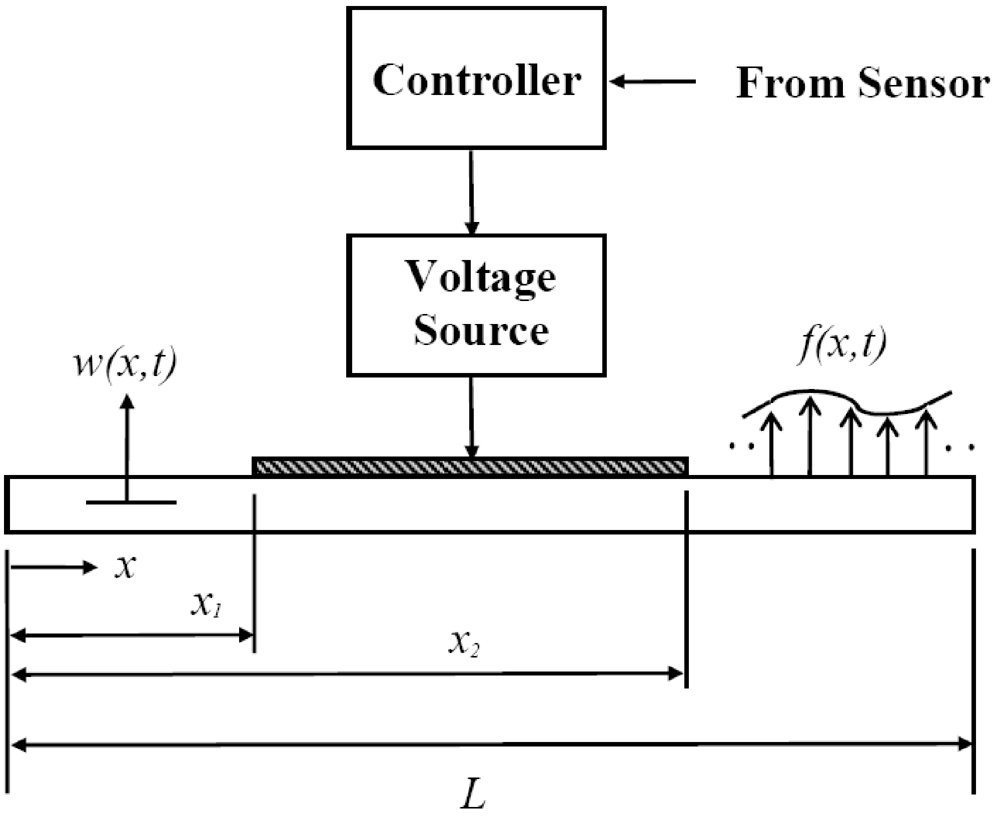
A Beam with a Surface Mounted Piezoelectric Element



Assumptions:

- The piezoelectric element is perfectly bonded
- The applied voltage is uniform along the beam, i.e., $v(x,t) = v(t)$
- The rotary inertia is negligible
- The shear deformation of the beam is negligible

Potential energies:

$$V_b = \frac{1}{2}\int_0^L E_b I_b \left(\frac{\partial^2 w}{\partial x^2}\right)^2 dx \qquad (1)$$

$$V_p = \frac{1}{2}\int_0^L E_p I_p \left(\frac{\partial^2 w}{\partial x^2}\right)^2 \left[H(x-x_1) - H(x-x_2)\right] dx \qquad (2)$$

where H is the Heaviside's function.

Kinetic energies:

$$T_b = \frac{1}{2}\int_0^L \rho_b A_b \left(\frac{\partial w}{\partial t}\right)^2 dx \qquad (3)$$

$$T_p = \frac{1}{2}\int_0^L \rho_p A_p \left(\frac{\partial w}{\partial t}\right)^2 [H(x-x_1) - H(x-x_2)] dx \qquad (4)$$

Virtual work:

$$\delta W_d = \int_0^L f(x,t)\delta w(x,t)dx \tag{5}$$

From the constitutive equation of the piezoelectric materials,

$$S_1 = s_{11}^E T_1 + d_{31} E_3 \tag{6}$$

$$T_1 = E_p (S_1 - d_{31} E_3) \tag{7}$$

$$\text{where } E_p = \frac{1}{s_{11}^E} \ , \qquad E_3 = \frac{v(t)}{t_p} \tag{8}$$

The virtual work done by the induced strain (force) is:

$$\delta W_p = \int_0^L E_p d_{31} b v(t) \delta\left(\frac{\partial u_p}{\partial x}\right)\left[H(x-x_1) - H(x-x_2)\right] dx \tag{9}$$

where b is the width of beam and piezo layer

$$u_p = -\left(\frac{t_b + t_p}{2}\right)\frac{\partial w}{\partial x} \tag{10}$$

Let $a = \dfrac{t_b + t_p}{2}$ (11)

$$\delta W_p = -\int_0^L E_p d_{31} a b v(t) \delta\left(\frac{\partial^2 w}{\partial x^2}\right)\left[H(x-x_1) - H(x-x_2)\right] dx \tag{12}$$

