Vibration

- Any motion that repeats itself after an interval of time is called *vibration* or *oscillation*.
- The minimum number of independent coordinates required to determine completely the positions of all parts of a system at any instant of time defines the *degree of freedom* of the system.

Single-Degree-of-Freedom (SDOF) Systems

Equation of motion:

- Free Vibration of Undamped Systems:

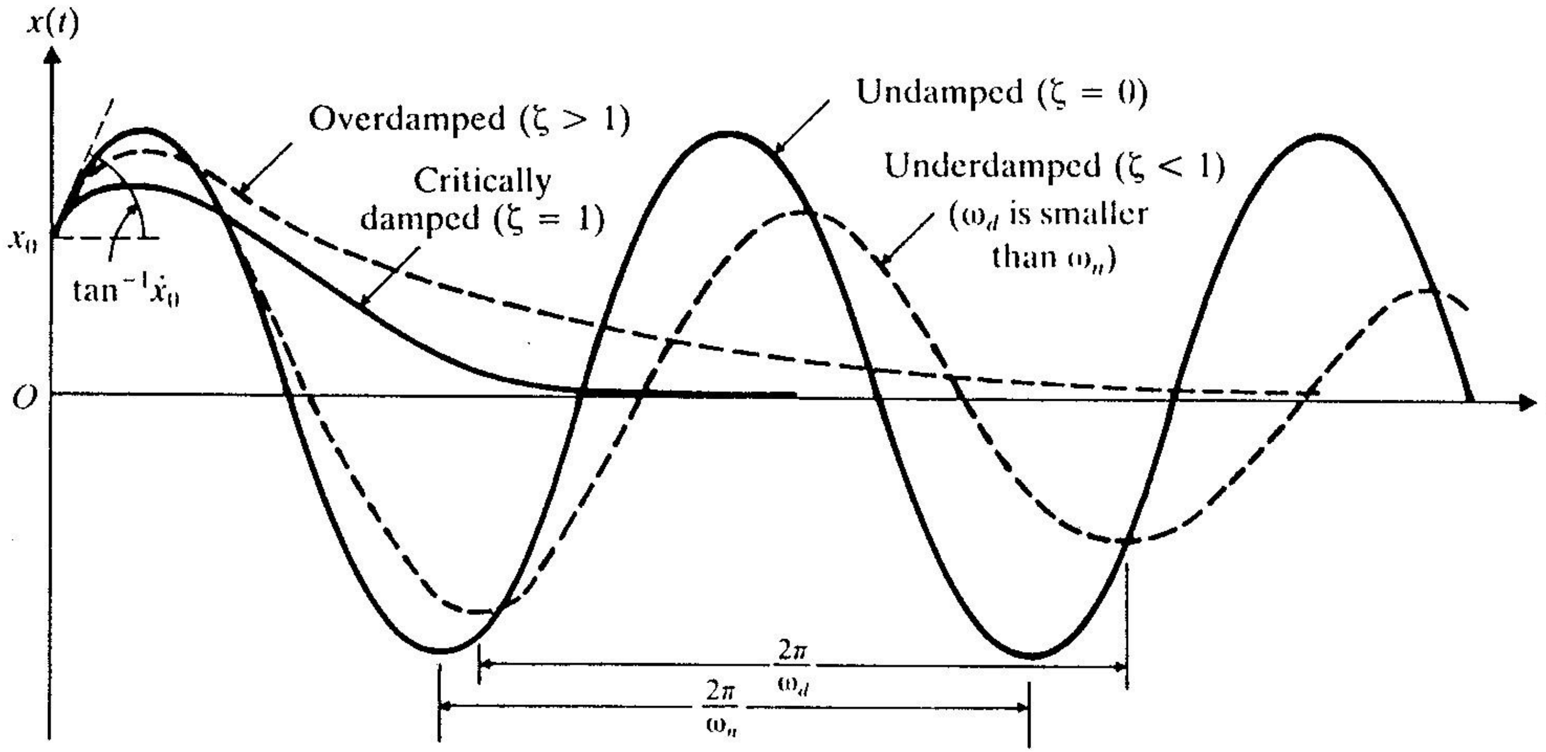
  Assume zero damping and external forces, The time taken to complete one cycle of motion defines the *period*

