Computational Mechanics

Chapter 1 Introduction

(Covers Chapter 1, and Sections 2.1, 2.2, 2.4 and 2.6 of *A first course in finite elements*)

Introduction to Computational Mechanics

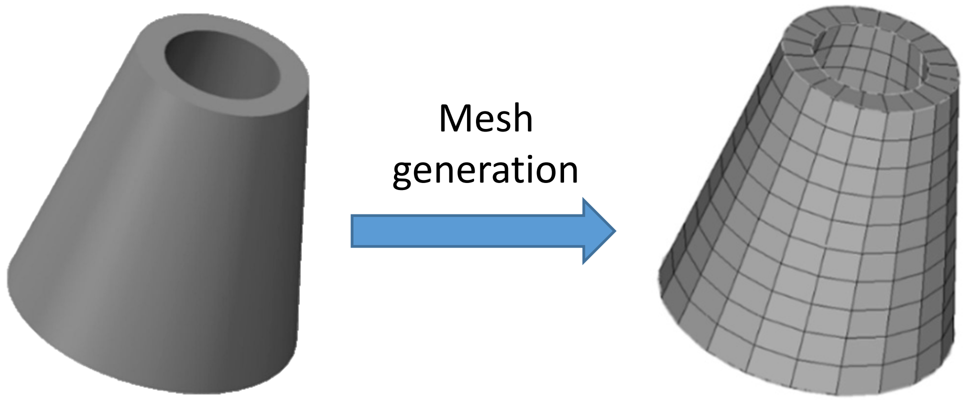
- Computational mechanics is the discipline concerned with the use of computational methods to study phenomena governed by the principles of mechanics[1].
- Computational mechanics is interdisciplinary:
  - Mechanics – Objective
  - Mathematics – Method
  - Computational science – Tool
- Common computational mechanics method:
  - Finite different method – direct approximation for partial differential equations, cannot handle complex geometry
  - Finite element method (FEM) – wide application areas, great for complicated engineering problems
    - ❖ Lagrangian mesh: deform with body, suitable for solid analysis
    - ❖ Eulerian mesh: fixed mesh, appropriate for fluid/field analysis

Similar equations for infinitesimal deformation

1. Ghaboussi J, Wu XS. Numerical Methods in Computational Mechanics. CRC Press; 2016 Nov 25.

Introduction to Finite Element Method (FEM)

- The FEM was developed in 1950s in the aerospace industry, from industry to academia
- FEM divides complex objectives/fields into regular finite elements:



Geometric model[1] Finite element model

1. Convert complex domains and boundary conditions into simple and unified ones

2. Change one set of mechanics equations to numerous ones – applicable to computers

- Commercial FEM software packages: Abaqus, LS-DYNA, NASTRAN, etc.

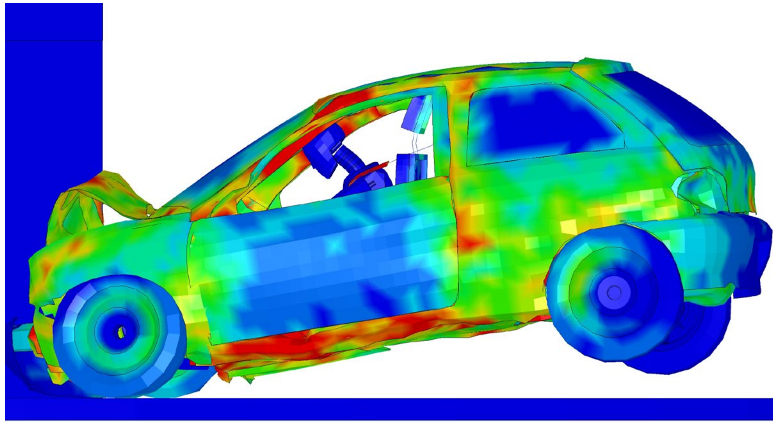
MATLAB is also a useful tool in development stage

1. Chen W, Zheng X, Liu S. Finite-element-mesh based method for modeling and optimization of lattice structures for additive manufacturing. Materials. 2018 Nov;11(11):2073.

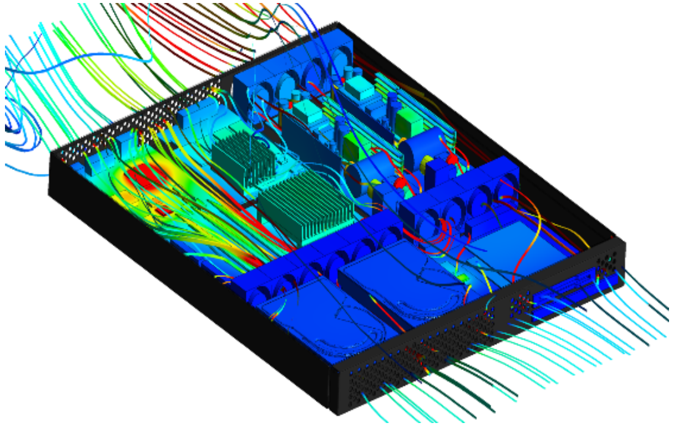
Applications of FEM

- The range of applications of FEM is very large:
  - Categorized based on field types – mechanical, thermal, electromagnetic, etc.
  - Categorized based on industries – automotive, consumer electronics, arms, etc.

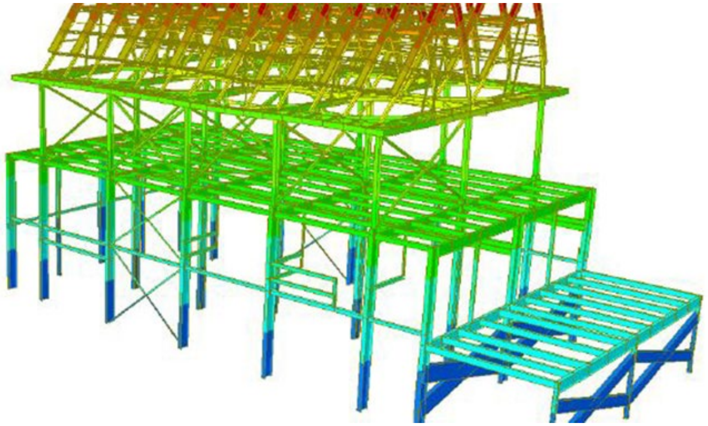
  ……
- Application examples of FEM include:



Automotive crash analysis[1]



Electronic chip performance analysis[2]



Seismic analysis for architectures[3]

1. https://ctag.com/en/servicios/tecnologia-seguridad-pasiva/simulacion-impacto_crash-simulation/
2. https://www.finiteelementanalysis.com.au/featured/dcir-analysis-of-pcb-in-ansys-siwave/
3. http://pepconsultingengineers.it/en/fem-analysis-e-seismic-design/

5-Step Analysis in FEM

- Preprocessing: subdividing the target domain into finite elements by automatic mesh generators.