Apply extended Hamilton's principle,

$$\int_{t_1}^{t_2} \left(\delta T - \delta V + \delta W_{NC}\right) dt = 0$$

$$\begin{aligned}
&\int_{t_1}^{t_2} (\delta\left\{\frac{1}{2}\int_0^L \rho_b A_b \left(\frac{\partial w}{\partial t}\right)^2 dx\right\} + \delta\left\{\frac{1}{2}\int_0^L \rho_p A_p \left(\frac{\partial w}{\partial t}\right)^2 [H(x-x_1)-H(x-x_2)]dx\right\} \\
&-\delta\left\{\frac{1}{2}\int_0^L E_b I_b \left(\frac{\partial^2 w}{\partial x^2}\right)^2 dx\right\} - \delta\left\{\frac{1}{2}\int_0^L E_p I_p \left(\frac{\partial^2 w}{\partial x^2}\right)^2 [H(x-x_1)-H(x-x_2)]dx\right\} \\
&+\int_0^L f(x,t)\delta w(x,t)dx - \int_0^L E_p d_{31} abv(t)\delta\left(\frac{\partial^2 w}{\partial x^2}\right)[H(x-x_1)-H(x-x_2)]dx)dt \\
&= 0
\end{aligned} \tag{13}$$

Integration (term by term)

- $\int_{t_1}^{t_2} \delta\left(\frac{1}{2}\int_0^L \rho A \left[\frac{\partial w}{\partial t}\right]^2 dx\right) dt$ $\quad (\delta a^2 = 2a\,\delta a)$

$$= \int_0^L \int_{t_1}^{t_2} \rho A \frac{\partial w}{\partial t} \delta\left(\frac{\partial w}{\partial t}\right) dt\, dx = \int_0^L \int_{t_1}^{t_2} \rho A \frac{\partial w}{\partial t} \frac{\partial(\delta w)}{\partial t} dt\, dx$$

Integration by parts w.r.t. t

$$= \int_0^L \left(\rho A \frac{\partial w}{\partial t}\right) \delta w \Big|_{t_1}^{t_2} dx - \int_0^L \int_{t_1}^{t_2} \frac{\partial}{\partial t}\left(\rho A \frac{\partial w}{\partial t}\right) \delta w\, dt\, dx$$

$$= -\int_0^L \int_{t_1}^{t_2} \rho A \frac{\partial^2 w}{\partial t^2} \delta w\, dt\, dx$$

- $$\int_{t_1}^{t_2} \delta\left\{\frac{1}{2}\int_0^L \rho_b A_b \left(\frac{\partial w}{\partial t}\right)^2 dx\right\} dt = -\int_{t_1}^{t_2}\int_0^L \rho_b A_b \left(\frac{\partial^2 w}{\partial t^2}\right) \delta w\, dx\, dt \qquad (14)$$

$$\bullet \int_{t_1}^{t_2} \delta \left(\frac{1}{2} \int_0^L EI(x) \left[\frac{\partial^2 w}{\partial x^2} \right]^2 dx \right) dt$$

$$= \int_{t_1}^{t_2} \int_0^L \frac{1}{2} EI \cdot 2 \left(\frac{\partial^2 w}{\partial x^2} \right) \delta \left(\frac{\partial^2 w}{\partial x^2} \right) dx\, dt$$

Integration by parts w.r.t. x

$$= \int_{t_1}^{t_2} EI \frac{\partial^2 w}{\partial x^2} \delta \left(\frac{\partial w}{\partial x} \right) dt \Big|_0^L - \int_{t_1}^{t_2} \int_0^L \frac{\partial}{\partial x} \left(EI \frac{\partial^2 w}{\partial x^2} \right) \frac{\partial (\delta w)}{\partial x} dx\, dt$$

Integration by parts again

$$= \int_{t_1}^{t_2} EI \frac{\partial^2 w}{\partial x^2} \delta \left(\frac{\partial w}{\partial x} \right) \Big|_0^L dt - \int_{t_1}^{t_2} \frac{\partial}{\partial x} \left(EI \frac{\partial^2 w}{\partial x^2} \right) \delta w \Big|_0^L dt$$

$$+ \int_{t_1}^{t_2} \int_0^L \frac{\partial^2}{\partial x^2} \left(EI \frac{\partial^2 w}{\partial x^2} \right) \delta w\, dx\, dt$$

Integration by parts again

$$= \int_{t_1}^{t_2} EI \frac{\partial^2 w}{\partial x^2} \delta\left(\frac{\partial w}{\partial x}\right)\Big|_0^L dt - \int_{t_1}^{t_2} \frac{\partial}{\partial x}\left(EI \frac{\partial^2 w}{\partial x^2}\right) \delta w \Big|_0^L dt$$

$$+ \int_{t_1}^{t_2}\int_0^L \frac{\partial^2}{\partial x^2}\left(EI \frac{\partial^2 w}{\partial x^2}\right) \delta w \, dx \, dt$$

For uniform beam (EI = const.)

$$= \int_{t_1}^{t_2} EI \frac{\partial^2 w}{\partial x^2} \delta\left(\frac{\partial w}{\partial x}\right)\Big|_0^L dt - \int_{t_1}^{t_2} EI \frac{\partial^3 w}{\partial x^3} \delta w \Big|_0^L dt + \int_{t_1}^{t_2}\int_0^L EI \frac{\partial^4 w}{\partial x^4} \delta w \, dx \, dt$$

