香港中文大學
The Chinese University of Hong Kong

THE CHINESE UNIVERSITY OF HONG KONG

DEPARTMENT OF MECHANICAL & AUTOMATION ENGINEERING

MAEG5080 Smart Materials & Structures

Assignment #4

by

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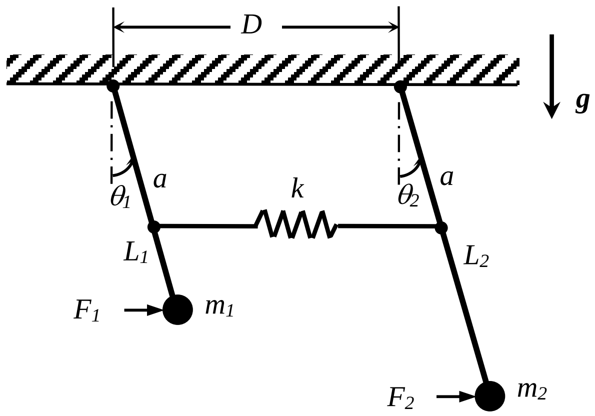
Problem 1

(40 points) Two plane pendulums, masses m_1 and m_2, are connected by respective massless rigid links L_1 and L_2 shown in the figure. Those two links are coupled via a spring of stiffness k at a distance a from the supports. The spring is unstretched when the links are vertical. The pendulums are respectively excited by external forces F_1(t) and F_2(t), which remain horizontal at all times. Derive the equations of motion for the system. Assume small motions on the plane.



Solution:

In terms of generalized coordinate q, the Lagrange's equation subject to a generalized force has the form

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}}\right)-\frac{\partial T}{\partial q}+\frac{\partial D}{\partial \dot{q}}+\frac{\partial U}{\partial q}=Q \tag{1}$$

Kinetic energy:

$$T=\frac{1}{2}m_1\left(L_1\dot{\theta}_1\right)^2+\frac{1}{2}m_2\left(L_2\dot{\theta}_2\right)^2 \tag{2}$$

Potential energy:

$$U=m_1gL_1\left(1-\cos\theta_1\right)+m_2gL_2\left(1-\cos\theta_2\right)+\frac{1}{2}k\left(2a\sin\left(\frac{\theta_1-\theta_2}{2}\right)\right)^2 \tag{3}$$

Rayleigh's damping (or dissipation) function:

$$D=0 \tag{4}$$

Generalized force:

$$Q=\sum_l F_l\cdot\frac{\partial r_l}{\partial q}=F_iL_i\cos\theta_i,\quad i=1,2 \tag{5}$$

For $q_1=\theta_1$,

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_1}\right)-\frac{\partial T}{\partial q_1}+\frac{\partial D}{\partial \dot{q}_1}+\frac{\partial U}{\partial q_1}=Q \tag{6}$$

$$\begin{aligned} &\frac{d}{dt}\left(m_1 L_1^2 \dot{\theta}_1\right) - 0 + 0 + m_1 g L_1 \sin\theta_1 \\ &\quad + \frac{1}{2} \times 2k\left(2a\sin\left(\frac{\theta_1-\theta_2}{2}\right)\right) \times 2a \times \frac{1}{2}\cos\left(\frac{\theta_1-\theta_2}{2}\right) = F_1 L_1 \cos\theta_1 \end{aligned} \tag{7}$$

$$m_1 L_1^2 \ddot{\theta}_1 + m_1 g L_1 \sin\theta_1 + k a^2 \sin\left(\theta_1 - \theta_2\right) = F_1 L_1 \cos\theta_1 \tag{8}$$

For $q_2 = \theta_2$,

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_2}\right) - \frac{\partial T}{\partial q_2} + \frac{\partial D}{\partial \dot{q}_2} + \frac{\partial U}{\partial q_2} = Q \tag{9}$$

$$\begin{aligned} &\frac{d}{dt}\left(m_2 L_2^2 \dot{\theta}_2\right) - 0 + 0 + m_2 g L_2 \sin\theta_2 \\ &\quad + \frac{1}{2} \times 2k\left(2a\sin\left(\frac{\theta_1-\theta_2}{2}\right)\right) \times 2a \times \left(-\frac{1}{2}\cos\left(\frac{\theta_1-\theta_2}{2}\right)\right) = F_2 L_2 \cos\theta_2 \end{aligned} \tag{10}$$

$$m_2 L_2^2 \ddot{\theta}_2 + m_2 g L_2 \sin\theta_2 - k a^2 \sin\left(\theta_1 - \theta_2\right) = F_2 L_2 \cos\theta_2 \tag{11}$$

Therefore, the equations of motion for the system is

$$\begin{cases} m_1 L_1^2 \ddot{\theta}_1 + m_1 g L_1 \sin\theta_1 + k a^2 \sin\left(\theta_1 - \theta_2\right) = F_1 L_1 \cos\theta_1 \\ m_2 L_2^2 \ddot{\theta}_2 + m_2 g L_2 \sin\theta_2 - k a^2 \sin\left(\theta_1 - \theta_2\right) = F_2 L_2 \cos\theta_2 \end{cases} \tag{12}$$

For the small motions on the plane, the equations of motion for the system is

$$\begin{cases} m_1 L_1^2 \ddot{\theta}_1 + m_1 g L_1 \theta_1 + k a^2 \left(\theta_1 - \theta_2\right) = F_1 L_1 \cos\theta_1 \\ m_2 L_2^2 \ddot{\theta}_2 + m_2 g L_2 \theta_2 - k a^2 \left(\theta_1 - \theta_2\right) = F_2 L_2 \cos\theta_2 \end{cases} \tag{13}$$