- Element formulation: development of equations for elements.

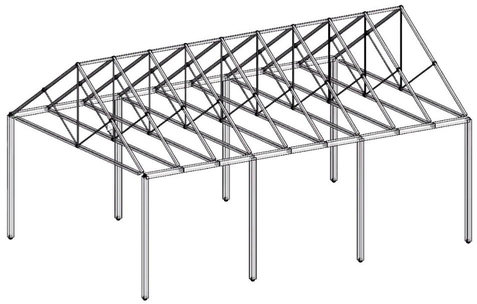
- Assembly: obtaining equations for the whole system by gathering ones at the element-level.

Easiest analysis with 1D bar elements and no partial differential equations

- Solving equations.

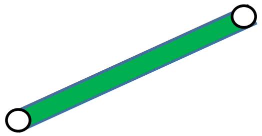
- Postprocessing: calculation results visualization and output.

Single Bar Element



Truss structure[1]

Mesh generation



Bar element

Bar elements are assumed to be thin:

1. Negligible torsion, bending and shear

2. Internal forces only along axes – spring

$$\boldsymbol{F}_n^e \leftrightarrow \boldsymbol{u}_n^e \text{ ?}$$

Nodal internal force

Nodal displacement

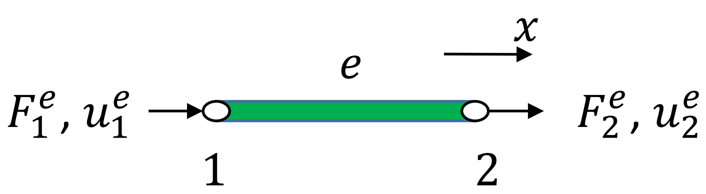
Notation:

- Element number – superscript
- Node number – subscript

1. https://www.karamba3d.com/tutorials/tutorials_workflow/export-gable-truss-to-revit-with-geometry-gym-rhynamo/

1D Single Bar Element – Stress and Deformation

- For simplification:
  1. Straight bar element
  2. Material obeying Hooke's Law
  3. Only axial loading
  4. Static analysis



- Equilibrium of the element (static analysis):

$$F_1^e + F_2^e = 0$$

- Calculation of axial stress:

$$\sigma^e = \frac{p^e}{A^e} = \frac{F_2^e}{A^e}$$

Sign of p and F?

- Hooke's Law:

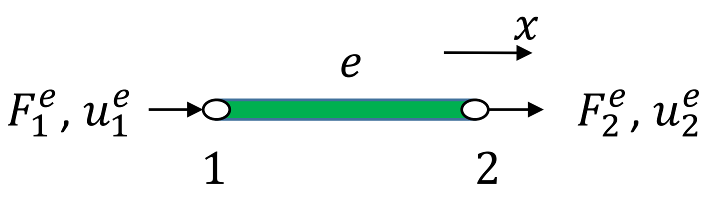
$$\sigma^e = E^e \varepsilon^e$$

- Compatible deformation – no gap or overlap

- Calculation of axial strain:

$$\varepsilon^e = \frac{\delta^e}{l^e} = \frac{l_{new}^e - l^e}{l^e} = \frac{u_2^e - u_1^e}{l^e}$$

1D Single Bar Element – Stiffness Matrix



- Definition of stiffness:

$$F_2^e = A^e\sigma^e = A^e E^e \varepsilon^e = \underbrace{\frac{A^e E^e}{l^e}}_{k^e}(u_2^e - u_1^e)$$

$$\Rightarrow F_2^e = k^e(u_2^e - u_1^e)$$

- Equilibrium condition:

$$F_1^e = -F_2^e = k^e(u_1^e - u_2^e)$$

- Force – displacement relationship written in the matrix form for computation:

$$\underbrace{\begin{bmatrix} F_1^e \\ F_2^e \end{bmatrix}}_{\mathbf{F}^e} = \underbrace{\begin{bmatrix} k^e & -k^e \\ -k^e & k^e \end{bmatrix}}_{\mathbf{K}^e} \underbrace{\begin{bmatrix} u_1^e \\ u_2^e \end{bmatrix}}_{\mathbf{d}^e}$$

$$\Rightarrow \mathbf{F}^e = \mathbf{K}^e \mathbf{d}^e$$

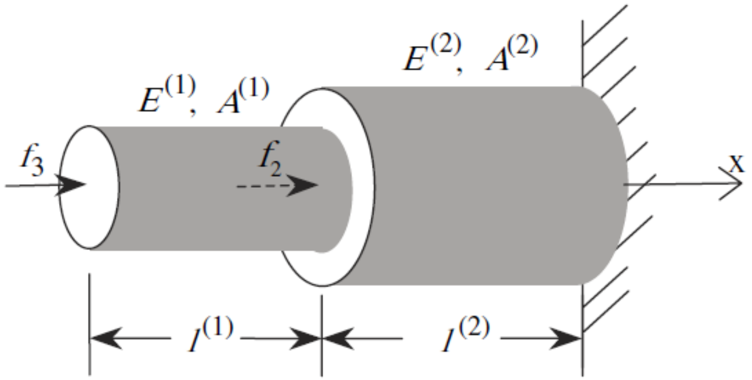
- Element stiffness matrix:

$$\mathbf{K}^e = \begin{bmatrix} k^e & -k^e \\ -k^e & k^e \end{bmatrix} = \frac{A^e E^e}{l^e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

1. Symmetric matrix
2. 1D element with constant A^e
3. Linearity for all ingredients

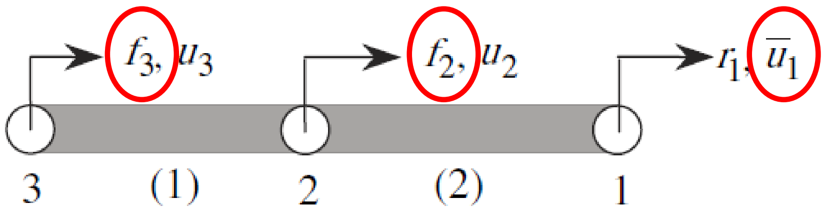
FEM Model for a 2-Bar System

- A system is made by several elements – starting from a 2-bar system:



2-bar system

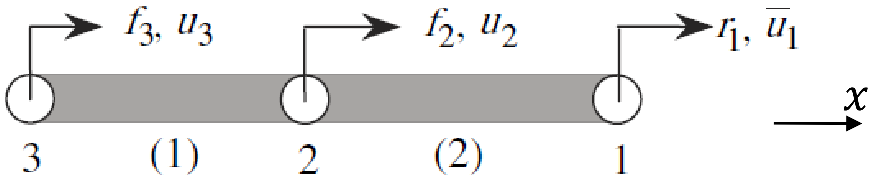
Cross-section size does not matter for 1D analysis