- $-\int_{t_1}^{t_2} \delta\left\{\frac{1}{2}\int_0^L E_b I_b \left(\frac{\partial^2 w}{\partial x^2}\right)^2 dx\right\} dt$

$$= -\int_{t_1}^{t_2} E_b I_b \left(\frac{\partial^2 w}{\partial x^2}\right) \delta\left(\frac{\partial w}{\partial x}\right)\Bigg|_0^L dt + \int_{t_1}^{t_2} E_b I_b \left(\frac{\partial^3 w}{\partial x^3}\right) \delta w \Bigg|_0^L dt - \int_{t_1}^{t_2}\int_0^L E_b I_b \left(\frac{\partial^4 w}{\partial x^4}\right) \delta w \, dx \, dt \qquad (15)$$

- $$\int_{t_1}^{t_2} \delta\left\{\frac{1}{2}\int_0^L \rho_b A_b \left(\frac{\partial w}{\partial t}\right)^2 dx\right\}dt = -\int_{t_1}^{t_2}\int_0^L \rho_b A_b \left(\frac{\partial^2 w}{\partial t^2}\right)\delta w dx dt \qquad (14)$$

- $$\int_{t_1}^{t_2} \delta\left\{\frac{1}{2}\int_0^L \rho_p A_p \left(\frac{\partial w}{\partial t}\right)^2 [H(x-x_1)-H(x-x_2)]dx\right\}dt$$

$$= -\int_{t_1}^{t_2}\int_0^L \rho_p A_p \left(\frac{\partial^2 w}{\partial t^2}\right)[H(x-x_1)-H(x-x_2)]\delta w dx dt \qquad (16)$$

- $$-\int_{t_1}^{t_2} \delta \left\{ \frac{1}{2} \int_0^L E_p I_p \left(\frac{\partial^2 w}{\partial x^2} \right)^2 [H(x-x_1) - H(x-x_2)] dx \right\} dt$$

$$= -\int_{t_1}^{t_2} \int_0^L E_p I_p \left(\frac{\partial^2 w}{\partial x^2} \right) [H(x-x_1) - H(x-x_2)] \delta \left(\frac{\partial^2 w}{\partial x^2} \right) dx dt$$

$$= -\int_{t_1}^{t_2} E_p I_p \left(\frac{\partial^2 w}{\partial x^2} \right) [H(x-x_1) - H(x-x_2)] \delta \left(\frac{\partial w}{\partial x} \right) \Bigg|_0^L dt$$

$$+ \int_{t_1}^{t_2} \int_0^L E_p I_p \left(\frac{\partial^3 w}{\partial x^3} \right) [H(x-x_1) - H(x-x_2)] \delta \left(\frac{\partial w}{\partial x} \right) dx dt \qquad (17)$$

$$+ \int_{t_1}^{t_2} \int_0^L E_p I_p \left(\frac{\partial^2 w}{\partial x^2} \right) [H'(x-x_1) - H'(x-x_2)] \delta \left(\frac{\partial w}{\partial x} \right) dx dt$$

$$= -\int_{t_1}^{t_2} E_p I_p \left(\frac{\partial^2 w}{\partial x^2}\right)[H(x-x_1)-H(x-x_2)]\delta\left(\frac{\partial w}{\partial x}\right)\Bigg|_0^L dt$$

$$+\int_{t_1}^{t_2}\int_0^L E_p I_p \left(\frac{\partial^3 w}{\partial x^3}\right)[H(x-x_1)-H(x-x_2)]\delta\left(\frac{\partial w}{\partial x}\right)dxdt \qquad (17)$$

$$+\int_{t_1}^{t_2}\int_0^L E_p I_p \left(\frac{\partial^2 w}{\partial x^2}\right)[H'(x-x_1)-H'(x-x_2)]\delta\left(\frac{\partial w}{\partial x}\right)dxdt$$

$$= \int_{t_1}^{t_2} E_p I_p \left(\frac{\partial^3 w}{\partial x^3}\right)[H(x-x_1)-H(x-x_2)]\delta w\Bigg|_0^L dt - \int_{t_1}^{t_2}\int_0^L E_p I_p \left(\frac{\partial^4 w}{\partial x^4}\right)[H(x-x_1)-H(x-x_2)]\delta w dxdt$$

$$-\int_{t_1}^{t_2}\int_0^L E_p I_p \left(\frac{\partial^3 w}{\partial x^3}\right)[H'(x-x_1)-H'(x-x_2)]\delta w dxdt + \int_{t_1}^{t_2} E_p I_p \left(\frac{\partial^2 w}{\partial x^2}\right)[H'(x-x_1)-H'(x-x_2)]\delta w\Bigg|_0^L dt$$

$$-\int_{t_1}^{t_2}\int_0^L E_p I_p \left(\frac{\partial^3 w}{\partial x^3}\right)[H'(x-x_1)-H'(x-x_2)]\delta w dxdt - \int_{t_1}^{t_2}\int_0^L E_p I_p \left(\frac{\partial^2 w}{\partial x^2}\right)[H''(x-x_1)-H''(x-x_2)]\delta w dxdt$$