Natural frequency

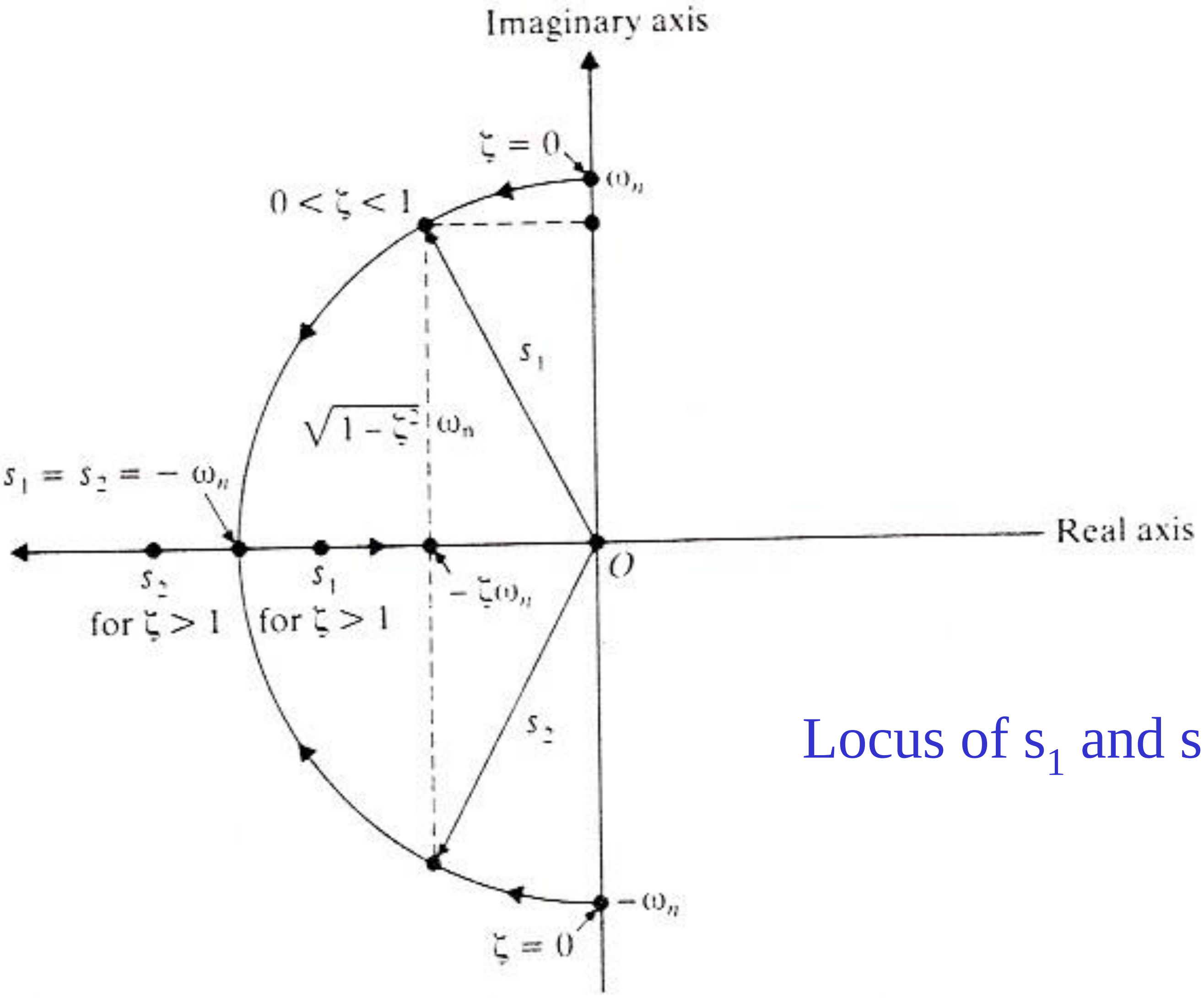
where *Hz* denotes *hertz*

- Free Vibration of Damped Systems:

  Let $F(t)=0$ and divide through by m,



Comparison of motions with different types of damping



Locus of s_1 and s_2

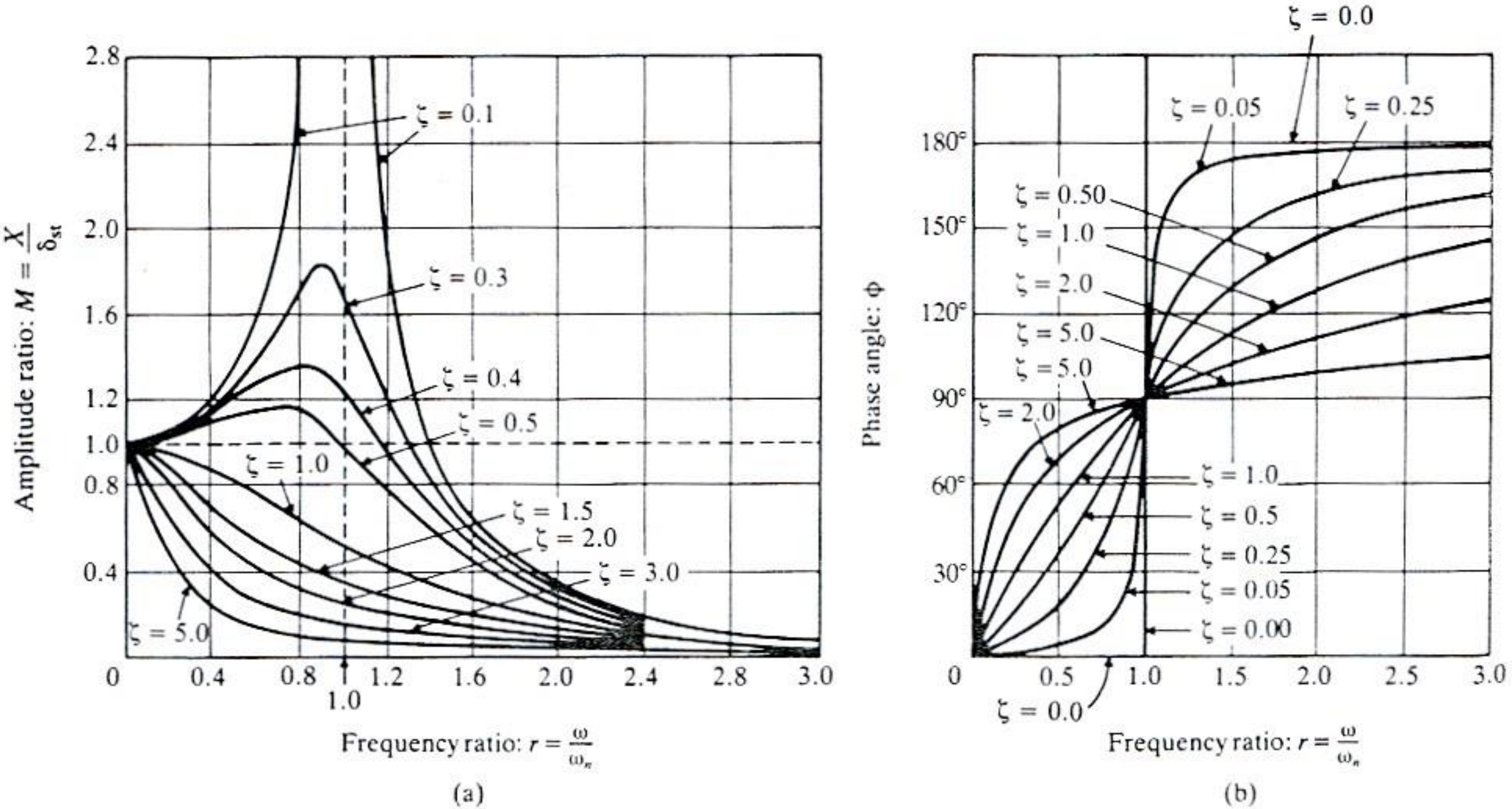
- Response to Harmonic Excitations

Excitation force