FEM model of the 2-bar system

- Global nodes and elements are not numbered in a specific order in FEM
- Constraint of FEM problems:
  - Boundary conditions are intentionally applied on nodes
  - Boundary conditions are either external forces or displacements, but not both to avoid over-constraint

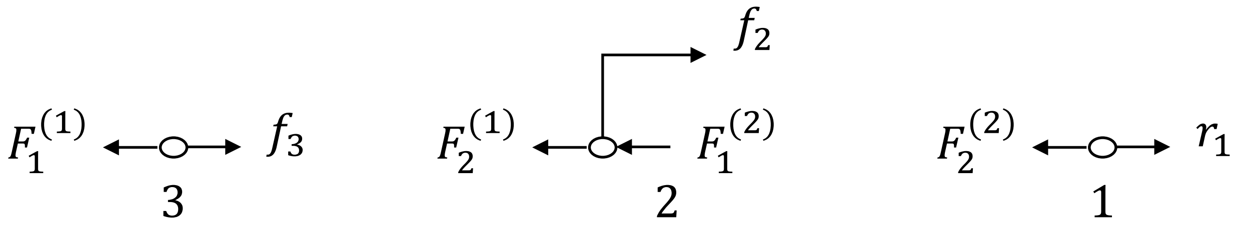
Force Equations for a 2-Bar System



- Internal forces and displacements for elements:

Local node numbering are unified to increase along x direction

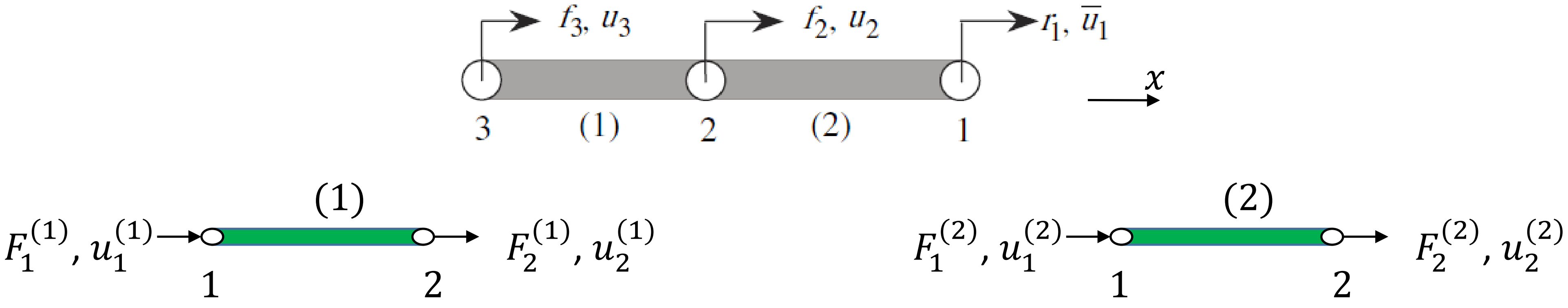
- Free-body diagrams of nodes:



$$\underbrace{\begin{bmatrix} 0 \\ F_2^{(1)} \\ F_1^{(1)} \end{bmatrix}}_{\tilde{\mathbf{F}}^{(1)}} + \underbrace{\begin{bmatrix} F_2^{(2)} \\ F_1^{(2)} \\ 0 \end{bmatrix}}_{\tilde{\mathbf{F}}^{(2)}} = \begin{bmatrix} r_1 \\ f_2 \\ f_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 \\ f_2 \\ f_3 \end{bmatrix}}_{\mathbf{f}} + \underbrace{\begin{bmatrix} r_1 \\ 0 \\ 0 \end{bmatrix}}_{\mathbf{r}}$$

Internal forces change directions because of Newton's 3rd law.

Stiffness Equations for a 2-Bar System (1/2)



- Utilize the 1D single bar stiffness equation:

Element (1):

$$\begin{bmatrix} F_1^{(1)} \\ F_2^{(1)} \end{bmatrix} = \begin{bmatrix} k^{(1)} & -k^{(1)} \\ -k^{(1)} & k^{(1)} \end{bmatrix} \begin{bmatrix} u_3 \\ u_2 \end{bmatrix}$$

Element (2):

$$\begin{bmatrix} F_1^{(2)} \\ F_2^{(2)} \end{bmatrix} = \begin{bmatrix} k^{(2)} & -k^{(2)} \\ -k^{(2)} & k^{(2)} \end{bmatrix} \begin{bmatrix} u_2 \\ u_1 \end{bmatrix}$$

Stiffness Equations for a 2-Bar System (2/2)

$$\underbrace{\begin{bmatrix}0\\F_2^{(1)}\\F_1^{(1)}\end{bmatrix}}_{\tilde{\mathbf{F}}^{(1)}}+\underbrace{\begin{bmatrix}F_2^{(2)}\\F_1^{(2)}\\0\end{bmatrix}}_{\tilde{\mathbf{F}}^{(2)}}=\begin{bmatrix}r_1\\f_2\\f_3\end{bmatrix}=\underbrace{\begin{bmatrix}0\\f_2\\f_3\end{bmatrix}}_{\mathbf{f}}+\underbrace{\begin{bmatrix}r_1\\0\\0\end{bmatrix}}_{\mathbf{r}}$$

$$\begin{bmatrix}0\\F_2^{(1)}\\F_1^{(1)}\end{bmatrix}=\begin{bmatrix}0&0&0\\0&k^{(1)}&-k^{(1)}\\0&-k^{(1)}&k^{(1)}\end{bmatrix}\begin{bmatrix}\bar{u}_1\\u_2\\u_3\end{bmatrix}$$

$$\begin{bmatrix}F_2^{(2)}\\F_1^{(2)}\\0\end{bmatrix}=\begin{bmatrix}k^{(2)}&-k^{(2)}&0\\-k^{(2)}&k^{(2)}&0\\0&0&0\end{bmatrix}\begin{bmatrix}\bar{u}_1\\u_2\\u_3\end{bmatrix}$$

$$\overbrace{\begin{bmatrix}0&0&0\\0&k^{(1)}&-k^{(1)}\\0&-k^{(1)}&k^{(1)}\end{bmatrix}}^{\tilde{\mathbf{K}}^{(1)}}\overbrace{\begin{bmatrix}u_1\\u_2\\u_3\end{bmatrix}}^{\mathbf{d}}+\overbrace{\begin{bmatrix}k^{(2)}&-k^{(2)}&0\\-k^{(2)}&k^{(2)}&0\\0&0&0\end{bmatrix}}^{\tilde{\mathbf{K}}^{(2)}}\overbrace{\begin{bmatrix}\bar{u}_1\\u_2\\u_3\end{bmatrix}}^{\mathbf{d}}$$