- $$-\int_{t_1}^{t_2}\int_0^L E_p d_{31} abv(t)\delta\left(\frac{\partial^2 w}{\partial x^2}\right)[H(x-x_1)-H(x-x_2)]dxdt$$

$$= -\int_{t_1}^{t_2}\int_0^L E_p d_{31} abv(t)[H''(x-x_1)-H''(x-x_2)]\delta w dxdt \tag{18}$$

Substituting eqs. (14) - (18) into eq. (13),

$$\begin{aligned}
&\int_{t_1}^{t_2}\{\int_0^L \Big(-\rho_b A_b\left(\frac{\partial^2 w}{\partial t^2}\right)-\rho_p A_p\left(\frac{\partial^2 w}{\partial t^2}\right)[H(x-x_1)-H(x-x_2)]-E_b I_b\left(\frac{\partial^4 w}{\partial x^4}\right)\\
&-E_p I_p\left(\frac{\partial^4 w}{\partial x^4}\right)[H(x-x_1)-H(x-x_2)]-2E_p I_p\left(\frac{\partial^3 w}{\partial x^3}\right)[H'(x-x_1)-H'(x-x_2)]\\
&-E_p I_p\left(\frac{\partial^2 w}{\partial x^2}\right)[H''(x-x_1)-H''(x-x_2)]+f(x,t)-E_p d_{31} abv(t)[H''(x-x_1)-H''(x-x_2)]\Big)\delta w dx\\
&-E_b I_b\left(\frac{\partial^2 w}{\partial x^2}\right)\delta\left(\frac{\partial w}{\partial x}\right)\Bigg|_0^L + E_b I_b\left(\frac{\partial^3 w}{\partial x^3}\right)\delta w\Bigg|_0^L\}dt = 0
\end{aligned} \tag{19}$$

For arbitrary δw in $0 < x < L$,

Equation of motion:

$$\rho_b A_b \left(\frac{\partial^2 w}{\partial t^2} \right) + E_b I_b \left(\frac{\partial^4 w}{\partial x^4} \right) + \left\{ \rho_p A_p \left(\frac{\partial^2 w}{\partial t^2} \right) + E_p I_p \left(\frac{\partial^4 w}{\partial x^4} \right) \right\} [H(x - x_1) - H(x - x_2)]$$

$$+ 2E_p I_p \left(\frac{\partial^3 w}{\partial x^3} \right) [H'(x - x_1) - H'(x - x_2)] + E_p I_p \left(\frac{\partial^2 w}{\partial x^2} \right) [H''(x - x_1) - H''(x - x_2)] \tag{20}$$

$$+ E_p d_{31} abv(t)[H''(x - x_1) - H''(x - x_2)] = f(x,t)$$

with boundary conditions:

$$\left(\frac{\partial^2 w}{\partial x^2} \right) \delta \left(\frac{\partial w}{\partial x} \right) \Bigg|_0^L = 0 \text{ and } \left(\frac{\partial^3 w}{\partial x^3} \right) \delta w \Bigg|_0^L = 0 \tag{21}$$

Galerkin's Method

Consider

$$M\left[\frac{\partial^2 w(x,t)}{\partial t^2}\right] + \mathcal{L}\left[w(x,t)\right] = f(x,t)$$

$M, \mathcal{L}$ are differential operators

e.g.

$$m(x)\frac{\partial^2 w}{\partial t^2} + \frac{\partial^2}{\partial x^2}\left(EI(x)\frac{\partial^2 w}{\partial x^2}\right) = f(x,t)$$

$$M = m$$

$$\mathcal{L} = \frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2}{\partial x^2}\right)$$

Let $\hat{w}(x,t) = \sum_{r=1}^{N} \phi_r(x) q_r(t)$

$q_r(t)$ — generalized coordinates

$\phi_r(x)$ — comparison functions

i.e. functions that satisfy all the B.C.s but not necessary the domain eq.

Sub. $\hat{w}$ into Eq. of motion,

$$M\left[\frac{\partial^2 \hat{w}}{\partial t^2}\right] + \mathcal{L}[\hat{w}] \simeq f(x,t)$$

$$\varepsilon = M\left[\frac{\partial^2 \hat{w}}{\partial t^2}\right] + \mathcal{L}[\hat{w}] - f(x,t)$$

How to minimize ε (error) ?

