Structural Dynamics

Newtonian

- Vector Calculus
- Newton's Law
- Space coordinates
- FBD – need to compute interacting forces

Lagrangian

- Calculus of Variations
- Hamilton's principle & Lagrange's eq.
- Generalized coordinates
- Treat the system as a whole
  – NOT individual bodies

Hamilton's Principle

Consider a system with N particles,

From d'Alembert's principle.

$$\sum_{n=1}^{N} (m_n \ddot{\underline{r}}_n - \underline{F}_n) \cdot \delta \underline{r}_n = 0 \qquad (1)$$

where

m_n is the mass of the nth particle

$\underline{r}_n$ is the position vector of the nth particle

$\underline{F}_n$ is the vector force applied to the nth particle

$\delta \underline{r}_n$ is the virtual displacement of the nth particle

If there is no movement,
no work is done.

Virtual displacement :

- infinitesimal, hypothetical change in the coordinate system from the particles actual path
- does NOT involve time

The <u>virtual work</u> is defined as

$$\delta W = \sum_{n=1}^{N} \underline{F}_n \cdot \delta \underline{r}_n \tag{2}$$

Consider the 1st term in eq (1),

$$\ddot{\underline{r}}_n \cdot \delta \underline{r}_n = \frac{d}{dt}\left(\dot{\underline{r}}_n \cdot \delta \underline{r}_n\right) - \delta\left(\frac{1}{2} \dot{\underline{r}}_n \cdot \dot{\underline{r}}_n\right) \tag{3}$$

Substitute eqs (2) and (3) into eq (1),

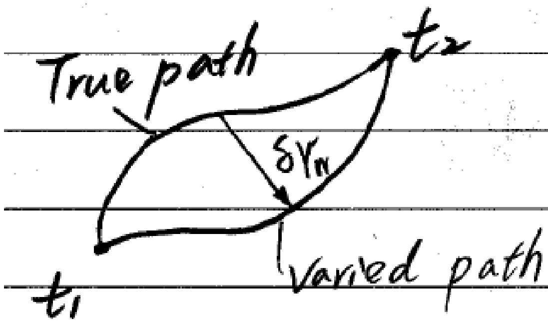
$$\sum_{n=1}^{N}\left[m_n\left(\frac{d}{dt}(\dot{\underline{r}}_n\cdot\delta\underline{r}_n)-\delta\left(\frac{1}{2}\dot{\underline{r}}_n\cdot\dot{\underline{r}}_n\right)\right)\right]-\delta W=0 \qquad (4)$$

Define the total kinetic energy of the N particles as

$$T=\sum_{n=1}^{N}\frac{1}{2}m_n\dot{\underline{r}}_n\cdot\dot{\underline{r}}_n \qquad (5)$$

(4) $\Rightarrow$

$$\delta T+\delta W=\sum_{n=1}^{N}m_n\frac{d}{dt}(\dot{\underline{r}}_n\cdot\delta\underline{r}_n) \qquad (6)$$



Choose

$$\delta \underline{r}_n(t_1) = \delta \underline{r}_n(t_2) = 0$$

Integrate eq. (6) from $t = t_1$ to t_2

$$\int_{t_1}^{t_2} (\delta T + \delta W)\, dt = \int_{t_1}^{t_2} \sum_{n=1}^{N} m_n \frac{d}{dt} (\dot{\underline{r}}_n \cdot \delta \underline{r}_n)\, dt$$

$$= \sum_{n=1}^{N} m_n (\dot{\underline{r}}_n \cdot \delta \underline{r}_n) \Big|_{t_1}^{t_2}$$

$$= 0 \qquad (7)$$

$$\boxed{\int_{t_1}^{t_2} (\delta T + \delta W)\, dt = 0}$$: Extended Hamilton's Principle (Generalized) (*)

The position vector of each particle can be expressed as a function of the generalized coordinates and time,

$$\underline{r}_i = \underline{r}_i(q_1, q_2, \cdots, q_n, t), \quad i = 1, 2, \cdots, p \qquad (2)$$