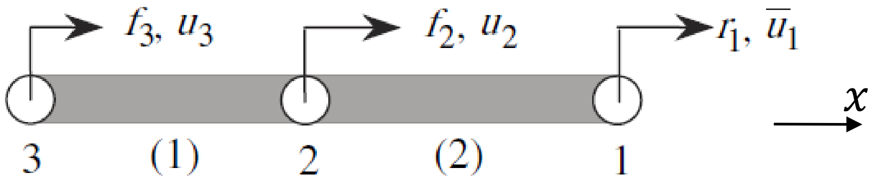
$$=\begin{bmatrix}0\\f_2\\f_3\end{bmatrix}+\begin{bmatrix}r_1\\0\\0\end{bmatrix}$$

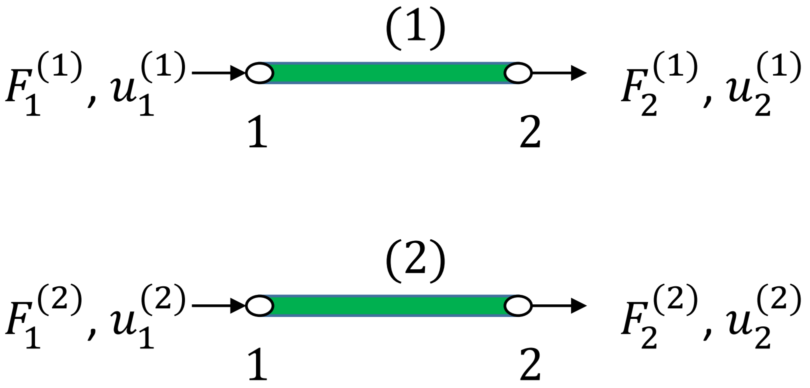
Equation for unknowns

$$\left(\tilde{\mathbf{K}}^{(1)}+\tilde{\mathbf{K}}^{(2)}\right)\mathbf{d}=\mathbf{K}\mathbf{d}=\mathbf{f}+\mathbf{r}$$

$$\mathbf{K}=\sum\tilde{\mathbf{K}}^{(i)}=\begin{bmatrix}k^{(2)}&-k^{(2)}&0\\-k^{(2)}&k^{(1)}+k^{(2)}&-k^{(1)}\\0&-k^{(1)}&k^{(1)}\end{bmatrix}$$

Direct Assembly of Global Stiffness Matrix





Element (1):

$$\begin{array}{cc} 3 & 2 \\ \left[\begin{array}{cc} k^{(1)} & -k^{(1)} \\ -k^{(1)} & k^{(1)} \end{array}\right] & \begin{array}{c} 3 \\ 2 \end{array} \end{array}$$

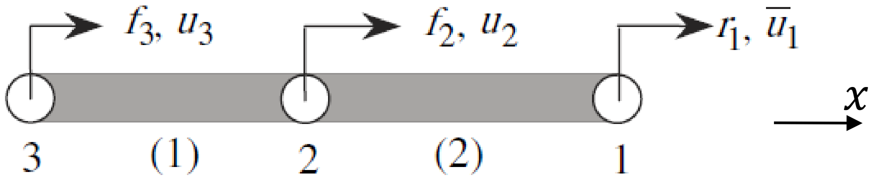
Element (2):

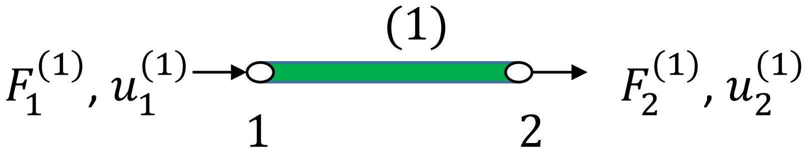
$$\begin{array}{cc} 2 & 1 \\ \left[\begin{array}{cc} k^{(2)} & -k^{(2)} \\ -k^{(2)} & k^{(2)} \end{array}\right] & \begin{array}{c} 2 \\ 1 \end{array} \end{array}$$

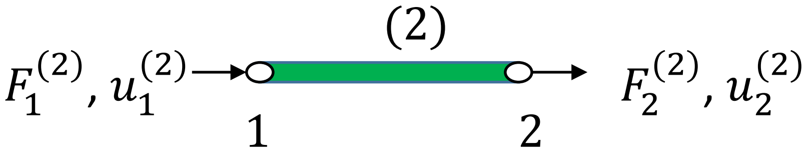
- Direct assembly – computer implementation:

$$\mathbf{K} = K_{mn} = \sum \widetilde{K}_{mn}^{(i)} = \begin{array}{cc} \begin{array}{ccc} 1 & 2 & 3 \end{array} & \\ \left[\begin{array}{ccc} k^{(2)} & -k^{(2)} & 0 \\ -k^{(2)} & k^{(1)} + k^{(2)} & -k^{(1)} \\ 0 & -k^{(1)} & k^{(1)} \end{array}\right] & \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \end{array}$$

Introduction of Gather Matrices







- Gather matrices are introduced to enforce compatibility among elements:

$$\mathbf{d}^{(1)} = \begin{bmatrix} u_1^{(1)} \\ u_2^{(1)} \end{bmatrix} = \begin{bmatrix} u_3 \\ u_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}}_{\mathbf{L}^{(1)}} \begin{bmatrix} \bar{u}_1 \\ u_2 \\ u_3 \end{bmatrix}, \qquad \mathbf{d}^{(2)} = \begin{bmatrix} u_1^{(2)} \\ u_2^{(2)} \end{bmatrix} = \begin{bmatrix} u_2 \\ u_1 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}}_{\mathbf{L}^{(2)}} \begin{bmatrix} \bar{u}_1 \\ u_2 \\ u_3 \end{bmatrix}$$

$$\mathbf{d}^e = \mathbf{L}^e \mathbf{d}$$

$$\mathbf{F}^e = \mathbf{K}^e \mathbf{d}^e \Rightarrow \mathbf{F}^e = \mathbf{K}^e \mathbf{L}^e \mathbf{d}$$

Application of Gather Matrices for Stiffness

$$\begin{bmatrix} 0 \\ F_2^{(1)} \\ F_1^{(1)} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_1^{(1)} \\ F_2^{(1)} \end{bmatrix} = \mathbf{L}^{(1)\mathrm{T}}\mathbf{F}^{(1)}$$

$$\begin{bmatrix} F_2^{(2)} \\ F_1^{(2)} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} F_1^{(2)} \\ F_2^{(2)} \end{bmatrix} = \mathbf{L}^{(2)\mathrm{T}}\mathbf{F}^{(2)}$$

$$\begin{bmatrix} 0 \\ F_2^{(1)} \\ F_1^{(1)} \end{bmatrix} + \begin{bmatrix} F_2^{(2)} \\ F_1^{(2)} \\ 0 \end{bmatrix} = \begin{bmatrix} r_1 \\ f_2 \\ f_3 \end{bmatrix} = \begin{bmatrix} 0 \\ f_2 \\ f_3 \end{bmatrix} + \begin{bmatrix} r_1 \\ 0 \\ 0 \end{bmatrix}$$

$$\sum \mathbf{L}^{(i)\mathrm{T}}\mathbf{F}^{(i)} = \mathbf{f} + \mathbf{r}$$

$$\mathbf{F}^{e} = \mathbf{K}^{e}\mathbf{L}^{e}\mathbf{d}$$

This is another approach to calculate the global stiffness matrix.