Problem 2

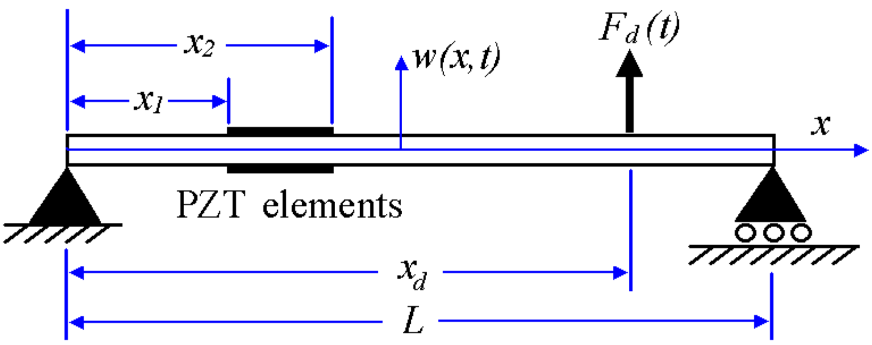
(60 points) Consider a simply-supported uniform beam with PZT actuators mounted on its top and bottom surfaces between $x = x_1$ and $x = x_2$. The patches are activated so as to produce pure bending in the beam. A discrete force $F_d(t)$ has been applied at $x = x_d$.

(a) Derive the partial differential equation for the transverse beam response $w(x,t)$ using extended Hamilton's Principle.

(b) Applying Galerkin's method with comparison functions (Hint: you may use $\phi_r(x) = \sin(r\pi x/L)$), determine the discretized ordinary differential equations (use three expansion terms, i.e., N = 3). Assume the damping matrix $C = \alpha M + \beta K$, where $\alpha = 0.6$ and $\beta = 1.2\times10^{-6}$.

(c) Changing the second order differential equations into the state space form, then use state feedback $u = -K_c x$ for the system, where control gain is given as Kc = [-55400 -22549 15848 -753 -249 174]. Under an impulse excitation for $F_d(t)$with magnitude 1/100 N.sec, plot the time response of transverse displacement at $x = 0.6L$ for the cases without and with control (0 to 0.5 sec). Also plot the corresponding voltage for the controlled case (0 to 0.5 sec).

The rectangular cross sections of the beam and PZT are given as: width b = 2 cm, thickness t_b = 2 mm, and t_p = 0.6 mm, respectively. Other parameters are given as follows: L = 50 cm, x_1 = 15 cm, x_2 = 24 cm, x_d = 30 cm, $d_{31} = -175\times10^{-12}$ m/V, $\rho_b = 2700$ kg/m^3, $\rho_p = 7600$ kg/m^3, $E_b = 7\times10^{10}$ N/m^2, $E_p = 6.5\times10^{10}$ N/m^2.



Solution:

(a) Potential energies:

$$V_b = \frac{1}{2}\int_0^L E_b I_b \left(\frac{\partial^2 w}{\partial x^2}\right)^2 dx \tag{14}$$

$$V_p = \int_0^L E_p I_p \left(\frac{\partial^2 w}{\partial x^2}\right)^2 \left[H(x - x_1) - H(x - x_2)\right] dx \tag{15}$$

where H is the Heaviside's function.

Kinetic energies:

$$T_b = \frac{1}{2}\int_0^L \rho_b A_b \left(\frac{\partial w}{\partial t}\right)^2 dx \tag{16}$$

$$T_p = \int_0^L \rho_p A_p \left(\frac{\partial w}{\partial t}\right)^2 \left[H\left(x-x_1\right) - H\left(x-x_2\right)\right] dx \tag{17}$$

Virtual work:

$$\delta W_d = \int_0^L f\left(x,t\right)\delta w\left(x,t\right) dx \tag{18}$$

From the constitutive equation of the piezoelectric materials

$$S_1 = s_{11}^E T_1 + d_{31} E_3 \tag{19}$$

$$T_1 = E_p \left(S_1 - d_{31} E_3\right) \tag{20}$$

where $E_p = \frac{1}{s_{11}^E}$, $E_3 = \frac{v(t)}{t_p}$.

The virtual work done by the induced strain (force) is:

$$\delta W_p = 2\int_0^L E_p d_{31} b v\left(t\right)\delta\left(\frac{\partial u_p}{\partial x}\right)\left[H\left(x-x_1\right) - H\left(x-x_2\right)\right] dx \tag{21}$$

where b is the width of beam and piezo layer. and

$$u_p = -\left(\frac{t_b + t_p}{2}\right)\frac{\partial w}{\partial x} \tag{22}$$

Let $a = \frac{t_b + t_p}{2}$,

$$\delta W_p = -2\int_0^L E_p d_{31} a b v\left(t\right)\delta\left(\frac{\partial^2 w}{\partial x^2}\right)\left[H\left(x-x_1\right) - H\left(x-x_2\right)\right] dx \tag{23}$$

Apply extended Hamilton's principle:

$$\int_{t_1}^{t_2} \left(\delta T - \delta V + \delta W_{NC}\right) dt = 0 \tag{24}$$

$$\begin{aligned}
&\int_{t_1}^{t_2}\left(\delta\left\{\frac{1}{2}\int_0^L \rho_b A_b\left(\frac{\partial w}{\partial t}\right)^2 dx\right\} + \delta\left\{\int_0^L \rho_p A_p\left(\frac{\partial w}{\partial t}\right)^2\left[H\left(x-x_1\right) - H\left(x-x_2\right)\right] dx\right\}\right) dt \\
&- \int_{t_1}^{t_2}\left(\delta\left\{\frac{1}{2}\int_0^L E_b I_b\left(\frac{\partial^2 w}{\partial x^2}\right)^2 dx\right\} + \delta\left\{\int_0^L E_p I_p\left(\frac{\partial^2 w}{\partial x^2}\right)^2\left[H\left(x-x_1\right) - H\left(x-x_2\right)\right] dx\right\}\right) dt \\
&+ \int_{t_1}^{t_2}\left(\int_0^L f\left(x,t\right)\delta w\left(x,t\right) dx - 2\int_0^L E_p d_{31} a b v\left(t\right)\delta\left(\frac{\partial^2 w}{\partial x^2}\right)\left[H\left(x-x_1\right) - H\left(x-x_2\right)\right] dx\right) dt = 0
\end{aligned} \tag{25}$$