where is the magnitude

is the excitation (or forcing) frequency

Equation of motion becomes



Variation of X and ϕ with frequency ratio r

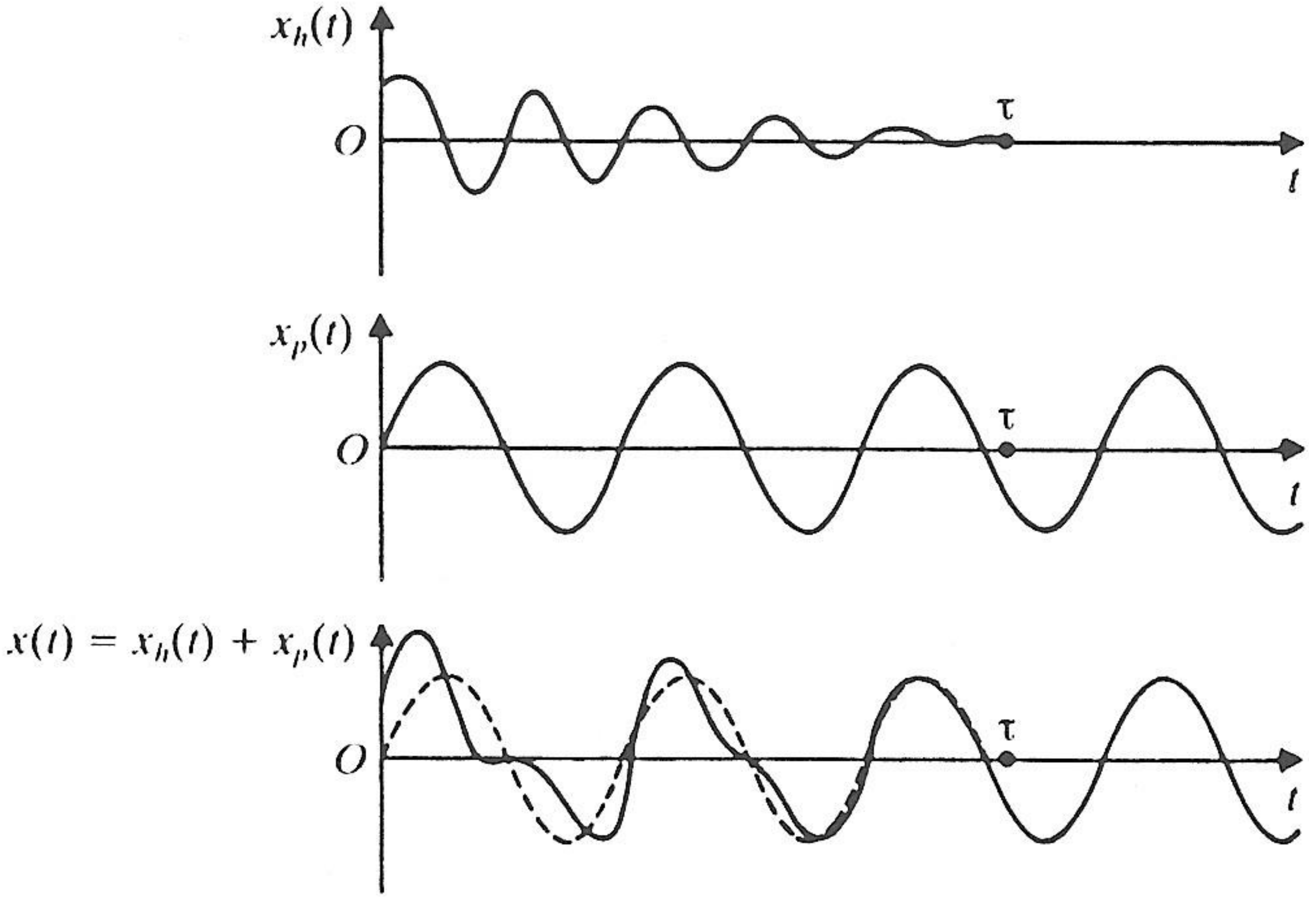
where is the solution of the homogeneous eq.