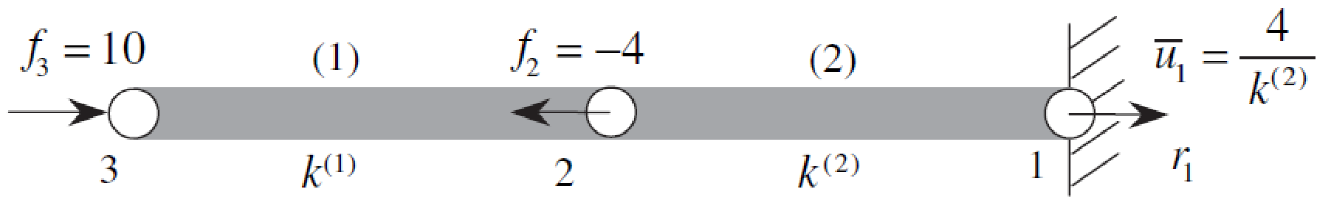
$$\mathbf{f} + \mathbf{r} = \sum \mathbf{L}^{(i)\mathrm{T}}\mathbf{F}^{(i)}$$

$$= \sum \mathbf{L}^{(i)\mathrm{T}}\mathbf{K}^{(i)}\mathbf{L}^{(i)}\mathbf{d}$$

$$= \mathbf{K}\mathbf{d}$$

Cumbersome but the symbolic expression is suitable for theoretical analysis

Solution Method – System Partition

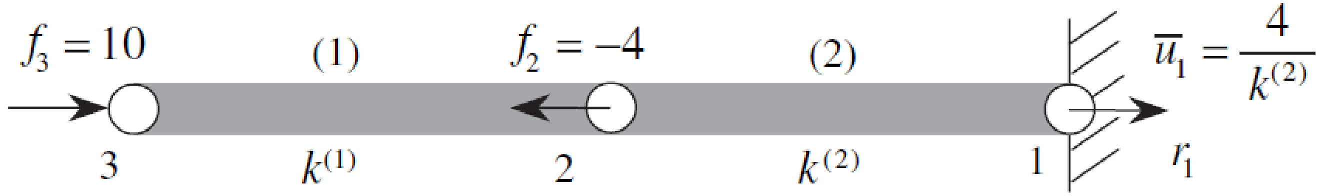


- System equation:

$$\mathbf{K} \cdot \mathbf{d} = \begin{bmatrix} k^{(2)} & -k^{(2)} & 0 \\ -k^{(2)} & k^{(1)} + k^{(2)} & -k^{(1)} \\ 0 & -k^{(1)} & k^{(1)} \end{bmatrix} \begin{bmatrix} \bar{u}_1 \\ u_2 \\ u_3 \end{bmatrix} = \mathbf{f} + \mathbf{r} = \begin{bmatrix} 0 \\ f_2 \\ f_3 \end{bmatrix} + \begin{bmatrix} r_1 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} k^{(2)} & -k^{(2)} & 0 \\ -k^{(2)} & k^{(1)} + k^{(2)} & -k^{(1)} \\ 0 & -k^{(1)} & k^{(1)} \end{bmatrix} \begin{bmatrix} 4/k^{(2)} \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} r_1 \\ -4 \\ 10 \end{bmatrix}$$

E-Nodes and F-Nodes



- 3 unknowns – 3 equations:

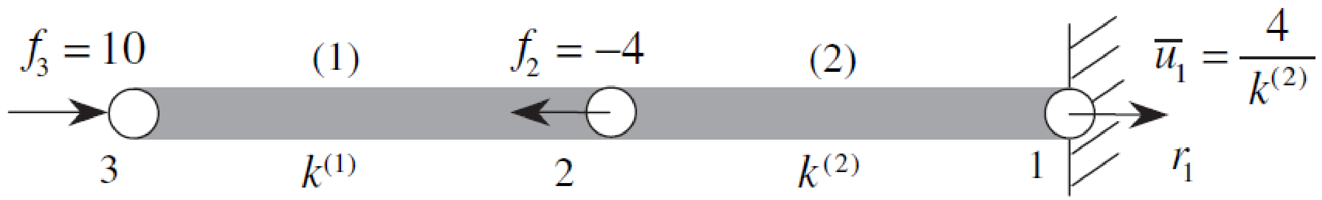
$$\begin{bmatrix} k^{(2)} & -k^{(2)} & 0 \\ -k^{(2)} & k^{(1)} + k^{(2)} & -k^{(1)} \\ 0 & -k^{(1)} & k^{(1)} \end{bmatrix} \begin{bmatrix} 4/k^{(2)} \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} r_1 \\ -4 \\ 10 \end{bmatrix}$$

- E-nodes – essential boundary conditions with known displacements and unknown external forces.

- F-nodes – free boundary conditions with unknown displacements and known external forces.

Nodes with no boundary conditions applied is F-nodes (0 external forces).

Matrix Partition



- 3 unknowns – 3 equations:

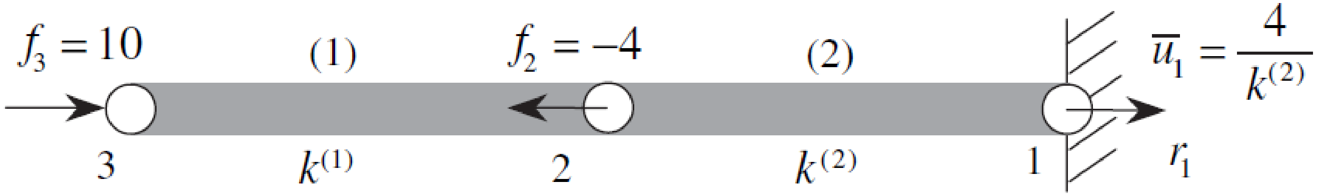
$$\mathbf{K}_\mathrm{E} \qquad \mathbf{K}_\mathrm{EF} \qquad \bar{\boldsymbol{d}}_\mathrm{E} \qquad \mathbf{r}_\mathrm{E}$$

$$\begin{bmatrix} k^{(2)} & -k^{(2)} & 0 \\ -k^{(2)} & k^{(1)} + k^{(2)} & -k^{(1)} \\ 0 & -k^{(1)} & k^{(1)} \end{bmatrix} \begin{bmatrix} 4/k^{(2)} \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} r_1 \\ -4 \\ 10 \end{bmatrix}$$

$$\mathbf{K}_\mathrm{EF}^\mathrm{T} \qquad \mathbf{K}_\mathrm{F} \qquad \mathbf{d}_\mathrm{F} \qquad \mathbf{f}_\mathrm{F}$$

- E-nodes can be numbered first for partition convenience.