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$$\begin{aligned}
&\int_{t_1}^{t_2}\delta\left\{\frac{1}{2}\int_0^L \rho_b A_b\left(\frac{\partial w}{\partial t}\right)^2 dx\right\}dt \\
&\quad=\int_0^L\int_{t_1}^{t_2}\rho_b A_b\frac{\partial w}{\partial t}\delta\left(\frac{\partial w}{\partial t}\right)dtdx \\
&\quad=\int_0^L\int_{t_1}^{t_2}\rho_b A_b\frac{\partial w}{\partial t}\frac{\partial\left(\delta w\right)}{\partial t}dtdx \\
&\quad=\int_0^L\left(\rho_b A_b\frac{\partial w}{\partial t}\right)\delta w|_{t_1}^{t_2}dx-\int_0^L\int_{t_1}^{t_2}\frac{\partial}{\partial t}\left(\rho_b A_b\frac{\partial w}{\partial t}\right)\delta w dtdx \\
&\quad=-\int_0^L\int_{t_1}^{t_2}\rho_b A_b\frac{\partial^2 w}{\partial t^2}\delta w dtdx
\end{aligned} \tag{26}$$

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$$\begin{aligned}
&\int_{t_1}^{t_2}\delta\left\{\frac{1}{2}\int_0^L E_b I_b\left(\frac{\partial^2 w}{\partial x^2}\right)^2 dx\right\}dt \\
&\quad=\int_{t_1}^{t_2}\int_0^L\frac{1}{2}E_b I_b\cdot 2\left(\frac{\partial^2 w}{\partial x^2}\right)\delta\left(\frac{\partial^2 w}{\partial x^2}\right)dxdt \\
&\quad=\int_{t_1}^{t_2}\left(E_b I_b\frac{\partial^2 w}{\partial x^2}\right)\delta\left(\frac{\partial w}{\partial x}\right)\Bigg|_0^L dt-\int_{t_1}^{t_2}\int_0^L\frac{\partial}{\partial x}\left(E_b I_b\frac{\partial^2 w}{\partial x^2}\right)\frac{\partial\left(\delta w\right)}{\partial x}dxdt \\
&\quad=\int_{t_1}^{t_2}\left(E_b I_b\frac{\partial^2 w}{\partial x^2}\right)\delta\left(\frac{\partial w}{\partial x}\right)\Bigg|_0^L dt-\int_{t_1}^{t_2}\frac{\partial}{\partial x}\left(E_b I_b\frac{\partial^2 w}{\partial x^2}\right)\delta w|_0^L dt \\
&\qquad\qquad+\int_{t_1}^{t_2}\int_0^L\frac{\partial^2}{\partial x^2}\left(E_b I_b\frac{\partial^2 w}{\partial x^2}\right)\delta w dxdt \\
&\quad=\int_{t_1}^{t_2}E_b I_b\frac{\partial^2 w}{\partial x^2}\delta\left(\frac{\partial w}{\partial x}\right)\Bigg|_0^L dt-\int_{t_1}^{t_2}E_b I_b\frac{\partial^3 w}{\partial x^3}\delta w|_0^L dt \\
&\qquad\qquad+\int_{t_1}^{t_2}\int_0^L E_b I_b\frac{\partial^4 w}{\partial x^4}\delta w dxdt \\
&\quad=\int_{t_1}^{t_2}\int_0^L E_b I_b\frac{\partial^4 w}{\partial x^4}\delta w dxdt
\end{aligned} \tag{27}$$

- Similar to the derivation of Equation (26), the second term in Equation (25) can be degenerated into

$$\begin{aligned}
&\int_{t_1}^{t_2}\delta\left\{\int_0^L\rho_p A_p\left(\frac{\partial w}{\partial t}\right)^2\left[H\left(x-x_1\right)-H\left(x-x_2\right)\right]dx\right\}dt \\
&\quad=-2\int_0^L\int_{t_1}^{t_2}\rho_p A_p\frac{\partial^2 w}{\partial t^2}\left[H\left(x-x_1\right)-H\left(x-x_2\right)\right]\delta w dtdx
\end{aligned} \tag{28}$$