represents *transient* response

free vibration dies out with time

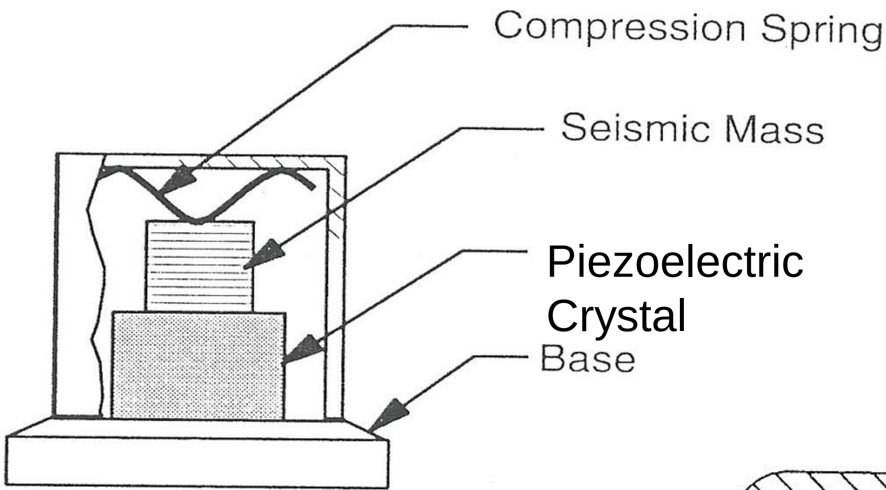
is the particular solution

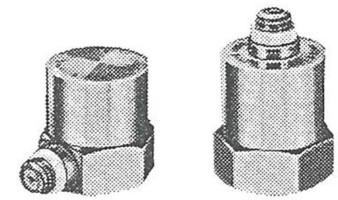
represents *steady-state* response

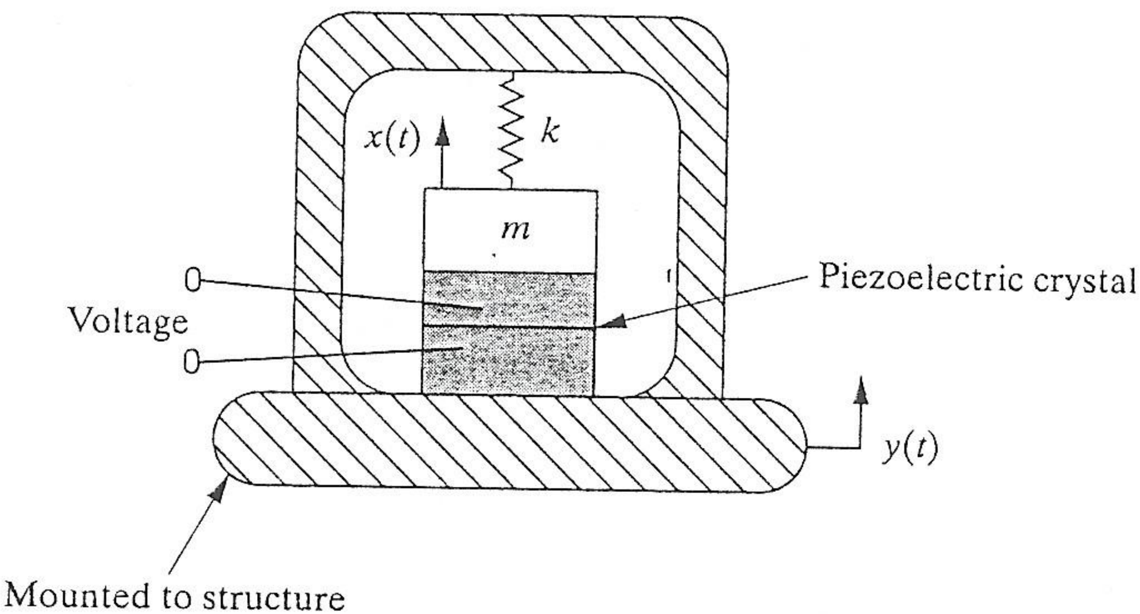


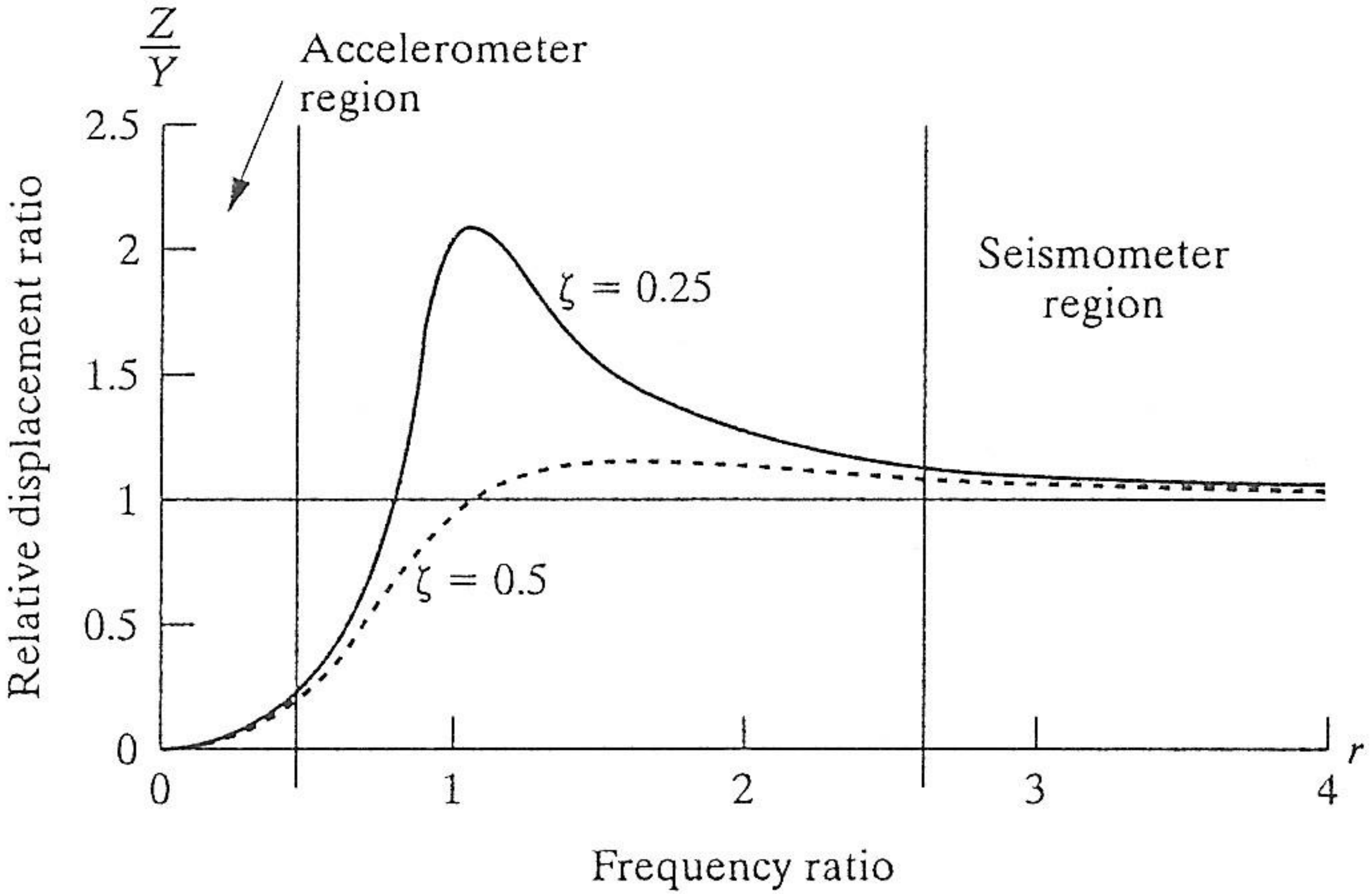
Homogenous, particular, and general solutions for an underdamped case