Inner product of two functions

$$\langle g(x), h(x) \rangle \triangleq \int_0^L g(x) h(x)\, dx \qquad \text{(definition)}$$

Galerkin :

Min. ε by $\langle \varepsilon, \phi_r \rangle = 0$

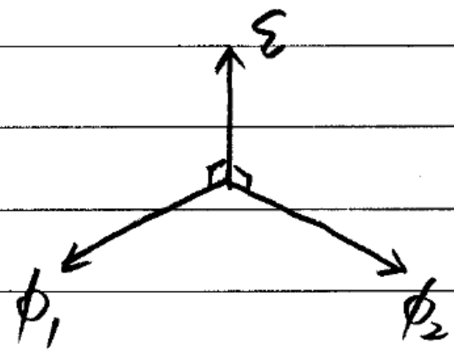
$$\Rightarrow \langle \varepsilon(x), \phi_r(x) \rangle = \int_0^L \varepsilon(x) \phi_r(x)\, dx = 0, \quad r = 1, 2, \cdots, N$$

Galerkin :

Min. ε by $\langle \varepsilon, \phi_r \rangle = 0$

$$\Rightarrow \langle \varepsilon(x), \phi_r(x) \rangle = \int_0^L \varepsilon(x)\phi_r(x)\,dx = 0, \quad r = 1, 2, \cdots, N$$

i.e., we make the error "orthogonal" to all the comparison functions used



- Plane (space) expanded by ϕ_r
- Force the error to have no "projection" on this space, i.e. ε has no influence on ϕ_r

Galerkin :

Min. ε by $\langle \varepsilon, \phi_r \rangle = 0$

$\Rightarrow \langle \varepsilon(x), \phi_r(x) \rangle = \int_0^L \varepsilon(x)\phi_r(x)\,dx = 0, \quad r = 1, 2, \cdots, N$

Do the inner product N times

$\Rightarrow$ Get N equations

(PDE discritize N ODE)

Discretization: (Galerkin's Method)

Assume

$$w(x,t)=\sum_{i=1}^{n}\phi_i(x)q_i(t) \tag{22}$$

where $\phi_i(x)$ satisfies all B.C.s

Sub. eq. (22) into eq. (20),

$$\rho_b A_b \sum_{i=1}^{n}\phi_i(x)\ddot{q}_i(t)+E_b I_b \sum_{i=1}^{n}\phi_i^{(4)}(x)q_i(t)$$

$$+\left[\rho_p A_p \sum_{i=1}^{n}\phi_i(x)\ddot{q}_i(t)+E_p I_p \sum_{i=1}^{n}\phi_i^{(4)}(x)q_i(t)\right]\left[H(x-x_1)-H(x-x_2)\right]$$

$$+2E_p I_p \sum_{i=1}^{n}\phi_i^{(3)}(x)q_i(t)\left[H'(x-x_1)-H'(x-x_2)\right]+E_p I_p \sum_{i=1}^{n}\phi_i''(x)q_i(t)\left[H''(x-x_1)-H''(x-x_2)\right]$$

$$+E_p d_{31} a b v(t)\left[H''(x-x_1)-H''(x-x_2)\right]-f(x,t)=\varepsilon \tag{23}$$

Min ε by $\left\langle \varepsilon, \phi_j \right\rangle = 0$

$$\Rightarrow \left\langle \varepsilon, \phi_j \right\rangle = \int_0^L \varepsilon(x,t)\phi_j(x)dx = 0 \qquad j = 1,2,\ldots,n \tag{24}$$

$$\Rightarrow \left[\rho_b A_b \left(\sum_{i=1}^{n} \int_0^L \phi_i(x)\phi_j(x)dx \right) + \rho_p A_p \left(\sum_{i=1}^{n} \int_0^L \phi_i(x)\phi_j(x)\left[H(x-x_1) - H(x-x_2)\right]dx \right) \right] \ddot{q}_i(t)$$

$$+ \left[E_b I_b \left(\sum_{i=1}^{n} \int_0^L \phi_i^{(4)}(x)\phi_j(x)dx \right) + E_p I_p \left(\sum_{i=1}^{n} \int_0^L \phi_i^{(4)}(x)\phi_j(x)\left[H(x-x_1) - H(x-x_2)\right]dx \right) \right] q_i(t)$$

$$+ \left[2E_p I_p \left(\sum_{i=1}^{n} \int_0^L \phi_i^{(3)}(x)\phi_j(x)\left[H'(x-x_1) - H'(x-x_2)\right]dx \right) \right] q_i(t)$$

$$+ \left[E_p I_p \left(\sum_{i=1}^{n} \int_0^L \phi_i''(x)\phi_j(x)\left[H''(x-x_1) - H''(x-x_2)\right]dx \right) \right] q_i(t)$$

$$+ E_p d_{31} abv(t) \left(\int_0^L \phi_j(x)\left[H''(x-x_1) - H''(x-x_2)\right]dx \right) - \int_0^L f(x,t)\phi_j(x)dx = 0 \tag{25}$$

- $$\int_0^L \phi_i^{(4)}(x)\phi_j(x)dx = \int_0^L \phi_i''(x)\phi_j''(x)dx \qquad (26)$$

- $$\int_0^L \phi_i^{(4)}(x)\phi_j(x)\left[H(x-x_1)-H(x-x_2)\right]dx \qquad (27)$$

- $$2\int_0^L \phi_i^{(3)}(x)\phi_j(x)\left[H'(x-x_1)-H'(x-x_2)\right]dx$$

$$= 2\phi_i^{(3)}(x)\phi_j(x)\left[H(x-x_1)-H(x-x_2)\right]\Big|_0^L - 2\int_0^L \phi_i^{(4)}(x)\phi_j(x)\left[H(x-x_1)-H(x-x_2)\right]dx$$

$$-2\int_0^L \phi_i^{(3)}\phi_j'(x)\left[H(x-x_1)-H(x-x_2)\right]dx \qquad (28)$$