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\begin{aligned}
-\int_{t_1}^{t_2} \delta \left\{ \int_0^L E_p I_p \left(\frac{\partial^2 w}{\partial x^2} \right)^2 \left[H\left(x-x_1\right) - H\left(x-x_2\right) \right] dx \right\} dt \\
&= -\int_{t_1}^{t_2} \int_0^L E_p I_p \cdot 2 \left(\frac{\partial^2 w}{\partial x^2} \right) \left[H\left(x-x_1\right) - H\left(x-x_2\right) \right] \delta \left(\frac{\partial^2 w}{\partial x^2} \right) dxdt \\
&= -2 \int_{t_1}^{t_2} \left(E_p I_p \frac{\partial^2 w}{\partial x^2} \left[H\left(x-x_1\right) - H\left(x-x_2\right) \right] \right) \delta \left(\frac{\partial w}{\partial x} \right) \Bigg|_0^L dt \\
&\quad + 2 \int_{t_1}^{t_2} \int_0^L \frac{\partial}{\partial x} \left(E_p I_p \frac{\partial^2 w}{\partial x^2} \left[H\left(x-x_1\right) - H\left(x-x_2\right) \right] \right) \frac{\partial\left(\delta w\right)}{\partial x} dxdt \\
&= -2 \int_{t_1}^{t_2} E_p I_p \left(\frac{\partial^2 w}{\partial x^2} \right) \left[H\left(x-x_1\right) - H\left(x-x_2\right) \right] \delta \left(\frac{\partial w}{\partial x} \right) \Bigg|_0^L dt \\
&\quad + 2 \int_{t_1}^{t_2} \int_0^L E_p I_p \left(\frac{\partial^3 w}{\partial x^3} \right) \left[H\left(x-x_1\right) - H\left(x-x_2\right) \right] \delta \left(\frac{\partial w}{\partial x} \right) dxdt \\
&\quad + 2 \int_{t_1}^{t_2} \int_0^L E_p I_p \left(\frac{\partial^2 w}{\partial x^2} \right) \left[H'\left(x-x_1\right) - H'\left(x-x_2\right) \right] \delta \left(\frac{\partial w}{\partial x} \right) dxdt \\
&= 2 \int_{t_1}^{t_2} E_p I_p \left(\frac{\partial^3 w}{\partial x^3} \right) \left[H\left(x-x_1\right) - H\left(x-x_2\right) \right] \delta w|_0^L dt \\
&\quad - 2 \int_{t_1}^{t_2} \int_0^L E_p I_p \left(\frac{\partial^4 w}{\partial x^4} \right) \left[H\left(x-x_1\right) - H\left(x-x_2\right) \right] \delta w dxdt \\
&\quad - 2 \int_{t_1}^{t_2} \int_0^L E_p I_p \left(\frac{\partial^3 w}{\partial x^3} \right) \left[H'\left(x-x_1\right) - H'\left(x-x_2\right) \right] \delta w dxdt \\
&\quad + 2 \int_0^L E_p I_p \left(\frac{\partial^2 w}{\partial x^2} \right) \left[H'\left(x-x_1\right) - H'\left(x-x_2\right) \right] \delta w|_0^L dt \\
&\quad - 2 \int_{t_1}^{t_2} \int_0^L E_p I_p \left(\frac{\partial^3 w}{\partial x^3} \right) \left[H'\left(x-x_1\right) - H'\left(x-x_2\right) \right] \delta w dxdt \\
&\quad - 2 \int_{t_1}^{t_2} \int_0^L E_p I_p \left(\frac{\partial^2 w}{\partial x^2} \right) \left[H''\left(x-x_1\right) - H''\left(x-x_2\right) \right] \delta w dxdt
\end{aligned} \tag{29}
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\begin{aligned}
-\int_{t_1}^{t_2} 2 \int_0^L E_p d_{31} a b v\left(t\right) \delta \left(\frac{\partial^2 w}{\partial x^2} \right) \left[H\left(x-x_1\right) - H\left(x-x_2\right) \right] dxdt \\
&= -2 \int_{t_1}^{t_2} \int_0^L E_p d_{31} a b v\left(t\right) \left[H''\left(x-x_1\right) - H''\left(x-x_2\right) \right] \delta w dxdt
\end{aligned} \tag{30}
$$

Substituting Equation (26)(27)(28)(29)(30) into Equation (25) yields that

$$\begin{aligned}\int_{t_1}^{t_2}\Bigg\{\int_0^L \Bigg(&-\rho_b A_b\left(\frac{\partial^2 w}{\partial t^2}\right)-2\rho_p A_p\left(\frac{\partial^2 w}{\partial t^2}\right)\left[H\left(x-x_1\right)-H\left(x-x_2\right)\right]\\&-E_b I_b\left(\frac{\partial^4 w}{\partial x^4}\right)-2E_p I_p\left(\frac{\partial^4 w}{\partial x^4}\right)\left[H\left(x-x_1\right)-H\left(x-x_2\right)\right]\\&-4E_p I_p\left(\frac{\partial^3 w}{\partial x^3}\right)\left[H'\left(x-x_1\right)-H'\left(x-x_2\right)\right]\\&-2E_p I_p\left(\frac{\partial^2 w}{\partial x^2}\right)\left[H''\left(x-x_1\right)-H''\left(x-x_2\right)\right]+f\left(x,t\right)\\&-2E_p d_{31}abv\left(t\right)\left[H''\left(x-x_1\right)-H''\left(x-x_2\right)\right]\Bigg)\delta w dx\\&-E_b I_b\left(\frac{\partial^2 w}{\partial x^2}\right)\delta\left(\frac{\partial w}{\partial x}\right)\Bigg|_0^L+E_b I_b\left(\frac{\partial^3 w}{\partial x^3}\right)\delta w\Big|_0^L\Bigg\}dt=0\end{aligned}\tag{31}$$