The kinetic energy of a system of p particles:

$$T = \frac{1}{2}\sum_{i=1}^{p} m_i \dot{\underline{r}}_i \cdot \dot{\underline{r}}_i \qquad (3)$$

where

$$\dot{\underline{r}}_i = \sum_{r=1}^{n} \frac{\partial \underline{r}_i}{\partial q_r}\dot{q}_r + \frac{\partial \underline{r}_i}{\partial t} \qquad (4)$$

Introducing eq. (4) into eq. (3), one obtains

$$T = T(q_1, q_2, \cdots, q_n, \dot{q}_1, \dot{q}_2, \cdots, \dot{q}_n, t) \qquad (5)$$

$$T = T(q_1, q_2, \cdots, q_n, \dot{q}_1, \dot{q}_2, \cdots, \dot{q}_n, t) \tag{5}$$

$\Rightarrow$

$$\delta T = \sum_{r=1}^{n} \frac{\partial T}{\partial q_r} \delta q_r + \sum_{r=1}^{n} \frac{\partial T}{\partial \dot{q}_r} \delta \dot{q}_r \tag{6}$$

where

$$\delta \dot{q}_r = \frac{d}{dt} \delta q_r \tag{7}$$

$$\delta W = \sum_{k=1}^{p} \underline{F}_k \cdot \delta \underline{r}_k \tag{8}$$

where

$$\delta \underline{r}_k = \sum_{r=1}^{n} \frac{\partial \underline{r}_k}{\partial q_r} \delta q_r \tag{9}$$

Substituting eq (9) into eq. (8), one obtains

$$\delta W = \sum_{k=1}^{p} \sum_{r=1}^{n} \underline{F}_k \cdot \frac{\partial \underline{r}_k}{\partial q_r} \delta q_r \qquad (10)$$

$$= \sum_{r=1}^{n} Q_r \, \delta q_r \qquad (11)$$

where

$$Q_r = \sum_{k=1}^{p} \underline{F}_k \cdot \frac{\partial \underline{r}_k}{\partial q_r} \qquad (12)$$

: Generalized Force

Sub. eq. (6) and eq. (11) into the extended Hamilton's principle,

$$\int_{t_1}^{t_2} (\delta T + \delta W)\,dt = \int_{t_1}^{t_2} \left(\sum_{r=1}^{n} \frac{\partial T}{\partial q_r} \delta q_r + \sum_{r=1}^{n} \frac{\partial T}{\partial \dot{q}_r} \delta \dot{q}_r \right) dt$$

$$+ \int_{t_1}^{t_2} \left(\sum_{r=1}^{n} Q_r \delta q_r \right) dt$$

Integration by parts,

$$= \int_{t_1}^{t_2} \left[\sum_{r=1}^{n} \frac{\partial T}{\partial q_r} \delta q_r - \sum_{r=1}^{n} \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_r} \right) \delta q_r \right] dt + \int_{t_1}^{t_2} \left(\sum_{r=1}^{n} Q_r \delta q_r \right) dt$$