Solution to F-Nodes



- 3 unknowns – 3 equations:

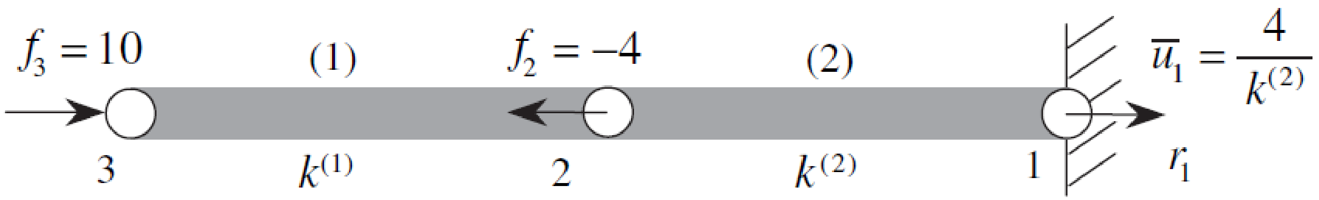
$$\begin{array}{cccc} \mathbf{K}_{\mathrm{E}} \quad\quad \mathbf{K}_{\mathrm{EF}} & & \bar{\mathbf{d}}_{\mathrm{E}} & \mathbf{r}_{\mathrm{E}} \\ \left[\begin{array}{c:cc} k^{(2)} & -k^{(2)} & 0 \\ \hdashline -k^{(2)} & k^{(1)}+k^{(2)} & -k^{(1)} \\ 0 & -k^{(1)} & k^{(1)} \end{array}\right] & & \left[\begin{array}{c} 4/k^{(2)} \\ \hdashline u_2 \\ u_3 \end{array}\right] = & \left[\begin{array}{c} r_1 \\ \hdashline -4 \\ 10 \end{array}\right] \\ \mathbf{K}_{\mathrm{EF}}^{\mathrm{T}} \quad\quad \mathbf{K}_{\mathrm{F}} & & \mathbf{d}_{\mathrm{F}} & \mathbf{f}_{\mathrm{F}} \end{array}$$

- Solve F-nodes part first:

$$\mathbf{K}_{\mathrm{EF}}^{\mathrm{T}} \cdot \bar{\mathbf{d}}_{\mathrm{E}} + \mathbf{K}_{\mathrm{F}} \cdot \mathbf{d}_{\mathrm{F}} = \mathbf{f}_{\mathrm{F}}$$

$$\Rightarrow \mathbf{d}_{\mathrm{F}} = \begin{bmatrix} u_2 \\ u_3 \end{bmatrix} = \mathbf{K}_{\mathrm{F}}^{-1} \cdot \left(\mathbf{f}_{\mathrm{F}} - \mathbf{K}_{\mathrm{EF}}^{\mathrm{T}} \cdot \bar{\mathbf{d}}_{\mathrm{E}}\right) = \begin{bmatrix} \dfrac{10}{k^{(2)}} \\ \dfrac{10}{k^{(1)}} + \dfrac{10}{k^{(2)}} \end{bmatrix}$$

Solution to E-Nodes



- 3 unknowns – 3 equations:

$$
\begin{array}{cccc}
\mathbf{K}_{\mathrm{E}} \quad \mathbf{K}_{\mathrm{EF}} & & \bar{\mathbf{d}}_{\mathrm{E}} & \mathbf{r}_{\mathrm{E}} \\
\left[\begin{array}{c|cc} k^{(2)} & -k^{(2)} & 0 \\ \hline -k^{(2)} & k^{(1)}+k^{(2)} & -k^{(1)} \\ 0 & -k^{(1)} & k^{(1)} \end{array}\right] & & \left[\begin{array}{c} 4/k^{(2)} \\ \hline u_2 \\ u_3 \end{array}\right] = & \left[\begin{array}{c} r_1 \\ \hline -4 \\ 10 \end{array}\right] \\
\mathbf{K}_{\mathrm{EF}}^{\mathrm{T}} \quad \mathbf{K}_{\mathrm{F}} & & \mathbf{d}_{\mathrm{F}} & \mathbf{f}_{\mathrm{F}}
\end{array}
$$

- Solve E-nodes part:

$$\mathbf{K}_{\mathrm{E}} \cdot \bar{\mathbf{d}}_{\mathrm{E}} + \mathbf{K}_{\mathrm{EF}} \cdot \mathbf{d}_{\mathrm{F}} = \mathbf{r}_{\mathrm{E}}$$

$$\Rightarrow \mathbf{r}_{\mathrm{E}} = [r_1] = [k^{(2)}][4/k^{(2)}] + [-k^{(2)} \quad 0]\begin{bmatrix} \dfrac{10}{k^{(2)}} \\ \dfrac{10}{k^{(1)}} + \dfrac{10}{k^{(2)}} \end{bmatrix} = -6$$

5-Step Analysis in FEM

- Preprocessing: subdividing the target domain into finite elements by automatic mesh generators.

- Element formulation: development of equations for elements.

- Assembly: obtaining equations for the whole system by gathering ones at the element-level.

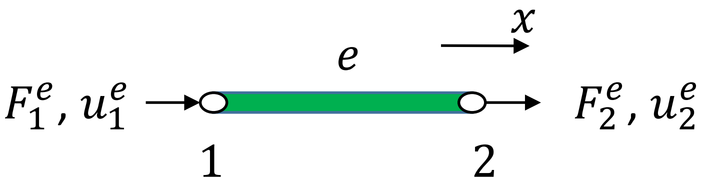
- Solving equations.

  - Only determined by element and node numbers
  - Applicable to other problems with different elements

- Postprocessing: calculation results visualization and output.

