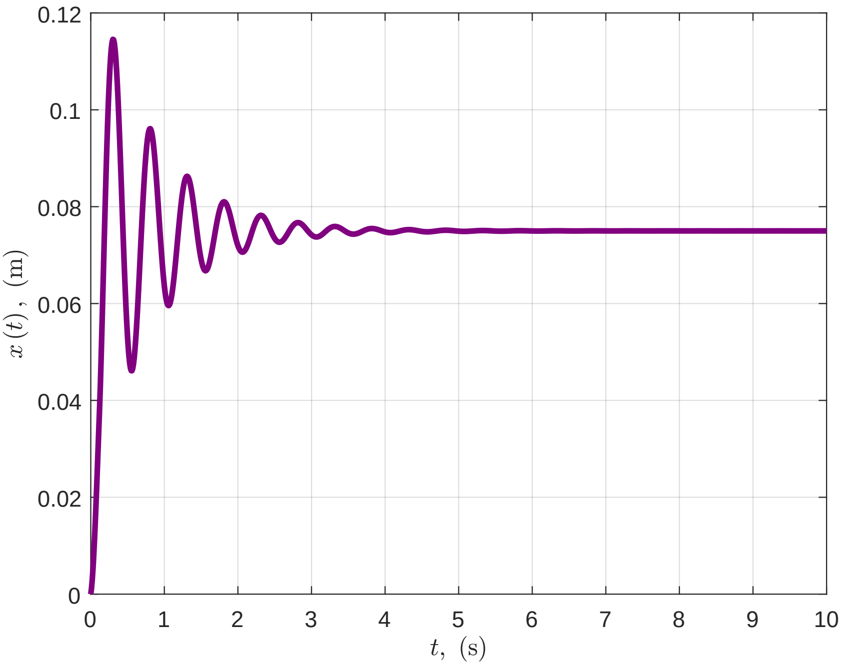


Figure 2: System motion diagram.

```
f(t,V) = f1+f2+f3+f4;
figure;
fplot(f,'color',[128, 0, 128]/256,'LineWidth',2.5)
xlim([0 10]);
grid on;
xlabel('$t, \mathrm{\ \left(s\right)}$','interpreter','latex');
ylabel('$x\left(t\right), \mathrm{\ \left(m\right)}$', ...
    'interpreter','latex');
a = get(gca,'XTickLabel');
set(gca,'XTickLabel',a,'FontName','Times','fontsize',12);
set(gcf,'renderer','painters');
filename = "Q2-1-Motion"+".pdf";
saveas(gcf,filename);
figure;
V = 0.01:0.01:3000;
sv = size(V);
Mx = zeros(1,sv(2));
for i = 1:sv(2)
    t = [0:0.01:2 3/V(i):0.01:(3/V(i)+2)];
    Mx(i) = max(f(t,V(i)));
end
plot(V,Mx,'color',[128, 0, 128]/256,'LineWidth',2.5);
grid on;
xlabel('$t, \mathrm{\ \left(s\right)}$','interpreter','latex');
ylabel('$x\left(t\right), \mathrm{\ \left(m\right)}$', ...
    'interpreter','latex');
a = get(gca,'XTickLabel');
```

```
set(gca,'XTickLabel',a,'FontName','Times','fontsize',12);
set(gcf,'renderer','painters');
filename = "Q2-1-Motion"+".pdf";
saveas(gcf,filename);
```

The results gotten from MATLAB are shown in Figure 3.