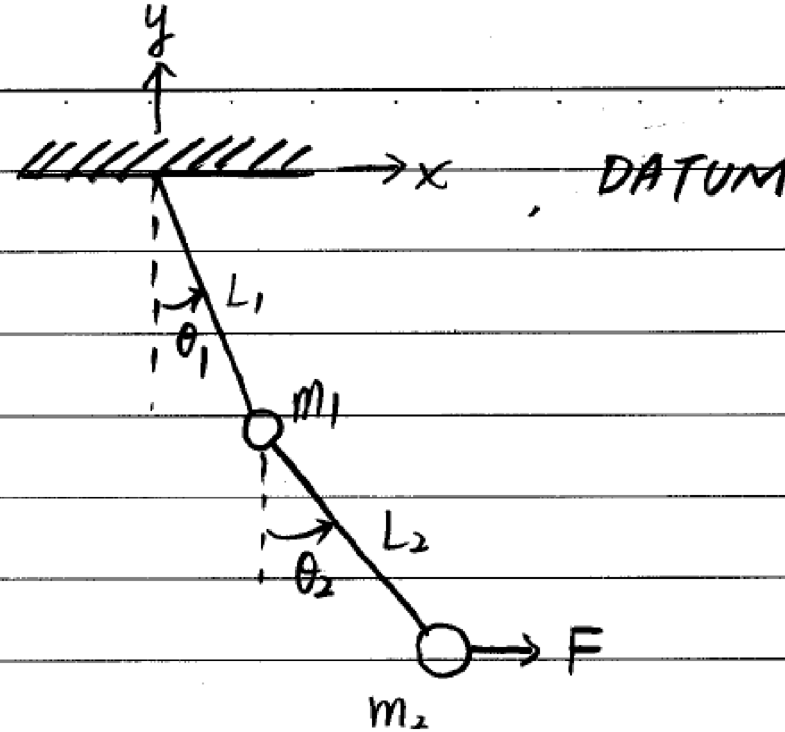
$$= - \sum_{r=1}^{n} \int_{t_1}^{t_2} \left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_r} \right) - \frac{\partial T}{\partial q_r} - Q_r \right] \delta q_r \, dt = 0$$

Since δq_r is arbitrary

$$\boxed{\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_r} \right) - \frac{\partial T}{\partial q_r} = Q_r} \quad , \quad r = 1, 2, \cdots, n \qquad (*)$$

Lagrange's equation

Ex.



$$\underline{r}_1 = L_1 \sin\theta_1 \hat{i} - L_1 \cos\theta_1 \hat{j}$$

$$\underline{V}_1 = \dot{\underline{r}}_1 = \dot{\theta}_1 L_1 \cos\theta_1 \hat{i} + \dot{\theta}_1 L_1 \sin\theta_1 \hat{j}$$

$$V_1^2 = \underline{V}_1 \cdot \underline{V}_1 = \dot{\theta}_1^2 L_1^2 \cos^2\theta_1 + \dot{\theta}_1^2 L_1^2 \sin^2\theta_1 = \dot{\theta}_1^2 L_1^2$$

$$\underline{r}_2 = (L_1 \sin\theta_1 + L_2 \sin\theta_2)\hat{i} - (L_1 \cos\theta_1 + L_2 \cos\theta_2)\hat{j}$$

$$\underline{V}_2 = (L_1\dot{\theta}_1 \cos\theta_1 + L_2\dot{\theta}_2 \cos\theta_2)\hat{i} + (L_1\dot{\theta}_1 \sin\theta_1 + L_2\dot{\theta}_2 \sin\theta_2)\hat{j}$$

$$V_2^2 = \underline{V}_2 \cdot \underline{V}_2 = L_1^2\dot{\theta}_1^2 + L_2^2\dot{\theta}_2^2 + 2L_1L_2\dot{\theta}_1\dot{\theta}_2 \cos\theta_1 \cos\theta_2 + 2L_1L_2\dot{\theta}_1\dot{\theta}_2 \sin\theta_1 \sin\theta_2$$

$$= L_1^2\dot{\theta}_1^2 + L_2^2\dot{\theta}_2^2 + 2L_1L_2\dot{\theta}_1\dot{\theta}_2 \cos(\theta_1 - \theta_2)$$

$$T = \frac{1}{2}m_1 V_1^2 + \frac{1}{2}m_2 V_2^2$$

$$V = m_1 g h_1 + m_2 g h_2$$

$$= -m_1 g L_1 \cos\theta_1 - m_2 g (L_1 \cos\theta_1 + L_2 \cos\theta_2)$$

$$L = T - V = \frac{1}{2}m_2\left[L_1^2\dot{\theta}_1^2 + L_2^2\dot{\theta}_2^2 + 2L_1L_2\dot{\theta}_1\dot{\theta}_2 \cos(\theta_1 - \theta_2)\right]$$

$$+ \frac{1}{2}m_1\dot{\theta}_1^2 L_1^2 + m_1 g L_1 \cos\theta_1 + m_2 g (L_1 \cos\theta_1 + L_2 \cos\theta_2)$$