- $\int_0^L \phi_i''(x)\phi_j(x)\left[H''(x-x_1)-H''(x-x_2)\right]dx$

$$=\phi_i''(x)\phi_j(x)\left[H'(x-x_1)-H'(x-x_2)\right]\Big|_0^L-\int_0^L \phi_i^{(3)}(x)\phi_j(x)\left[H'(x-x_1)-H'(x-x_2)\right]dx$$

$$-\int_0^L \phi_i''(x)\phi_j'(x)\left[H'(x-x_1)-H'(x-x_2)\right]dx \qquad (29)$$

$$=\int_0^L \phi_i^{(4)}(x)\phi_j(x)\left[H(x-x_1)-H(x-x_2)\right]dx+\int_0^L \phi_i^{(3)}(x)\phi_j'(x)\left[H(x-x_1)-H(x-x_2)\right]dx$$

$$+\int_0^L \phi_i^{(3)}(x)\phi_j'(x)\left[H(x-x_1)-H(x-x_2)\right]dx+\int_0^L \phi_i''(x)\phi_j''(x)\left[H(x-x_1)-H(x-x_2)\right]dx$$

- (27)+(28)+(29) = $\int_0^L \phi_i''(x)\phi_j''(x)\left[H(x-x_1)-H(x-x_2)\right]dx$ (30)

- $\int_0^L \phi_j(x)\left[H''(x-x_1)-H''(x-x_2)\right]dx$

$$= \phi_j(x)\left[H'(x-x_1)-H'(x-x_2)\right]\Big|_0^L - \int_0^L \phi_j'(x)\left[H'(x-x_1)-H'(x-x_2)\right]dx$$

$$= -\phi_j'(x)\left[H(x-x_1)-H(x-x_2)\right]\Big|_0^L + \int_0^L \phi_j''(x)\left[H(x-x_1)-H(x-x_2)\right]dx$$

$$= \int_0^L \phi_j''(x)\left[H(x-x_1)-H(x-x_2)\right]dx = \phi_j'(x_2)-\phi_j'(x_1) \qquad (31)$$

Substituting (26), (30), (31) into (25),

$$\left[\rho_b A_b\left(\sum_{i=1}^{n}\int_0^L \phi_i(x)\phi_j(x)dx\right)+\rho_p A_p\left(\sum_{i=1}^{n}\int_0^L \phi_i(x)\phi_j(x)\left[H(x-x_1)-H(x-x_2)\right]dx\right)\right]\ddot{q}_i(t)$$

$$+\left[E_b I_b\left(\sum_{i=1}^{n}\int_0^L \phi_i''(x)\phi_j''(x)dx\right)+E_p I_p\left(\sum_{i=1}^{n}\int_0^L \phi_i''(x)\phi_j''(x)\left[H(x-x_1)-H(x-x_2)\right]dx\right)\right]q_i(t)$$

$$+E_p d_{31}abv(t)\left(\phi_j'(x_2)-\phi_j'(x_1)\right)=\int_0^L f(x,t)\phi_j(x)dx \qquad (32)$$

$$\sum_{i=1}^{n} m_{ij}\ddot{q}_i(t) + \sum_{i=1}^{n} k_{ij}q_i(t) = f_{c_j}(t) + f_{d_j}(t)\,, \qquad j = 1,\, 2,\, \ldots\,,\, n$$

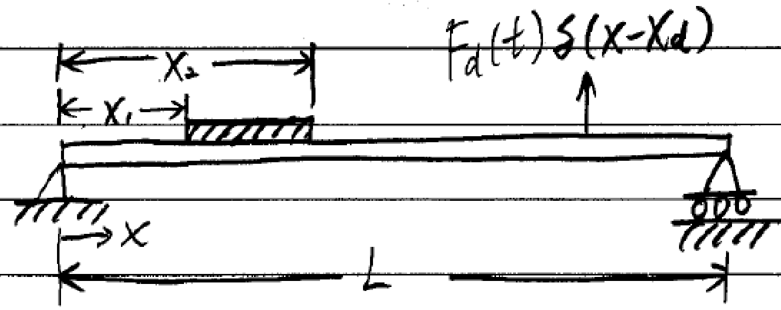
where $m_{ij} = \rho_b A_b \int_0^L \phi_i(x)\phi_j(x)dx + \rho_p A_p \int_{x_1}^{x_2} \phi_i(x)\phi_j(x)dx$

$$k_{ij} = E_b I_b \int_0^L \phi_i''(x)\phi_j''(x)dx + E_p I_p \int_{x_1}^{x_2} \phi_i''(x)\phi_j''(x)dx$$

$$f_{c_j} = E_p d_{31} abv(t)\left[\left(\phi_j'(x_1) - \phi_j'(x_2)\right)\right]$$

$$f_{d_j} = \int_0^L f(x,t)\phi_j(x)dx$$

e.g. A Simply Supported Beam with a Piezoelectric Element



Choose $\phi_r(x) = \sin\left(\frac{r\pi x}{L}\right)$, $r = 1, 2, \cdots, N$

satisfy all B.C.s:

$x = 0$: $w = 0$, $w'' = 0$

$x = L$: $w = 0$, $w'' = 0$

Assume

$$w(x,t) \simeq \sum_{r=1}^{N} \phi_r(x)\, q_r(t)$$

⇒ Discretized eq. of motion :

$$[M]\{\ddot{q}\} + [C]\{\dot{q}\} + [K]\{q\} = \{f_c\} + \{f_d\}$$

where

$$m_{sr} = \int_0^L \rho_b A_b \phi_s(x)\phi_r(x)\,dx + \int_{x_1}^{x_2} \rho_p A_p \phi_s(x)\phi_r(x)\,dx$$

$$k_{sr} = \int_0^L E_b I_b \phi_s''(x)\phi_r''(x)\,dx + \int_{x_1}^{x_2} E_p I_p \phi_s''(x)\phi_r''(x)\,dx$$

$$f_{c_s} = E_p d_{31}\, ab\, v(t)\,[\phi_s'(x_1) - \phi_s'(x_2)]$$

$$f_{d_s} = F_d(t)\,\phi_s(x_d)$$

$$[C] = \alpha[M] + \beta[K]$$

for $s = r$:

$$m_{rr} = \frac{\rho_b A_b L}{2} + \rho_p A_p \left(\frac{X_2 - X_1}{2}\right) + \frac{\rho_p A_p L}{4\pi r}\left[\sin\left(\frac{2\pi r X_1}{L}\right) - \sin\left(\frac{2\pi r X_2}{L}\right)\right]$$

$$k_{rr} = \left(\frac{\pi r}{L}\right)^4 \left\{\frac{E_b I_b L}{2} + E_p I_p \left(\frac{X_2 - X_1}{2}\right) + \frac{E_p I_p L}{4\pi r}\left[\sin\left(\frac{2\pi r X_1}{L}\right) - \sin\left(\frac{2\pi r X_2}{L}\right)\right]\right\}$$

for $s \neq r$:

$$m_{sr} = \frac{\rho_p A_p L}{\pi}\left[\frac{r \sin\left(\frac{s\pi X}{L}\right)\cos\left(\frac{r\pi X}{L}\right)}{(s^2 - r^2)} + \frac{s \cos\left(\frac{s\pi X}{L}\right)\sin\left(\frac{r\pi X}{L}\right)}{(r^2 - s^2)}\right]_{X_1}^{X_2}$$