For arbitrary δw in $0 < x < L$, the equation of motion is

$$\begin{aligned}\rho_b A_b\left(\frac{\partial^2 w}{\partial t^2}\right)&+\left[2\rho_p A_p\left(\frac{\partial^2 w}{\partial t^2}\right)+2E_p I_p\left(\frac{\partial^4 w}{\partial x^4}\right)\right]\left[H\left(x-x_1\right)-H\left(x-x_2\right)\right]\\&+E_b I_b\left(\frac{\partial^4 w}{\partial x^4}\right)+4E_p I_p\left(\frac{\partial^3 w}{\partial x^3}\right)\left[H'\left(x-x_1\right)-H'\left(x-x_2\right)\right]\\&+2E_p I_p\left(\frac{\partial^2 w}{\partial x^2}\right)\left[H''\left(x-x_1\right)-H''\left(x-x_2\right)\right]\\&+2E_p d_{31}abv\left(t\right)\left[H''\left(x-x_1\right)-H''\left(x-x_2\right)\right]=f\left(x,t\right)\end{aligned}\tag{32}$$

with boundary conditions

At $x = 0$, $\delta w = 0$, $\frac{\partial^2 w}{\partial x^2} = 0$

At $x = L$, $\delta w = 0$, $\frac{\partial^2 w}{\partial x^2} = 0$

(b) Assume

$$w\left(x,t\right)=\sum_{i=1}^{n}\phi_i\left(x\right)q_i\left(t\right)\tag{33}$$

where $\phi_i\left(x\right)$ satisfies all boundary conditions.

Substituting Equation (33) into Equation (32) yields that

$$\begin{aligned}
&\rho_b A_b \sum_{i=1}^{n} \phi_i(x)\ddot{q}_i(t) + E_b I_b \sum_{i=1}^{n} \phi_i^{(4)}(x) q_i(t) \\
&\quad + \left[2\rho_p A_p \sum_{i=1}^{n} \phi_i(x)\ddot{q}_i(t) + 2E_p I_p \sum_{i=1}^{n} \phi_i^{(4)}(x) q_i(t)\right]\left[H(x-x_1) - H(x-x_2)\right] \\
&\quad + 4E_p I_p \sum_{i=1}^{n} \phi_i^{(3)}(x) q_i(t)\left[H'(x-x_1) - H'(x-x_2)\right] \\
&\quad + 2E_p I_p \sum_{i=1}^{n} \phi_i''(x) q_i(t)\left[H''(x-x_1) - H''(x-x_2)\right] \\
&\quad + 2E_p d_{31} a b v(t)\left[H''(x-x_1) - H''(x-x_2)\right] - f(x,t) = \varepsilon
\end{aligned} \tag{34}$$

Min ε by $\langle \varepsilon, \phi_j \rangle = 0$,

$$\Longrightarrow \langle \varepsilon, \phi_j \rangle = \int_0^L \varepsilon(x,t)\phi_j(x)\,dx = 0 \qquad j = 1, 2, \ldots, n \tag{35}$$

Equation (34) becomes

$$\begin{aligned}
&\left[\rho_b A_b \left(\sum_{i=1}^{n} \int_0^L \phi_i(x)\phi_j(x)\,dx\right)\right. \\
&\left.+ 2\rho_p A_p \left(\sum_{i=1}^{n} \int_0^L \phi_i(x)\phi_j(x)\left[H(x-x_1) - H(x-x_2)\right]dx\right)\right]\ddot{q}_i(t) \\
&+ \left[E_b I_b \left(\sum_{i=1}^{n} \int_0^L \phi_i^{(4)}(x)\phi_j(x)\,dx\right)\right. \\
&\quad \left.+ 2E_p I_p \left(\sum_{i=1}^{n} \int_0^L \phi_i^{(4)}(x)\phi_j(x)\left[H(x-x_1) - H(x-x_2)\right]dx\right)\right] q_i(t) \\
&+ \left[4E_p I_p \left(\sum_{i=1}^{n} \int_0^L \phi_i^{(3)}(x)\phi_j(x)\left[H'(x-x_1) - H'(x-x_2)\right]dx\right)\right] q_i(t) \\
&+ \left[2E_p I_p \left(\sum_{i=1}^{n} \int_0^L \phi_i''(x)\phi_j(x)\left[H''(x-x_1) - H''(x-x_2)\right]dx\right)\right] q_i(t) \\
&+ 2E_p d_{31} a b v(t) \int_0^L \phi_j(x)\left[H''(x-x_1) - H''(x-x_2)\right]dx \\
&- \int_0^L f(x,t)\phi_j(x)\,dx = 0
\end{aligned} \tag{36}$$

- $$\int_0^L \phi_i^{(4)}\phi_j(x)\,dx = \int_0^L \phi_i''(x)\phi_j''(x)\,dx \tag{37}$$

- $$\int_0^L \phi_i^{(4)}(x)\phi_j(x)\left[H(x-x_1) - H(x-x_2)\right]dx \tag{38}$$

- $$\begin{aligned}&\int_0^L \phi_i^{(3)}(x)\,\phi_j(x)\,[H'(x-x_1)-H'(x-x_2)]\,dx\\&\quad=\phi_i^{(3)}(x)\,\phi_j(x)\,\left[H(x-x_1)-H(x-x_2)\right]\big|_0^L\\&\qquad-\int_0^L \phi_i^{(4)}(x)\,\phi_j(x)\,[H(x-x_1)-H(x-x_2)]\,dx\\&\qquad-\int_0^L \phi_i^{(3)}(x)\,\phi_j'(x)\,[H(x-x_1)-H(x-x_2)]\,dx\end{aligned}\tag{39}$$

- $$\begin{aligned}&\int_0^L \phi_i''(x)\,\phi_j(x)\,[H''(x-x_1)-H''(x-x_2)]\,dx\\&\quad=\phi_i''(x)\,\phi_j(x)\,\left[H'(x-x_1)-H'(x-x_2)\right]\big|_0^L\\&\qquad-\int_0^L \phi_i^{(3)}(x)\,\phi_j(x)\,[H'(x-x_1)-H'(x-x_2)]\,dx\\&\qquad-\int_0^L \phi_i''(x)\,\phi_j'(x)\,[H'(x-x_1)-H'(x-x_2)]\,dx\\&\quad=\int_0^L \phi_i^{(4)}(x)\,\phi_j(x)\,[H(x-x_1)-H(x-x_2)]\,dx\\&\qquad+\int_0^L \phi_i^{(3)}(x)\,\phi_j'(x)\,[H(x-x_1)-H(x-x_2)]\,dx\\&\qquad+\int_0^L \phi_i^{(3)}(x)\,\phi_j'(x)\,[H(x-x_1)-H(x-x_2)]\,dx\\&\qquad+\int_0^L \phi_i''(x)\,\phi_j''(x)\,[H(x-x_1)-H(x-x_2)]\,dx\end{aligned}\tag{40}$$