Piezoelectric Accelerometer







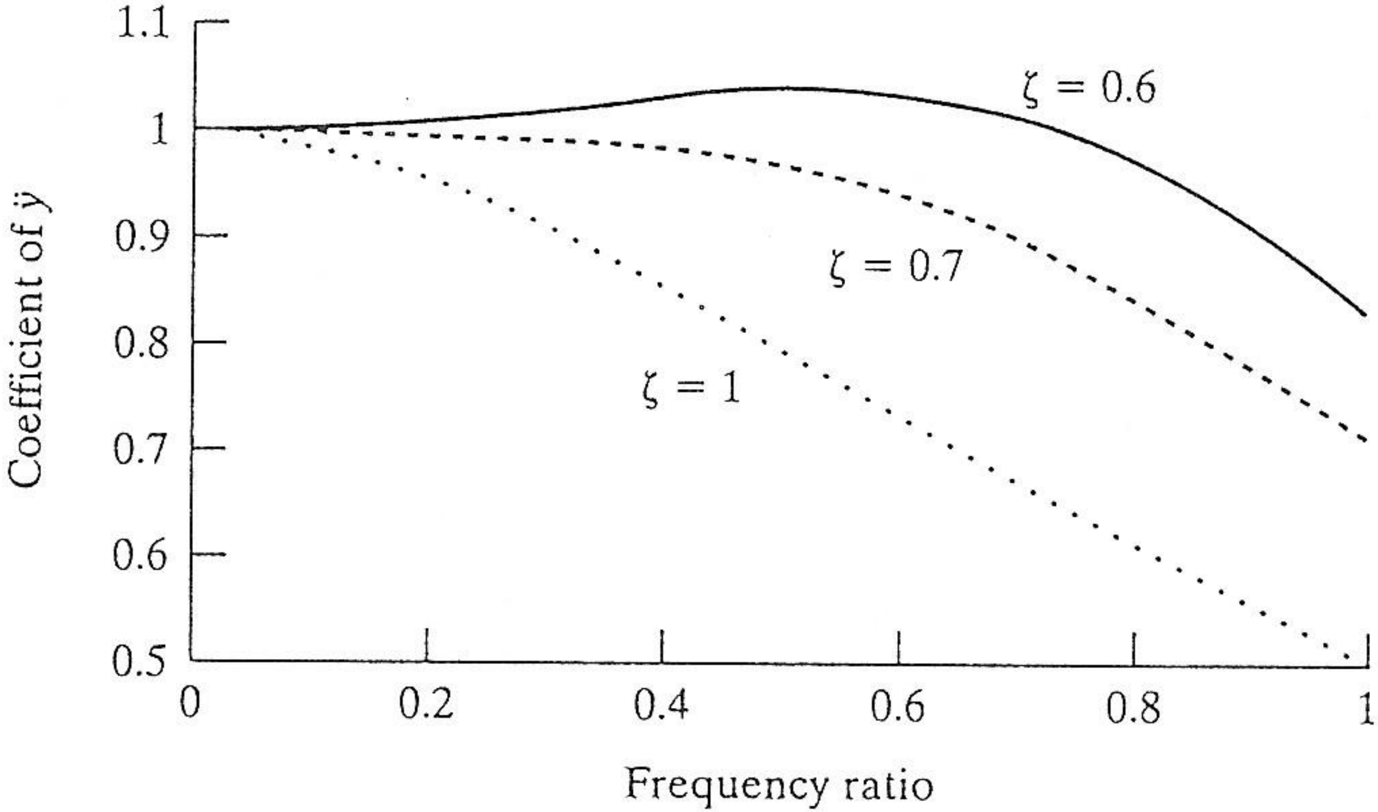


Magnitude versus frequency of the relative displacement for a transducer

For larger values of r

- relative displacement and the displacement of the base have the same amplitude
- can be used to measure harmonic base displacement