Let $q_1 = \theta_1$, $q_2 = \theta_2$

$$\delta W_{NC} = F\,\delta(L_1 \sin\theta_1 + L_2 \sin\theta_2)$$

$$= F(L_1 \cos\theta_1\,\delta\theta_1 + L_2 \cos\theta_2\,\delta\theta_2)$$

$\Rightarrow Q_{1NC} = F L_1 \cos\theta_1$, $Q_{2NC} = F L_2 \cos\theta_2$

$$L = T - V = \frac{1}{2} m_2 \left[L_1^2 \dot{\theta}_1^2 + L_2^2 \dot{\theta}_2^2 + 2 L_1 L_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2) \right]$$

$$+ \frac{1}{2} m_1 \dot{\theta}_1^2 L_1^2 + m_1 g L_1 \cos\theta_1 + m_2 g \left(L_1 \cos\theta_1 + L_2 \cos\theta_2 \right)$$

$$\frac{\partial L}{\partial \dot{\theta}_1} = m_2 L_1^2 \dot{\theta}_1 + m_2 L_1 L_2 \dot{\theta}_2 \cos(\theta_1 - \theta_2) + m_1 L_1^2 \dot{\theta}_1$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) = (m_1 + m_2) L_1^2 \ddot{\theta}_1 + m_2 L_1 L_2 \ddot{\theta}_2 \cos(\theta_1 - \theta_2) - m_2 L_1 L_2 (\dot{\theta}_1 - \dot{\theta}_2) \dot{\theta}_2 \sin(\theta_1 - \theta_2)$$

$$\frac{\partial L}{\partial \theta_1} = - m_2 L_1 L_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) - g (m_1 + m_2) L_1 \sin\theta_1$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) - \frac{\partial L}{\partial \theta_1} = Q_{INC}$$

$$\Rightarrow (m_1 + m_2) L_1^2 \ddot{\theta}_1 + m_2 L_1 L_2 \ddot{\theta}_2 \cos(\theta_1 - \theta_2) + m_2 L_1 L_2 \dot{\theta}_2^2 \sin(\theta_1 - \theta_2)$$

$$+ (m_1 + m_2) g L_1 \sin\theta_1 = F L_1 \cos\theta_1$$

$$L = T - V = \frac{1}{2} m_2 \left[L_1^2 \dot{\theta}_1^2 + L_2^2 \dot{\theta}_2^2 + 2 L_1 L_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2) \right]$$

$$+ \frac{1}{2} m_1 \dot{\theta}_1^2 L_1^2 + m_1 g L_1 \cos\theta_1 + m_2 g \left(L_1 \cos\theta_1 + L_2 \cos\theta_2 \right)$$

$$\frac{\partial L}{\partial \dot{\theta}_2} = m_2 L_2^2 \dot{\theta}_2 + m_2 L_1 L_2 \dot{\theta}_1 \cos(\theta_1 - \theta_2)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) = m_2 L_2^2 \ddot{\theta}_2 + m_2 L_1 L_2 \ddot{\theta}_1 \cos(\theta_1 - \theta_2) - m_2 L_1 L_2 \dot{\theta}_1 (\dot{\theta}_1 - \dot{\theta}_2) \sin(\theta_1 - \theta_2)$$

$$\frac{\partial L}{\partial \theta_2} = m_2 L_1 L_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) - m_2 g L_2 \sin\theta_2$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) - \frac{\partial L}{\partial \theta_2} = Q_{2NC}$$

$$\Rightarrow m_2 L_2^2 \ddot{\theta}_2 + m_2 L_1 L_2 \ddot{\theta}_1 \cos(\theta_1 - \theta_2) - m_2 L_1 L_2 \dot{\theta}_1^2 \sin(\theta_1 - \theta_2)$$

$$+ m_2 g L_2 \sin\theta_2 = F L_2 \cos\theta_2$$