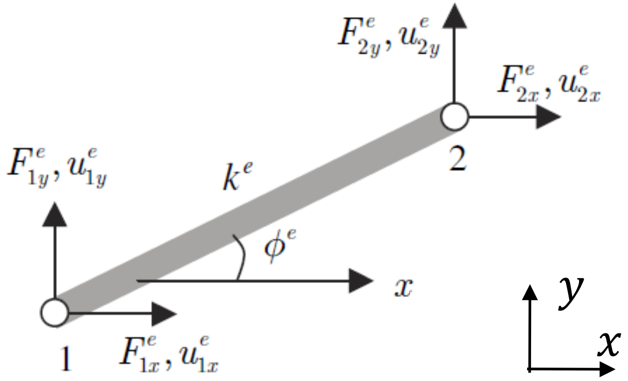
2D Truss System

- Expand the single bar element from 1D space to 2D space:



$$\mathbf{F}^e = \begin{bmatrix} F_1^e \\ F_2^e \end{bmatrix} = [F_1^e \quad F_2^e]^{\mathrm{T}}$$

$$\mathbf{d}^e = \begin{bmatrix} u_1^e \\ u_2^e \end{bmatrix} = [u_1^e \quad u_2^e]^{\mathrm{T}}$$

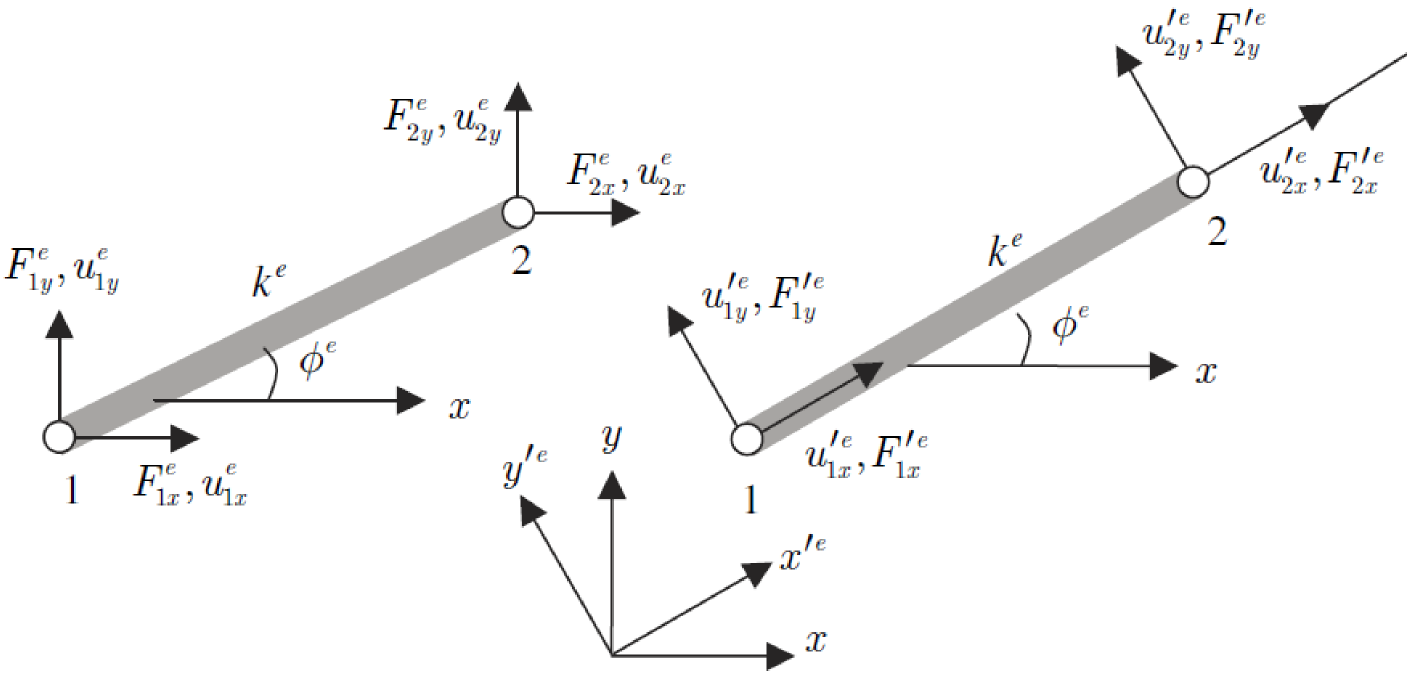


$$\mathbf{F}^e = [F_{1x}^e \quad F_{1y}^e \quad F_{2x}^e \quad F_{2y}^e]^{\mathrm{T}}$$

$$\mathbf{d}^e = [u_{1x}^e \quad u_{1y}^e \quad u_{2x}^e \quad u_{2y}^e]^{\mathrm{T}}$$

Local Coordinate for 2D Single Bar Elements

- Local coordinate facilitates force-displacement analysis:



- Along bar direction, similar to the 1D element:

Transverse displacement has negligible effect

$$\begin{bmatrix} F'^e_{1x} \\ F'^e_{2x} \end{bmatrix} = \begin{bmatrix} k^e & -k^e \\ -k^e & k^e \end{bmatrix} \begin{bmatrix} u'^e_{1x} \\ u'^e_{2x} \end{bmatrix}$$

- In transverse direction, bar elements are assumed to have 0 shear stiffness:

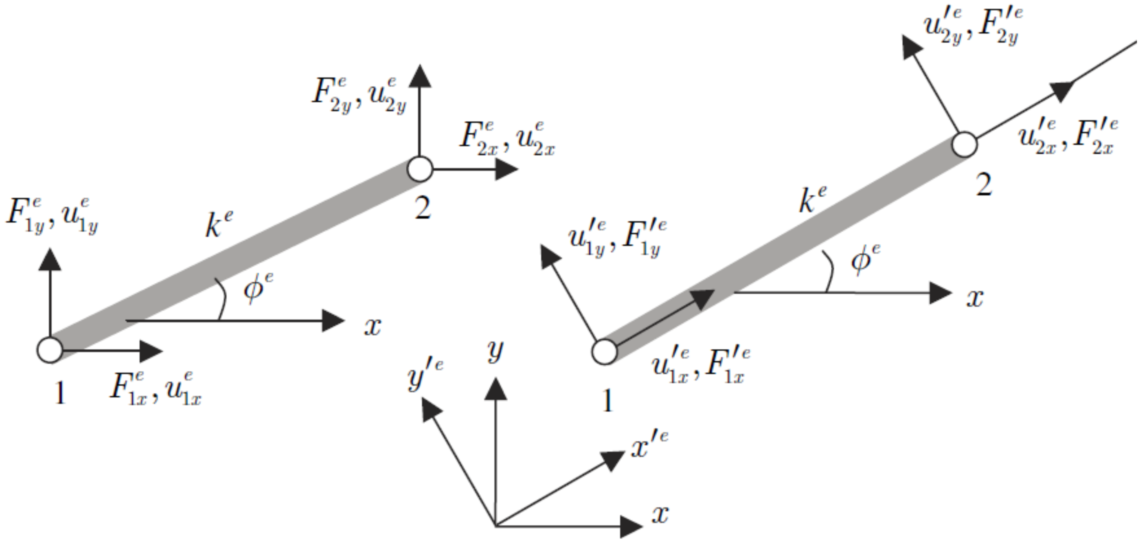
$$F'^e_{1y} = F'^e_{2y} = 0$$

$$\Rightarrow$$

$$\underbrace{\begin{bmatrix} F'^e_{1x} \\ F'^e_{1y} \\ F'^e_{2x} \\ F'^e_{2y} \end{bmatrix}}_{\mathbf{F}'^e} = \underbrace{k^e \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{\mathbf{K}'^e} \underbrace{\begin{bmatrix} u'^e_{1x} \\ u'^e_{1y} \\ u'^e_{2x} \\ u'^e_{2y} \end{bmatrix}}_{\mathbf{d}'^e}$$