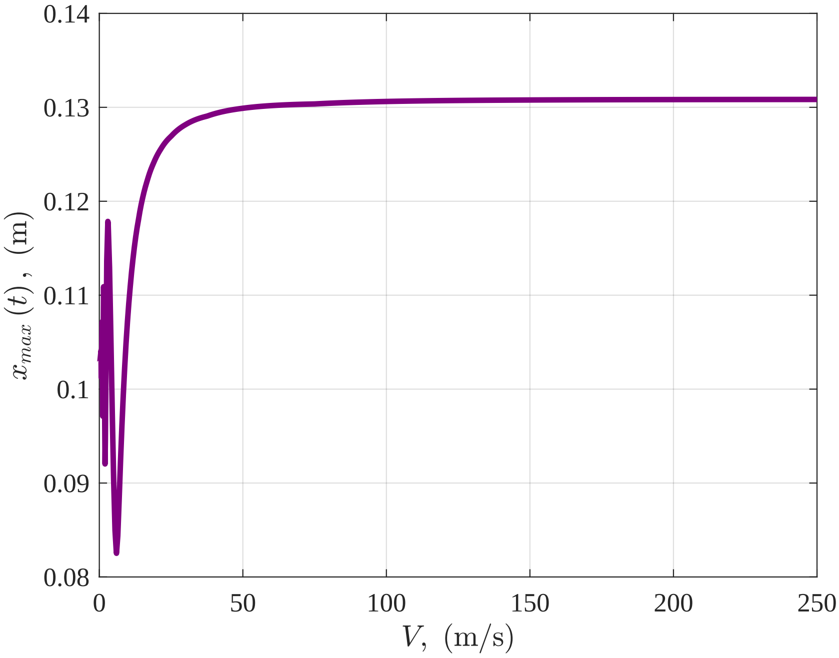


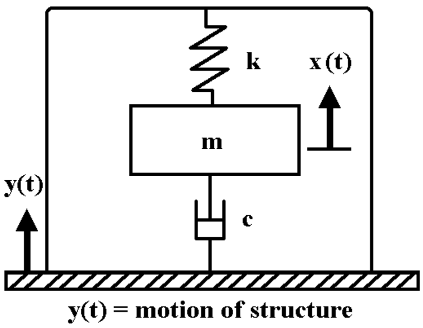
Figure 3: The maximum displacement versus the velocity.

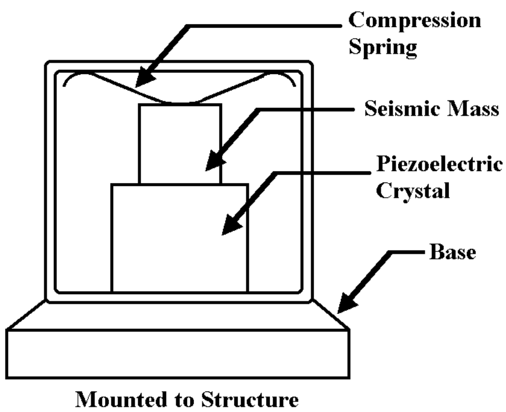
From Figure 3, it can be concluded that

- The maximum height attainable by the vehicle at any speed is 0.1309 m, with the velocity equal to 2560 m/s.
- The residual suspension motion (when both wheels are on the higher part of the road) is minimized when the velocity is equal to 11.9450 m/s. At this speed, the minimum displacement is equal to 0.0825 m

Problem 2

An accelerometer has a suspended mass of 0.01 kg with a damped natural frequency of vibration of 150 Hz. When mounted on an engine undergoing an acceleration of 1 g at an operating speed of 6000 rpm, the acceleration is recorded as 9.5 m/s2 by the instrument. Find the damping constant and the spring stiffness of the accelerometer (Choose damping ratio close to 0.7 if possible).





Solution:

The amplitude ratio of the accelerometer is equal to (Inman & Singh, 1994)

$$M = \frac{g_m}{g_t} = \frac{1}{\sqrt{\left(1 - r^2\right)^2 + (2\zeta r)^2}} = \frac{9.5}{9.8} \tag{7}$$

In addition, the forcing function frequency and the damped natural frequency of the system are equal to

$$\omega = \frac{2\pi \times 6000}{60} = 200\pi \text{ rad/s} \tag{8}$$

and

$$\omega_d = \omega_n \sqrt{1 - \zeta^2} = 2\pi f = 300\pi \text{ rad/s} \tag{9}$$

Solving Equation (7), (8), and (9) to obtain ζ and ω_n, we can know that $\zeta = 0.72$ and $\omega_n = 1363.9 \text{ rad/s}$. Therefore, the damping constant and the spring stiffness of the accelerometer are equal to

$$c = 2m\zeta\omega_n = 19.72 \text{ N/(m/s)} \tag{10}$$

and

$$k = m\omega_n^2 = 1.86 \times 10^4 \text{ N/m} \tag{11}$$

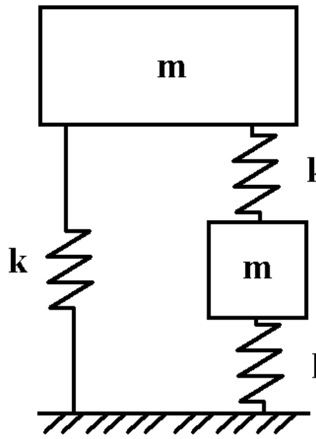
Problem 3

A spring-mass system (see figure), which is constrained to move in the vertical direction only.