Seismometer: instrument with low natural frequency compared to the excitation frequency



Effect of damping on the constant of proportionality between base acceleration and the relative displacement (voltage) for an accelerometer

For small values of *r*,

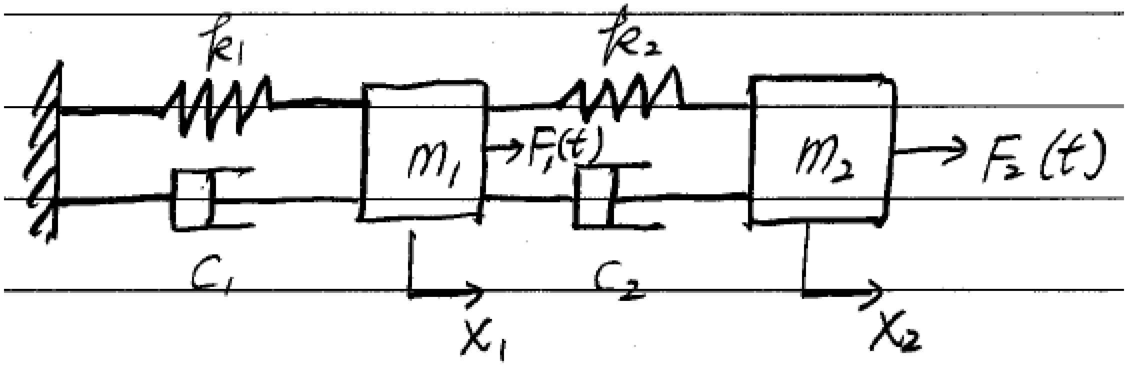
- the relative position is proportional to the base acceleration
- the accelerometer can be used in the range

Accelerometer: a high natural-frequency instrument that measures the acceleration of a vibrating body

e.g., a mass resting on a piezoelectric ceramic crystal,

up to can be measured

Multi-DOF Systems



Write equations in matrix form

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} c_1+c_2 & -c_2 \\ -c_2 & c_2 \end{bmatrix} \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} + \begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

- Free Vibration of an Undamped System

Equations of motion:

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

or $M\ddot{X} + KX = 0$

Synchronous motion

(All coordinates execute the same motion in time)

$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} e^{i\omega t}$$

$$\Rightarrow -\omega^2 \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} e^{i\omega t} + \begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} e^{i\omega t} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$\therefore \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix}$ must satisfy homogeneous linear equation:

$$\begin{bmatrix} (k_1+k_2)-m_1\omega^2 & -k_2 \\ -k_2 & k_2-m_2\omega^2 \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

det = 0 for nonzero A_1, A_2

$\Rightarrow$ Characteristic equation:

$$[(k_1+k_2)-m_1\omega^2][k_2-m_2\omega^2]-k_2^2=0$$

Natural frequencies ω_1 and ω_2

Let $\omega_r = \sqrt{\lambda_r}$

$$k_1 k_2 + k_2^2 - \lambda m_1 k_2 - \lambda m_2 k_1 - \lambda m_2 k_2 + \lambda^2 m_1 m_2 - k_2^2 = 0$$

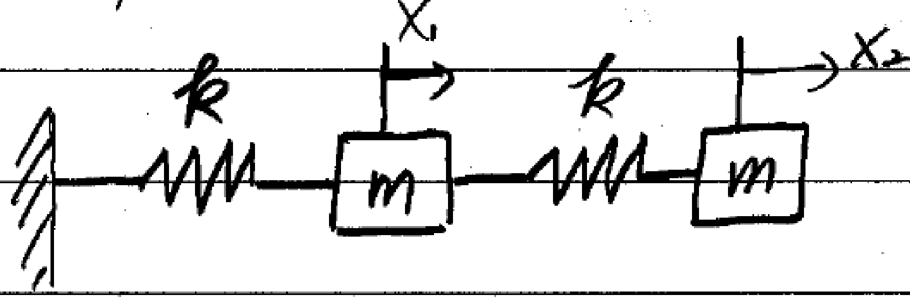
$$\lambda^2 (m_1 m_2) + \lambda\,(-m_1 k_2 - m_2 k_1 - m_2 k_2) + (k_1 k_2) = 0$$

roots

$$\lambda_r = \frac{m_1 k_2 + m_2 k_1 + m_2 k_2 \pm \sqrt{k_2^2 m_1^2 - 2 k_1 k_2 m_1 m_2 + 2 k_2^2 m_1 m_2 + k_1^2 m_2^2 + 2 k_1 k_2 m_2^2 + k_2^2 m_2^2}}{2 m_1 m_2}$$

(cont.)

