Computational Mechanics

Chapter 10 Constitutive Models – Elasticity

Characteristic of 1D Elasticity

- Stress depends only on current strain (deformation) state:
  - Loading and unloading stress-strain curves are identical
  - Original shape is fully recovered upon unloading
  - Reversible deformation
  - One-to-one correspondence between stress and strain
- Multiple dimension elasticity may not satisfy (such as hypoelasticity)

1D Elasticity with Small Strain

- Expression with true stress and true strain:

$$\sigma_x = s(\varepsilon_x)$$

- Path-independence implies no energy dissipation and existence of potential energy:

$$\sigma_x = s(\varepsilon_x) = \frac{dw(\varepsilon_x)}{d\varepsilon_x}$$

Strain energy density per unit volume

$$\Rightarrow dw(\varepsilon_x) = \sigma_x d\varepsilon_x \Rightarrow w = \int_0^{\varepsilon_x} \sigma_x d\varepsilon_x$$

- Linear elasticity at the initial stage:

$$\sigma_x = E\varepsilon_x \Rightarrow w = \frac{1}{2} E\varepsilon_x^2$$

1D Elasticity with Large Strain

- Introduction of strain from deformation gradient, such as Green strain $E_x^G = \frac{1}{2}(\boldsymbol{F}^T \cdot \boldsymbol{F} - \boldsymbol{I})_x$

- Existence of potential energy – stress need to form work conjugate:

$$S_x = \frac{dw}{dE_x^G}$$

2nd Piola-Kirchhoff stress (work conjugate)

Hypoelastic:
- Simple implementation
- Path-independent *approximation* at small strain for multiaxial problem

- Hyperelastic – stress obtained from a potential function of the strain:

1D simplest example $w = \frac{1}{2}E^{SE}(E_x^G)^2 \Rightarrow S_x = \frac{dw}{dE_x^G} = E^{SE}E_x^G$

- Hypoelastic – relate rate of Cauchy stress $\dot{\sigma}_x$ to rate-of-deformation D_x:

1D linear example $\dot{\sigma}_x = ED_x = E\frac{\dot{\lambda}_x}{\lambda_x} \Rightarrow \sigma_x = Eln\lambda_x$

Multiaxial Nonlinear Elasticity

- 2 situations for material constitutive laws:

  - Different constitutive behaviors for different materials;

  - Same material behaviors described by different measurement of stress and deformation.

Kirchhoff Material

- Extension of linear elastic laws – good for small strains and large rotations:

$$S_{ij} = C_{ijkl}E^G_{kl}, \qquad \boldsymbol{S} = \boldsymbol{C}:\boldsymbol{E}^G$$

Tensor expression for constitutive laws

- Tangent modulus tensor for rate relationship:

$$\dot{\boldsymbol{S}} = \boldsymbol{C}^{SE}:\dot{\boldsymbol{E}}^G$$

- Minor symmetry of $\boldsymbol{C}$ from stress and strain symmetry:

$$C_{ijkl} = C_{jikl} = C_{ijlk}$$

- Path-independent material – strain energy per unit volume:

$$w = \int S_{ij}dE^G_{ij} = \int C_{ijkl}E^G_{kl}dE^G_{ij} = \frac{1}{2}C_{ijkl}E^G_{kl}E^G_{ij} = \frac{1}{2}\boldsymbol{E}^G:\boldsymbol{C}:\boldsymbol{E}^G \Rightarrow S_{ij} = \frac{\partial w}{\partial E^G_{ij}} \ or\ \boldsymbol{S} = \frac{\partial w}{\partial \boldsymbol{E}^G}$$

$$\Rightarrow C_{ijkl} = \frac{\partial^2 w}{\partial E^G_{ij}\partial E^G_{kl}} = \frac{\partial^2 w}{\partial E^G_{kl}\partial E^G_{ij}} = C_{klij}$$