- $$\begin{aligned}&\int_0^L \phi_j(x)\,[H''(x-x_1)-H''(x-x_2)]\,dx\\&\quad=\phi_j(x)\,\left[H'(x-x_1)-H'(x-x_2)\right]\big|_0^L\\&\qquad-\int_0^L \phi_j'(x)\,[H'(x-x_1)-H'(x-x_2)]\,dx\\&\quad=-\phi_j(x)\,\left[H(x-x_1)-H(x-x_2)\right]\big|_0^L\\&\qquad+\int_0^L \phi_j''(x)\,[H(x-x_1)-H(x-x_2)]\,dx\\&\quad=\int_0^L \phi_j''(x)\,[H(x-x_1)-H(x-x_2)]\,dx=\phi_j'(x_2)-\phi_j'(x_1)\end{aligned}\tag{41}$$

From Equation (38), (39), and (40), we can get that

$$\begin{aligned} &2\int_0^L \phi_i^{(4)}(x)\,\phi_j(x)\left[H(x-x_1)-H(x-x_2)\right]dx \\ &+4\int_0^L \phi_i^{(3)}(x)\,\phi_j(x)\left[H'(x-x_1)-H'(x-x_2)\right]dx \\ &+2\int_0^L \phi_i''(x)\,\phi_j(x)\left[H''(x-x_1)-H''(x-x_2)\right]dx \\ &=2\int_0^L \phi_i''(x)\,\phi_j''(x)\left[H(x-x_1)-H(x-x_2)\right]dx \end{aligned} \tag{42}$$

Substituting Equation (37), (41), (42) into (36) yields

$$\begin{aligned} &\left[\rho_b A_b\left(\sum_{i=1}^{n}\int_0^L \phi_i(x)\,\phi_j(x)\,dx\right)\right. \\ &\left.+2\rho_p A_p\left(\sum_{i=1}^{n}\int_0^L \phi_i(x)\,\phi_j(x)\left[H(x-x_1)-H(x-x_2)\right]dx\right)\right]\ddot{q}_i(t) \\ &+\left[E_b I_b\left(\sum_{i=1}^{n}\int_0^L \phi_i''(x)\,\phi_j''(x)\,dx\right)\right. \\ &\left.+2E_p I_p\left(\sum_{i=1}^{n}\int_0^L \phi_i''(x)\,\phi_j''(x)\left[H(x-x_1)-H(x-x_2)\right]dx\right)\right]q_i(t) \\ &+2E_p d_{31} abv(t)\left(\phi_j'(x_2)-\phi_j'(x_1)\right)=\int_0^L f(x,t)\,\phi_j(x)\,dx \end{aligned} \tag{43}$$

Therefore, we can get that

$$\sum_{r=1}^{n} m_{sr}\ddot{q}_r(t)+\sum_{s=1}^{n} k_{sr}q_r(t)=f_{c_s}(t)+f_{d_s}(t),\quad s=1,2,\ldots,n \tag{44}$$

where
$$\begin{aligned} m_{sr}&=\rho_b A_b\int_0^L \phi_r(x)\,\phi_s(x)\,dx+2\rho_p A_p\int_{x_1}^{x_2}\phi_r(x)\,\phi_s(x)\,dx \\ k_{sr}&=E_b I_b\int_0^L \phi_r''(x)\,\phi_s''(x)\,dx+2E_p I_p\int_{x_1}^{x_2}\phi_r''(x)\,\phi_s''(x)\,dx \\ f_{c_s}(t)&=2E_p d_{31} abv(t)\left(\phi_s'(x_1)-\phi_s'(x_2)\right) \\ f_{d_s}(t)&=\int_0^L f(x,t)\,\phi_s(x)\,dx=\int_0^L F_d(t)\,\phi_s(x)\,\delta(x-x_d)\,dx=F_d(t)\,\phi_s(x_d) \end{aligned}$$

Choosing $\phi_r=\sin(r\pi x/L), r=1,2,\ldots,n$ satisfy all boundary conditions

At $x=0,\quad \delta w=0,\quad \frac{\partial^2 w}{\partial x^2}=0$

At $x=L,\quad \delta w=0,\quad \frac{\partial^2 w}{\partial x^2}=0$

Because $C=\alpha M+\beta K$, where $\alpha=0.6$ and $\beta=1.2\times10^{-6}$

$$M\{\ddot{q}\}+C\{\dot{q}\}+K\{q\}=\{f_c\}+\{f_d\} \tag{45}$$

for $s=r$,

$$m_{rr}=\frac{\rho_b A_b L}{2}+\rho_p A_p(x_2-x_1)+\frac{\rho_p A_p L}{2\pi r}\left[\sin\left(\frac{2\pi r x_1}{L}\right)-\sin\left(\frac{2\pi r x_2}{L}\right)\right] \tag{46}$$

$$k_{rr}=\left(\frac{\pi r}{L}\right)^4\left\{\frac{E_b I_b L}{2}+E_p I_p(x_2-x_1)+\frac{E_p I_p L}{2\pi r}\left[\sin\left(\frac{2\pi r x_1}{L}\right)-\sin\left(\frac{2\pi r x_2}{L}\right)\right]\right\} \tag{47}$$