Transformation Law

- Local equations should be transformed to the global coordinate for system-level analysis:



- Displacement vector coordinate transformation:

$$\underbrace{\begin{bmatrix} u'^e_{1x} \\ u'^e_{1y} \\ u'^e_{2x} \\ u'^e_{2y} \end{bmatrix}}_{\mathbf{d}'^e} = \underbrace{\begin{bmatrix} \cos\Phi^e & \sin\Phi^e & 0 & 0 \\ -\sin\Phi^e & \cos\Phi^e & 0 & 0 \\ 0 & 0 & \cos\Phi^e & \sin\Phi^e \\ 0 & 0 & -\sin\Phi^e & \cos\Phi^e \end{bmatrix}}_{\mathbf{R}^e} \underbrace{\begin{bmatrix} u^e_{1x} \\ u^e_{1y} \\ u^e_{2x} \\ u^e_{2y} \end{bmatrix}}_{\mathbf{d}^e}$$

$$(\mathbf{R}^e)^{-1} = \mathbf{R}^{e\mathbf{T}} \Rightarrow \mathbf{R}^{e\mathbf{T}}\mathbf{d}'^e = \mathbf{R}^{e\mathbf{T}}\mathbf{R}^e\mathbf{d}'^e = \mathbf{d}^e$$

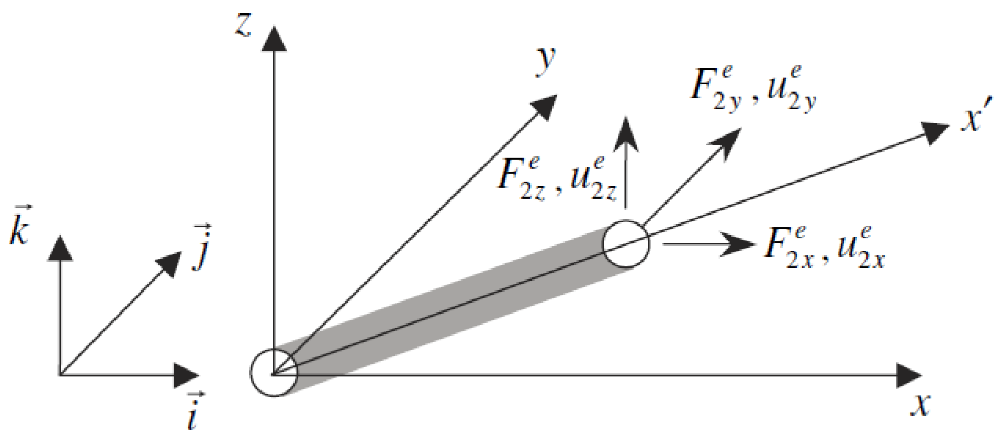
- Similarly for force vectors:

$$\mathbf{R}^e\mathbf{F}^e = \mathbf{F}'^e, \qquad \mathbf{R}^{e\mathbf{T}}\mathbf{F}'^e = \mathbf{F}^e$$

$$\Rightarrow \mathbf{F}^e = \mathbf{R}^{e\mathbf{T}}\mathbf{F}'^e = \mathbf{R}^{e\mathbf{T}}\mathbf{K}'^e\mathbf{d}'^e = \underbrace{\mathbf{R}^{e\mathbf{T}}\mathbf{K}'^e\mathbf{R}^e}_{\mathbf{K}^e}\mathbf{d}^e$$

3D Truss System

- Further expand the single bar element from 1D space to 3D space:



$$\mathbf{F}^e = \begin{bmatrix} F_{1x}^e & F_{1y}^e & F_{1z}^e & F_{2x}^e & F_{2y}^e & F_{2z}^e \end{bmatrix}^{\mathrm{T}}$$

$$\mathbf{d}^e = \begin{bmatrix} u_{1x}^e & u_{1y}^e & u_{1z}^e & u_{2x}^e & u_{2y}^e & u_{2z}^e \end{bmatrix}^{\mathrm{T}}$$

- Transformation from local to global coordinates:

$$\vec{\imath}' = \frac{1}{l^e}\left(x_{21}^e \vec{\imath} + y_{21}^e \vec{\jmath} + z_{21}^e \vec{k}\right)$$

$$(x_{21}^e = x_2^e - x_1^e, \text{ etc.})$$

$$\overrightarrow{u_I}^e = u'^e_{Ix}\vec{\imath}' + u'^e_{Iy}\vec{\jmath}' + u'^e_{Iz}\vec{k}' = u_{Ix}^e\vec{\imath} + u_{Iy}^e\vec{\jmath} + u_{Iz}^e\vec{k}$$

$$\Rightarrow u'^e_{Ix} = u_{Ix}^e\vec{\imath}\cdot\vec{\imath}' + u_{Iy}^e\vec{\jmath}\cdot\vec{\imath} + u_{Iz}^e\vec{k}\cdot\vec{\imath}$$

$$\Rightarrow u'^e_{Ix} = \frac{1}{l^e}\left(x_{21}^e u_{Ix}^e + y_{21}^e u_{Iy}^e + z_{21}^e u_{Iz}^e\right)$$

$$\Rightarrow \begin{bmatrix} u'^e_{1x} \\ u'^e_{2x} \end{bmatrix} = \underbrace{\frac{1}{l^e}\begin{bmatrix} x_{21}^e & y_{21}^e & z_{21}^e & & \mathbf{0} & \\ & \mathbf{0} & & x_{21}^e & y_{21}^e & z_{21}^e \end{bmatrix}}_{\mathbf{R}^e}\mathbf{d}^e$$

$$\Rightarrow \mathbf{K}_{6\times6}^e = \mathbf{R}_{6\times2}^{e\mathrm{T}}\mathbf{K}'^e_{2\times2}\mathbf{R}_{2\times6}^e$$

Local stiffness matrix is 2X2, but coordinate transformation add dimension

The End