Major symmetry

Constitutive Law Components

- General 4th order tensor component number:

$$3^4 = 81$$

- Minor symmetry:

$$6 \times 6 = 36$$

- Major symmetry:

$$\frac{(6+1) \times 6}{2} = 21$$

- Voigt notation:

$$\begin{bmatrix} S_{11} \\ S_{22} \\ S_{33} \\ S_{23} \\ S_{13} \\ S_{12} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ & & C_{33} & C_{34} & C_{35} & C_{36} \\ & & & C_{44} & C_{45} & C_{46} \\ & symmetry & & & C_{55} & C_{56} \\ & & & & & C_{66} \end{bmatrix} \begin{bmatrix} E_{11}^G \\ E_{22}^G \\ E_{33}^G \\ 2E_{23}^G \\ 2E_{13}^G \\ 2E_{12}^G \end{bmatrix}$$

Fully anisotropic material

- Expression with Voigt notation:

$$\{\boldsymbol{S}\} = [\boldsymbol{C}]\{\boldsymbol{E}^G\}, \qquad S_a = C_{ab} E_b^G$$

- Isotropic elastic material:

$$C_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu \left(\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk} \right)$$

Lamé constants

$$\mu = \frac{E}{2(1+v)}, \qquad \lambda = \frac{vE}{(1+v)(1-2v)}$$

$$S_{ij} = C_{ijkl} E_{kl}^G = \lambda \delta_{ij} \delta_{kl} E_{kl}^G + \mu \left(\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk} \right) E_{kl}^G$$

$$\Rightarrow S_{ij} = \lambda \delta_{ij} E_{kk}^G + \mu E_{ij}^G + \mu E_{ji}^G = \lambda \delta_{ij} E_{kk}^G + 2\mu E_{ij}^G$$

Hypoelasticity

- Objective relation between objective stress and strain rates:

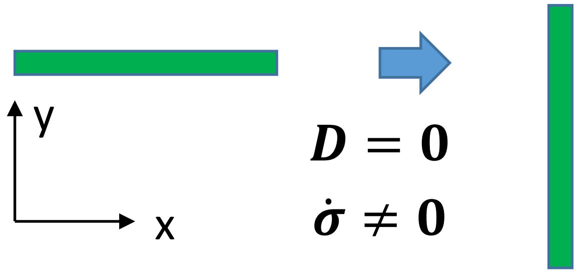
$$\boldsymbol{\sigma}^{\nabla} = \boldsymbol{f}(\boldsymbol{\sigma}, \boldsymbol{D}) \Rightarrow hypoelastic\ \boldsymbol{C}:\boldsymbol{D}$$

$$D_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)$$

- Physical meaning of $\boldsymbol{D}$ – objective:

$$\dot{(ds^2)} = \dot{(d\boldsymbol{x} \cdot d\boldsymbol{x})} = d\boldsymbol{v} \cdot d\boldsymbol{x} + d\boldsymbol{x} \cdot d\boldsymbol{v} = 2d\boldsymbol{x} \cdot \boldsymbol{D} \cdot d\boldsymbol{x}$$

- Objective rate of Cauchy stress:



$$\boldsymbol{D} = \boldsymbol{0}$$

$$\dot{\boldsymbol{\sigma}} \neq \boldsymbol{0}$$

$$\dot{\boldsymbol{\sigma}} \neq \boldsymbol{C}:\boldsymbol{D}$$

- Commonly used rates:
  - Jaumann rate $\boldsymbol{\sigma}^{\nabla J}$
  - Truesdell rate $\boldsymbol{\sigma}^{\nabla T}$
  - Green-Naghdi rate $\boldsymbol{\sigma}^{\nabla G}$
- Different stress rates correspond to different $\boldsymbol{C}$s

Constraint of hypoelasticity:

- Non-conservative energy in large deformation
- Works for small elastic deformation, such as elastic-plastic

Hyperelastic Materials

- Strain energy potential independent of the load path:

$$\boldsymbol{S} = \frac{\partial w(\boldsymbol{E})}{\partial \boldsymbol{E}} = 2\frac{\partial \psi(\boldsymbol{C})}{\partial \boldsymbol{C}}, \qquad \boldsymbol{E} = \frac{1}{2}(\boldsymbol{C} - \boldsymbol{I})$$