Example



Equatione of motion :

$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} 2k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Characteristic equation

$$\det \begin{vmatrix} 2k-\lambda m & -k \\ -k & k-\lambda m \end{vmatrix} = \lambda^2(m^2) + \lambda(-3mk) + k^2 = 0$$

Eigenvalues $\lambda_1 = \frac{3-\sqrt{5}}{2}\frac{k}{m}$

$$\lambda_2 = \frac{3+\sqrt{5}}{2}\frac{k}{m}$$

Natural frequencies $\omega_r = \sqrt{\lambda_r}$

$$\omega_1 = \sqrt{\frac{3-\sqrt{5}}{2}}\sqrt{\frac{k}{m}} \simeq 0.618\sqrt{\frac{k}{m}}$$

$$\omega_2 = \sqrt{\frac{3+\sqrt{5}}{2}}\sqrt{\frac{k}{m}} \simeq 1.618\sqrt{\frac{k}{m}}$$

Modal vectors

$$[K - \lambda_r M]\{A\}_r = \{0\}$$

Mode 1 : $\lambda_1 = \frac{3-\sqrt{5}}{2}\frac{k}{m}$

$$\begin{bmatrix} 2k - \left(\frac{3-\sqrt{5}}{2}\right)k & -k \\ -k & k - \left(\frac{3-\sqrt{5}}{2}\right)k \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix}_1 = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

absolute magnitudes of the elements of the modal vector are not unique

$$\begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix}_1 = \alpha_1 \begin{Bmatrix} 1 \\ \frac{1+\sqrt{5}}{2} \end{Bmatrix} \simeq \alpha_1 \begin{Bmatrix} 1.000 \\ 1.618 \end{Bmatrix}$$

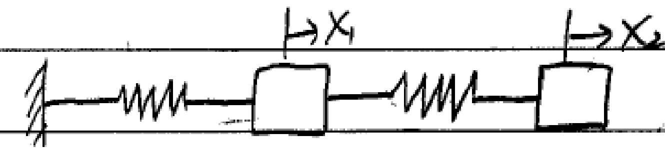
Mode 2 :

$$\lambda_2 = \frac{3+\sqrt{5}}{2}\,\frac{k}{m}$$

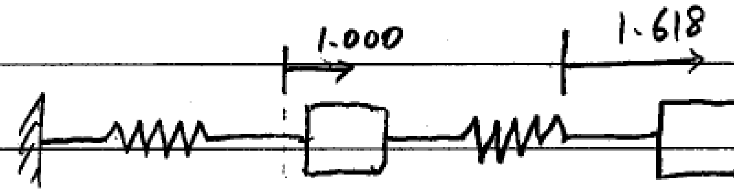
$$\begin{bmatrix} 2k-\left(\frac{3+\sqrt{5}}{2}\right)k & -k \\ -k & k-\left(\frac{3+\sqrt{5}}{2}\right)k \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix}_2 = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix}_2 = \alpha_2 \begin{Bmatrix} 1 \\ \frac{1-\sqrt{5}}{2} \end{Bmatrix} \simeq \alpha_2 \begin{Bmatrix} 1.000 \\ -0.618 \end{Bmatrix}$$

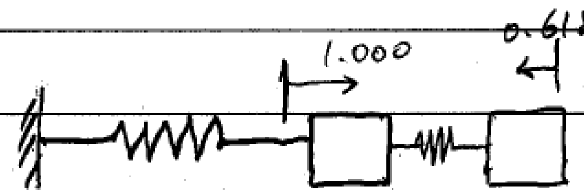
Summary – Natural Modes of Vibration:



Natural Frequencies	Modal Vectors
$\omega_1 = \sqrt{\frac{3-\sqrt{5}}{2}}\sqrt{\frac{k}{m}}$	$\begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix}_1 = \alpha_1 \begin{Bmatrix} 1 \\ \frac{1+\sqrt{5}}{2} \end{Bmatrix}$
$\omega_2 = \sqrt{\frac{3+\sqrt{5}}{2}}\sqrt{\frac{k}{m}}$	$\begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix}_2 = \alpha_2 \begin{Bmatrix} 1 \\ \frac{1-\sqrt{5}}{2} \end{Bmatrix}$



move same direction



move opposite direction