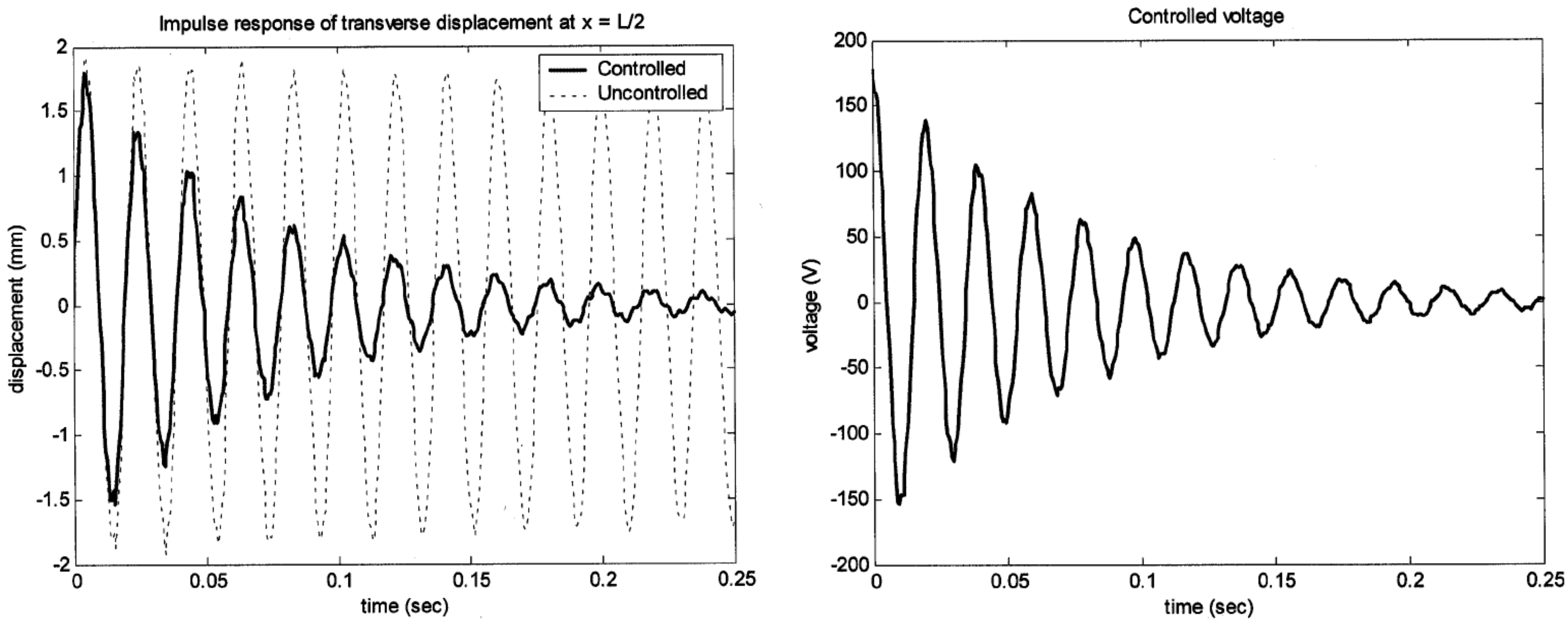
$$k_{sr} = \frac{E_p I_p L}{\pi}\left(\frac{s r \pi^2}{L^2}\right)^2\left[\frac{r \sin\left(\frac{s\pi X}{L}\right)\cos\left(\frac{r\pi X}{L}\right)}{(s^2 - r^2)} + \frac{s \cos\left(\frac{s\pi X}{L}\right)\sin\left(\frac{r\pi X}{L}\right)}{(r^2 - s^2)}\right]_{X_1}^{X_2}$$

```
% Simply supported beam with a PZT element

clear

% initialize

b=1.27e-2;
L=0.3;
x1=0.1; x2=0.16;
xd=0.18;

Eb=7.1e10;
pb=2700;  tb=2.286e-3;

Ec=6.49e10;
pc=7600;  tc=0.762e-3;
d31=-175e-12;

Ac=b*tc;  Ab=b*tb;
Ib=b*tb^3/12;  Ic=b*tc^3/12;
a=(tb+tc)/2;
```

```
% stiffness and mass matrices

N=5; % no. of expansion terms

K=zeros(N);
M=zeros(N);
C=zeros(N);
Fc=zeros(N,1); Fd=zeros(N,1);

for r=1:N;
    for s=1:N;
        if r == s
    K(r,s)=(pi*r/L)^4*(Eb*Ib*L/2+Ec*Ic*(x2-x1)/2+...
        Ec*Ic*L/(4*pi*r)*(sin(2*pi*r*x1/L)-sin(2*pi*r*x2/L)));
    M(r,s)=pb*Ab*L/2+pc*Ac*(x2-x1)/2+...
        pc*Ac*L/(4*pi*r)*(sin(2*pi*r*x1/L)-sin(2*pi*r*x2/L));
        else
    K(r,s)=Ec*Ic*L/pi*(pi^2*r*s/L^2)^2*...
        ((r*sin(s*pi*x2/L)*cos(r*pi*x2/L))/(s^2-r^2)+...
        (s*sin(r*pi*x2/L)*cos(s*pi*x2/L))/(r^2-s^2)-...
        ((r*sin(s*pi*x1/L)*cos(r*pi*x1/L))/(s^2-r^2)+...
        (s*sin(r*pi*x1/L)*cos(s*pi*x1/L))/(r^2-s^2)));
    M(r,s)=pc*Ac*L/pi*((r*sin(s*pi*x2/L)*cos(r*pi*x2/L))/(s^2-r^2)+...
        (s*sin(r*pi*x2/L)*cos(s*pi*x2/L))/(r^2-s^2)-...
        ((r*sin(s*pi*x1/L)*cos(r*pi*x1/L))/(s^2-r^2)+...
        (s*sin(r*pi*x1/L)*cos(s*pi*x1/L))/(r^2-s^2)));
        end;
    end;
```

```
    % due to voltage input
    Fc(r)=-a*Ec*d31*b*(pi*r/L)*(cos(r*pi*x2/L)-cos(r*pi*x1/L));
    % due to discrete force with magnitude 1/100
    Fd(r)=1/100*sin(r*pi*xd/L);
end;

% add internal damping

C=0.64*M+1.2e-6*K;

% state-space model

AL=-inv(M)*K;
AR=-inv(M)*C;
A=[zeros(N) eye(N);...
   AL AR];
BL1=inv(M)*Fc; BL2=inv(M)*Fd;
B1=[zeros(N,1);BL1];
B2=[zeros(N,1);BL2];
for r=1:N;
    CCw(1,r)=sin(r*pi/2);  % displacement w at midpoint (x=L/2)
end;
CC=[CCw zeros(1,N)];
D=[0];
```

```
% control gain
Kc=1.0e+004*[-1.4035 -0.1594 0.5005 0.2457 -0.1712...
        -0.0289 -0.0033 0.0076 0.0045 -0.0030];
Ac=A-B1*Kc;

% impulse response

t=0:0.0005:0.25;
IU=1;
[y,x,t]=impulse(A,B2,CC,D,IU,t); % uncontrolled response
[yc,x,t]=impulse(Ac,B2,CC,D,IU,t); % controlled response
u=-Kc*x'; % controlled voltage

% plot results

figure(1),plot(t,yc*1000,t,y*1000,':') % unit (mm)
title('Impulse response of transverse displacement at x = L/2')
xlabel('time (sec)')
ylabel('displacement (mm)')

figure(2),plot(t,u)
title('Controlled voltage')
xlabel('time (sec)')
ylabel('voltage (V)')
```


Impulse response of transverse displacement at x = L/2
Controlled
Uncontrolled
displacement (mm)
time (sec)
Controlled voltage
voltage (V)
time (sec)