- Approximation for rubber-like materials under large elastic strain

- Demonstration for path-independence:

$$\int_{\boldsymbol{E}_2}^{\boldsymbol{E}_1} \boldsymbol{S} d\boldsymbol{E} = w(\boldsymbol{E}_1) - w(\boldsymbol{E}_2) \Rightarrow \oint \boldsymbol{S} d\boldsymbol{E} = 0$$

Isotropic Hyperelastic Materials

- Requirement for isotropy – ψ is a function of principal invariants $I_1, I_2, and\ I_3$:

$$I_1 = trace(\boldsymbol{C}) = C_{ii} \Rightarrow \frac{\partial I_1}{\partial \boldsymbol{C}} = \boldsymbol{I}$$

$$\frac{\partial}{\partial C_{ij}}\left[\frac{1}{2}(C_{ii})^2 - \frac{1}{2}C_{ij}C_{ji}\right] = C_{ii}\delta_{ij} - C_{ji}$$

$$I_2 = \frac{1}{2}\left[\left(trace(\boldsymbol{C})\right)^2 - trace(\boldsymbol{C}\cdot\boldsymbol{C})\right] = \frac{1}{2}\left[(C_{ii})^2 - C_{ij}C_{ji}\right] \Rightarrow \frac{\partial I_2}{\partial \boldsymbol{C}} = I_1\boldsymbol{I} - \boldsymbol{C}^T$$

$$I_3 = C_{1i}c_{1i} = C_{2i}c_{2i} = C_{3i}c_{3i} \Rightarrow \frac{\partial I_3}{\partial C_{ij}} = c_{ij}$$

$$I_3 = \det(\boldsymbol{C}) \Rightarrow \frac{\partial I_3}{\partial \boldsymbol{C}} = I_3\boldsymbol{C}^{-T}$$

$$\boldsymbol{C}^{-1} = \frac{\boldsymbol{C}^*}{I_3} \Rightarrow \boldsymbol{C}^{-T} = \frac{\boldsymbol{C}^{*T}}{I_3} = \frac{\boldsymbol{c}}{I_3} \Rightarrow \frac{\partial I_3}{\partial C_{ij}} = I_3\boldsymbol{C}^{-T}$$

- Stress calculation:

$$\boldsymbol{S} = 2\frac{\partial\psi(\boldsymbol{C})}{\partial \boldsymbol{C}} = 2\left(\frac{\partial\psi}{\partial I_i}\frac{\partial I_i}{\partial \boldsymbol{C}}\right) = 2\left(\frac{\partial\psi}{\partial I_1} + I_1\frac{\partial\psi}{\partial I_2}\right)\boldsymbol{I} - 2\frac{\partial\psi}{\partial I_2}\boldsymbol{C} + 2I_3\frac{\partial\psi}{\partial I_3}\boldsymbol{C}^{-1}$$

Neo-Hookean Material

- Extension of isotropic linear law with large deformation:

$$\psi(\boldsymbol{C}) = \frac{1}{2}\lambda_0(\ln J)^2 - \mu_0 \ln J + \frac{1}{2}\mu_0(trace(\boldsymbol{C}) - 3)$$

$$J = \det(\boldsymbol{F}) = \sqrt{\det(\boldsymbol{C})}$$

$$\Rightarrow \psi(\boldsymbol{C}) = \frac{1}{2}\lambda_0\left(\frac{1}{2}\ln I_3\right)^2 - \frac{\mu_0}{2}\ln I_3 + \frac{1}{2}\mu_0(I_1 - 3)$$

- Stress calculation:

$$\boldsymbol{S} = 2\frac{\partial\psi(\boldsymbol{C})}{\partial\boldsymbol{C}} = \lambda_0 \ln J\, \boldsymbol{C}^{-1} + \mu_0(\boldsymbol{I} - \boldsymbol{C}^{-1})$$

The End