for $s \neq r$,

$$m_{sr} = \frac{2\rho_p A_p L}{\pi} \left[\frac{r \sin\left(\frac{s\pi x}{L}\right) \cos\left(\frac{r\pi x}{L}\right)}{s^2 - r^2} + \frac{s \cos\left(\frac{s\pi x}{L}\right) \sin\left(\frac{r\pi x}{L}\right)}{r^2 - s^2} \right] \Bigg|_{x_1}^{x_2} \tag{48}$$

$$k_{sr} = \frac{2E_p I_p L}{\pi} \left(\frac{sr\pi^2}{L^2}\right)^2 \left[\frac{r \sin\left(\frac{s\pi x}{L}\right) \cos\left(\frac{r\pi x}{L}\right)}{s^2 - r^2} + \frac{s \cos\left(\frac{s\pi x}{L}\right) \sin\left(\frac{r\pi x}{L}\right)}{r^2 - s^2} \right] \Bigg|_{x_1}^{x_2} \tag{49}$$

And

$$f_c(t) = 2E_p d_{31} abv(t) \left(\frac{\pi s}{L}\right) \left(\cos\left(\frac{s\pi x_1}{L}\right) - \cos\left(\frac{s\pi x_2}{L}\right)\right) \tag{50}$$

$$f_d(t) = F_d(t)\, \phi_s(x_d) = F_d(t) \sin\left(\frac{s\pi x_d}{L}\right) \tag{51}$$

(c) Let

$$x(t) = \begin{bmatrix} q(t) \\ \dot{q}(t) \end{bmatrix} \tag{52}$$

Equation (45) becomes

$$\{\ddot{q}\} = -M^{-1}C\{\dot{q}\} - M^{-1}K\{q\} + M^{-1}\{f_c\} + M^{-1}\{f_d\} \tag{53}$$

In state-space form:

$$\begin{aligned} \dot{x} &= Ax + Bu + \hat{B}u_d \\ y &= C_0 x + Du \end{aligned} \tag{54}$$

where

$$A = \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}C \end{bmatrix} \tag{55}$$

$$B = \begin{bmatrix} 0 \\ M^{-1} \end{bmatrix} 2E_p d_{31} ab \begin{bmatrix} \frac{\pi}{L}\left(\cos\left(\frac{\pi}{L}x_1\right) - \cos\left(\frac{\pi}{L}x_2\right)\right) \\ \frac{2\pi}{L}\left(\cos\left(\frac{2\pi}{L}x_1\right) - \cos\left(\frac{2\pi}{L}x_2\right)\right) \\ \vdots \\ \frac{n\pi}{L}\left(\cos\left(\frac{n\pi}{L}x_1\right) - \cos\left(\frac{n\pi}{L}x_2\right)\right) \end{bmatrix} \tag{56}$$

$$\hat{B} = \begin{bmatrix} 0 \\ M^{-1} \end{bmatrix} \begin{bmatrix} \sin\left(\frac{\pi}{L}x_d\right) \\ \sin\left(\frac{2\pi}{L}x_d\right) \\ \vdots \\ \sin\left(\frac{n\pi}{L}x_d\right) \end{bmatrix} \tag{57}$$

Because $y = w(x,t) = \sum_{i=1}^{n} \phi_r(x_0)\, q_r(t)$ where $x_0 = 0.6L$

$$\begin{aligned} C_0 &= \begin{bmatrix} \phi_1(x_0) & \phi_2(x_0) & \cdots & \phi_n(x_0) & 0 & 0 & \cdots & 0 \end{bmatrix}_{1\times 2n} \\ &= \begin{bmatrix} \sin(0.6\pi) & \sin(1.2\pi) & \cdots & \sin(0.6n\pi) & 0 & 0 & \cdots & 0 \end{bmatrix}_{1\times 2n} \end{aligned} \tag{58}$$

and

$$D = 0 \tag{59}$$

If we use the state feedback $u = -K_c x$, the state-space of the sytem is

$$\begin{aligned} \dot{x} &= (A - BK_c)\, x + \hat{B} u_d \\ y &= (C_0 - DK_c)\, x \end{aligned} \tag{60}$$

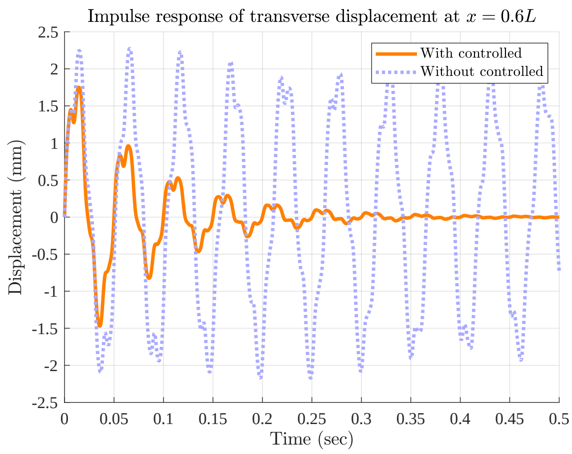
The following codes are used to simulate the response of the system:

```
clc; clf; clear all;

% initialize

b = 2e-2;
L = 0.50;
x1 = 0.15; x2 = 0.24;
xd = 0.30;

Eb = 7e10;
pb = 2700; tb = 2e-3;

Ec = 6.5e10;
pc = 7600; tc = 0.6e-3;
d31 = -175e-12;

Ac = b*tc;  Ab = b*tb;
Ib = b*tb^3/12; Ic = b*tc^3/12+Ac*(tb+tc)^2/4;
a = (tb+tc)/2;

% stiffness and mass matrices

N = 3; % no. of expansion terms

K = zeros(N);
M = zeros(N);
C = zeros(N);
Fc = zeros(N,1); Fd = zeros(N,1);

for r = 1:N
    for s = 1:N
        if r == s
            K(r,s) = (pi*r/L)^4*(Eb*Ib*L/2+Ec*Ic*(x2-x1)+...
                Ec*Ic*L/(2*pi*r)*(sin(2*pi*r*x1/L)-sin(2*pi*r*x2/L)));
            M(r,s) = pb*Ab*L/2+pc*Ac*(x2-x1)+...
                pc*Ac*L/(2*pi*r)*(sin(2*pi*r*x1/L)-sin(2*pi*r*x2/L));
        else
            K(r,s) = 2*Ec*Ic*L/pi*(pi^2*r*s/L^2)^2*...
                ((r*sin(s*pi*x2/L)*cos(r*pi*x2/L))/(s^2-r^2)+...
                (s*sin(r*pi*x2/L)*cos(s*pi*x2/L))/(r^2-s^2)-...
```