(a) Derive the equations of motion.

(b) Find the natural frequencies.

(c) Find and sketch the mode shapes.



Solution:

(a) We set the coordinate for the upper block as x_1 and the lower block as x_2. Then, the equations of motion is derived as follows:

$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{12}$$

(b) We can use determinate to calculate the natural frequencies.

$$\begin{aligned} \det\left([k] - [m]\,\omega^2\right) = 0 &\Rightarrow \left(2k - m\omega^2\right)\left(2k - m\omega^2\right) - k^2 = 0 \\ &\Rightarrow m^2\omega^4 - 4km\omega^2 + 3k^2 = 0 \\ &\Rightarrow \omega_1 = \sqrt{\frac{k}{m}}, \omega_2 = \sqrt{\frac{3k}{m}} \end{aligned} \tag{13}$$

(c) For Mode 1 ($\omega_1 = \sqrt{\frac{k}{m}}$),

$$\begin{bmatrix} 2k - m\omega^2 & -k \\ -k & 2k - m\omega^2 \end{bmatrix} \begin{bmatrix} u_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow u_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{14}$$

For Mode 2 ($\omega_1 = \sqrt{\frac{3k}{m}}$),

$$\begin{bmatrix} 2k - m\omega^2 & -k \\ -k & 2k - m\omega^2 \end{bmatrix} \begin{bmatrix} u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow u_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \tag{15}$$

Based on this, the mode shapes are drawn as shown in Figure 4.

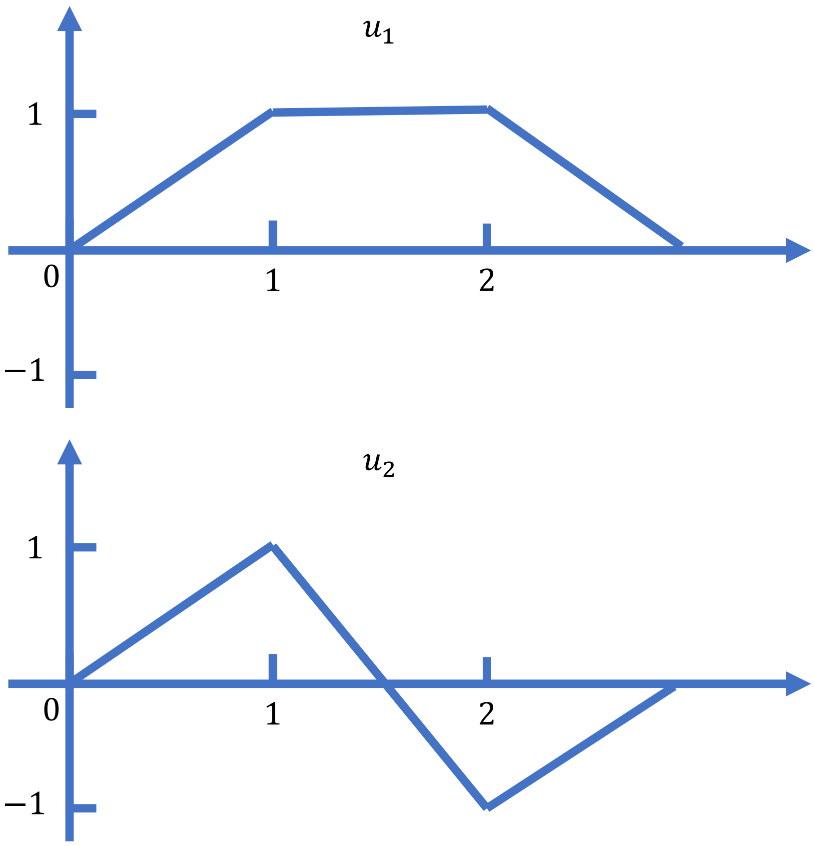


Figure 4: Mode shapes for the system.

References

Inman, D. J. & Singh, R. C. (1994). *Engineering vibration*, volume 3. Prentice Hall Englewood Cliffs, NJ.