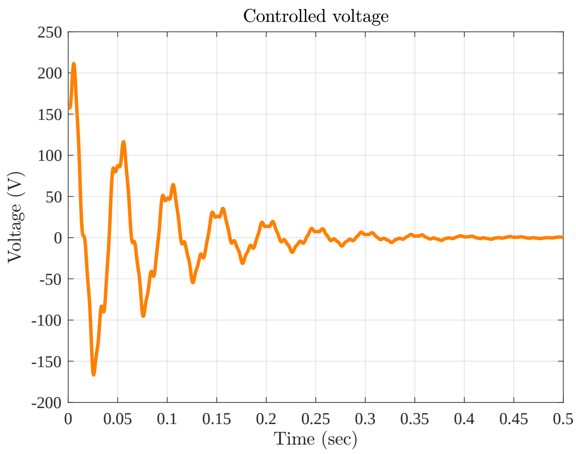
```
                    ((r*sin(s*pi*x1/L)*cos(r*pi*x1/L))/(s^2-r^2)+...
                    (s*sin(r*pi*x1/L)*cos(s*pi*x1/L))/(r^2-s^2)));
                M(r,s) = 2*pc*Ac*L/pi*((r*sin(s*pi*x2/L)*cos(r*pi*x2/L))/(s^2-r^2)+...
                    (s*sin(r*pi*x2/L)*cos(s*pi*x2/L))/(r^2-s^2)-...
                    ((r*sin(s*pi*x1/L)*cos(r*pi*x1/L))/(s^2-r^2)+...
                    (s*sin(r*pi*x1/L)*cos(s*pi*x1/L))/(r^2-s^2)));
            end
        end

    % due to voltage input
    Fc(r) = 2*a*Ec*d31*b*(pi*r/L)*(cos(r*pi*x1/L)-cos(r*pi*x2/L));
    % due to discrete force with magnitude 1/100
    Fd(r) = 1/100*sin(r*pi*xd/L);
end

% add internal damping

C = 0.6*M+1.2e-6*K;

% state-space model

AL = -inv(M)*K;
AR = -inv(M)*C;
A = [zeros(N) eye(N);...
   AL AR];
BL1 = inv(M)*Fc; BL2 = inv(M)*Fd;
B1 = [zeros(N,1);BL1];
B2 = [zeros(N,1);BL2];
for r = 1:N
    CCw(1,r) = sin(r*pi*0.6);  % displacement w at midpoint (x=L/2)
end
CC = [CCw zeros(1,N)];
D = [0];

% control gain
Kc = [-55400 -22549 15848 -753 -249 174];
Ac=A-B1*Kc;

% impulse response

t = 0:0.0005:0.5;
IU = 1;
[y,x,t] = impulse(A,B2,CC,D,IU,t); % uncontrolled response
[yc,x,t] = impulse(Ac,B2,CC,D,IU,t); % controlled response
u = -Kc*x'; % controlled voltage

```

```
% plot results

figure(1);
hold on;
plot(t,yc*1000,'color',[1 0.5 0],'LineWidth',2.5)
plot(t,y*1000,':','color',[0.667 0.667 1],'LineWidth',2.5) % unit (mm)
hold off;
grid on;
title('Impulse response of transverse displacement at $x = 0.6L$' ...
    ,'interpreter','latex');
xlabel('Time (sec)','interpreter','latex');
ylabel('Displacement (mm)','interpreter','latex');
legend('With controlled','Without controlled','interpreter','latex');
a = get(gca,'XTickLabel');
set(gca,'XTickLabel',a,'FontName','Times','fontsize',12);
set(gca,'position',[0.15 0.20 0.6 0.6]);
set(gcf,'position',[100 100 800 600]);
set(gcf,'renderer','painters');
filename = "Q4-2-tyyc"+".pdf";
saveas(gcf,filename);

figure(2);
plot(t,u,'color',[1 0.5 0],'LineWidth',2.5);
grid on;
title('Controlled voltage','interpreter','latex');
xlabel('Time (sec)','interpreter','latex');
ylabel('Voltage (V)','interpreter','latex');
a = get(gca,'XTickLabel');
set(gca,'XTickLabel',a,'FontName','Times','fontsize',12);
set(gca,'position',[0.15 0.20 0.6 0.6]);
set(gcf,'position',[100 100 800 600]);
set(gcf,'renderer','painters');
filename = "Q4-2-tu"+".pdf";
saveas(gcf,filename);
```

The simulation results are shown in Figure 1.



(a)



(b)

Figure 1: Simulation results for the response of the system.