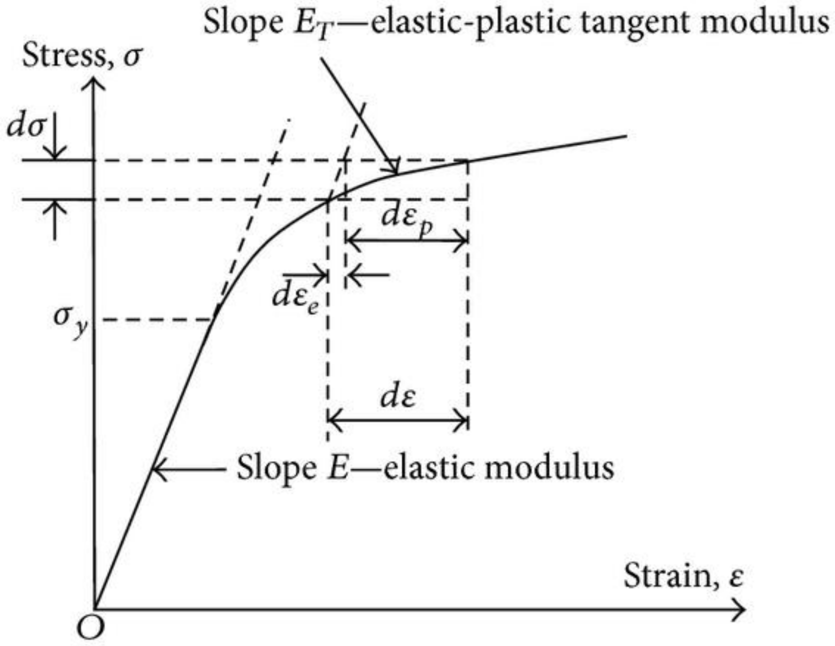
Computational Mechanics

Chapter 11 Constitutive Models – Plasticity

Characteristic of Plasticity

- Plasticity: permanent strains are developed upon unloading
- Yield strength: stress when plastic strains are developed
- Major ingredients of plasticity theory:
  - Each increment of strain is decomposed into elastic and plastic parts
  - A yield function which governs the onset and continuance of plastic deformation
  - A flow rule which governs the plastic flow and determines the plastic strain increment
  - Evolution equations for inner properties (such as strain-hardening) of materials

1D Rate-Independent Plasticity (1/3)



Stress-strain curve for typical elastic-plastic material [1]

- Decomposition of strain increments:

$$d\varepsilon = d\varepsilon^e + d\varepsilon^p$$

- Strain decomposition:

$$d\varepsilon = d\varepsilon^e + d\varepsilon^p \; or \; \dot{\varepsilon} = \dot{\varepsilon}^e + \dot{\varepsilon}^p$$

Incremental form — Rate form (preferred for simplicity)

- Stress calculation:

$$d\sigma = E d\varepsilon^e \; or \; \dot{\sigma} = E\dot{\varepsilon}^e$$

- Stress from whole strain:

$$d\sigma = E^{tan} d\varepsilon, \qquad \dot{\sigma} = \underline{E}^{tan}\dot{\varepsilon}$$

Elastic-plastic tangent modulus

- Plastic flow potential for evolution of plastic strain:

$$\dot{\varepsilon}^p = \dot{\lambda}\frac{\partial \Psi}{\partial \sigma}$$

Plastic rate parameter ($\dot{\lambda}$)

Direction for plastic strain evolution ($\partial\sigma$)

[1] https://www.hindawi.com/journals/cpis/2013/267095/fig8/

1D Rate-Independent Plasticity (2/3)

$$\dot{\varepsilon}^p = \dot{\lambda}\frac{\partial \Psi}{\partial \sigma}$$

- Plastic flow potential example:

$$\Psi = |\sigma| = \bar{\sigma} = \sigma sign(\sigma)$$

1D effective stress

$$\Rightarrow \frac{\partial \Psi}{\partial \sigma} = sign(\sigma)$$

Effective plastic strain, one *internal variable*

- Yield condition:

$$f = \bar{\sigma} - \sigma_Y(\bar{\varepsilon}) = 0$$

Yield strength

Isotropic hardening – same tension and compression strength

- Work/strain hardening:

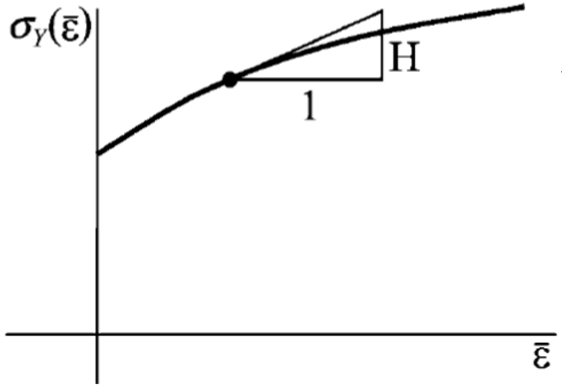
$$f = 0 \Rightarrow \sigma_Y(\bar{\varepsilon}) = \bar{\sigma}$$

$$\bar{\sigma}, \bar{\varepsilon} \uparrow \Rightarrow \sigma_Y \uparrow$$

- $\bar{\varepsilon}$ calculation for metal:

$$\bar{\varepsilon} = \int \dot{\bar{\varepsilon}} dt = \int \sqrt{\dot{\varepsilon}^p \dot{\varepsilon}^p} dt$$

- Plastic modulus:



$$H = \frac{d\sigma_Y(\bar{\varepsilon})}{d\bar{\varepsilon}}$$

1D Rate-Independent Plasticity (3/3)

$$\Psi = |\sigma| = \bar{\sigma} = \sigma sign(\sigma)$$

$$\Rightarrow \frac{\partial \Psi}{\partial \sigma} = sign(\sigma) = \frac{\partial\left(\bar{\sigma} - \sigma_Y(\bar{\varepsilon})\right)}{\partial \sigma} = \frac{\partial f}{\partial \sigma}$$

Associate plasticity for *this specific model* – plastic strain evolution direction normal to the yield surface.

$$\dot{\bar{\varepsilon}} = \sqrt{\dot{\varepsilon}^p \dot{\varepsilon}^p} = \sqrt{\dot{\lambda}^2 sign(\sigma)^2} = \dot{\lambda}$$

$$\Rightarrow \dot{\varepsilon}^p = \dot{\lambda}\frac{\partial \Psi}{\partial \sigma} = \dot{\bar{\varepsilon}}\frac{\partial f}{\partial \sigma} = sign(\sigma)\dot{\bar{\varepsilon}}$$

- When plastic deformation happens, $f = \bar{\sigma} - \sigma_Y(\bar{\varepsilon}) = 0$, leading to consistency condition:

$$\dot{f} = \dot{\bar{\sigma}} - \frac{d\sigma_Y(\bar{\varepsilon})}{d\bar{\varepsilon}}\dot{\bar{\varepsilon}} = 0$$

$$\Rightarrow \dot{\bar{\sigma}} = \frac{d\sigma_Y(\bar{\varepsilon})}{d\bar{\varepsilon}}\dot{\bar{\varepsilon}} = H\dot{\bar{\varepsilon}}$$

$$\dot{\bar{\sigma}} = \dot{\sigma} sign(\sigma) = H\dot{\bar{\varepsilon}} = H\dot{\varepsilon}^p sign(\sigma)$$

$$\dot{\varepsilon} = \dot{\varepsilon}^e + \dot{\varepsilon}^p, \qquad \dot{\sigma} = E^{tan}\dot{\varepsilon}, \qquad \dot{\sigma} = E\dot{\varepsilon}^e$$

$$\Rightarrow \frac{1}{E^{tan}} = \frac{1}{E} + \frac{1}{H}$$

Plastic switch parameter

- Plastic switch to change between purely elastic loading/unloading and plastic loading:

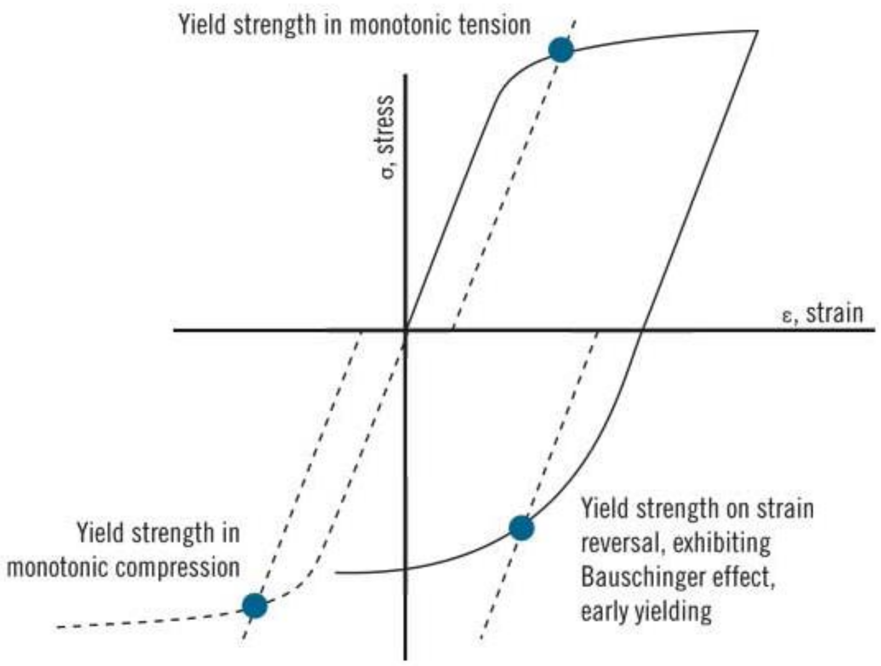
$$\frac{1}{E^{tan}} = \frac{1}{E} + \frac{1}{H} \Rightarrow E^{tan} = E - \frac{E^2}{E+H} = E - \beta \frac{E^2}{E+H} \begin{cases} \beta = 1, & plastic \\ \beta = 0, & purely\ elastic \end{cases}$$

- Kuhn-Tucker conditions for switch between elastic/plastic response:

  - $\dot{\lambda} \geq 0$ – non-negative plastic rate parameter following the stress direction
  - $f \leq 0$ – stress state must on or within the yield surface
  - $\dot{\lambda} f = 0$ –
    1) During plastic loading when $\dot{\lambda} > 0$, stress must be on the yield surface ($f = 0$)
    2) During elastic response when $\dot{\lambda} = 0$, there is no plastic flow ($f < 0$)

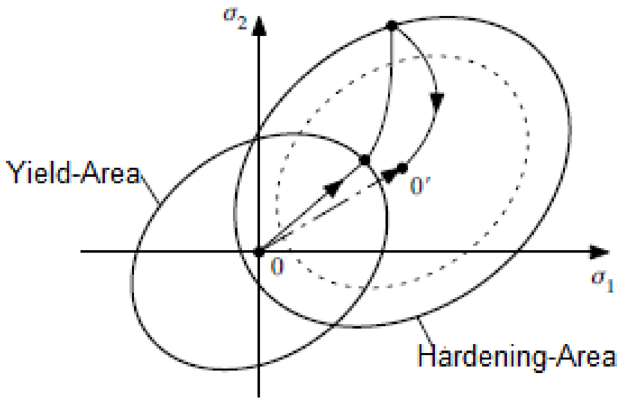
Kinematic Hardening (1/2)

- Bauschinger effect for cyclic plastic loading – center of yield surface moves with plastic flow:



Stress-strain curve for Bauschinger effect [1]

- Visualization of kinematic and isotropic hardening:



2D yield surfaces for mixed hardening [2]

- Consideration of 1D kinematic hardening:

$$\dot{\varepsilon}^p = \dot{\lambda}\frac{\partial \Psi}{\partial \sigma}, \qquad \Psi = |\sigma - \alpha|$$

Backstress, internal variable

$$f = |\sigma - \alpha| - \sigma_Y(\bar{\varepsilon}) = 0$$

Effective plastic strain, another internal variable

[1] https://mechanical-engg.com/blogs/entry/481-what-is-bauschinger-effect/

[2] https://www.dlubal.com/en/support-and-learning/support/knowledge-base/001479

Kinematic Hardening (2/2)

$$\Psi = |\sigma - \alpha|, \qquad f = |\sigma - \alpha| - \sigma_Y(\bar{\varepsilon})$$

$$\Rightarrow \frac{\partial \Psi}{\partial \sigma} = sign(\sigma - \alpha) = \frac{\partial f}{\partial \sigma}$$

Associate plasticity

$$\dot{\bar{\varepsilon}} = \sqrt{\dot{\varepsilon}^p \dot{\varepsilon}^p} = \sqrt{\dot{\lambda}^2 sign(\sigma - \alpha)^2} = \dot{\lambda}$$

$$\Rightarrow \dot{\varepsilon}^p = \dot{\lambda}\frac{\partial \Psi}{\partial \sigma} = \dot{\bar{\varepsilon}}\frac{\partial f}{\partial \sigma} = sign(\sigma - \alpha)\dot{\bar{\varepsilon}}$$

- Linear kinematic hardening:

$$\dot{\alpha} = \kappa \dot{\varepsilon}^p$$

- Consistency condition when yield happens:

$$\dot{f} = (\dot{\sigma} - \dot{\alpha}) sign(\sigma - \alpha) - H\dot{\bar{\varepsilon}} = 0$$

$$\Rightarrow \dot{\bar{\varepsilon}} = \frac{1}{H}(\dot{\sigma} - \dot{\alpha}) sign(\sigma - \alpha)$$

$$\dot{\sigma} = E(\dot{\varepsilon} - \dot{\varepsilon}^p) = E\left(\dot{\varepsilon} - \dot{\bar{\varepsilon}} sign(\sigma - \alpha)\right)$$

$$\Rightarrow (\dot{\sigma} - \dot{\alpha}) = E\left(\dot{\varepsilon} - \dot{\bar{\varepsilon}} sign(\sigma - \alpha)\right) - \dot{\alpha}$$
$$= E\left(\dot{\varepsilon} - \dot{\bar{\varepsilon}} sign(\sigma - \alpha)\right) - \kappa \dot{\varepsilon}^p$$

$$\Rightarrow \dot{\bar{\varepsilon}} = \frac{E\dot{\varepsilon} sign(\sigma - \alpha)}{E + H + \kappa}$$

$$\Rightarrow E^{tan} = \frac{\dot{\sigma}}{\dot{\varepsilon}} = \frac{E(H + \kappa)}{E + H + \kappa}$$

Summary for 1D Rate-Independent Plasticity

- Strain rate:

$$\dot{\varepsilon} = \dot{\varepsilon}^e + \dot{\varepsilon}^p$$

- Stress rate:

$$\dot{\sigma} = E\dot{\varepsilon}^e = E(\dot{\varepsilon} - \dot{\varepsilon}^p)$$

- Plastic flow rule (*special case*):

$$\dot{\varepsilon}^p = \dot{\lambda}\frac{\partial\Psi}{\partial\sigma} = \dot{\bar{\varepsilon}}\frac{\partial|\sigma - \alpha|}{\partial\sigma}$$

- Backstress (*linear kinematic hardening*):

$$\dot{\alpha} = \kappa\dot{\varepsilon}^p$$

- Yield condition:

$$f = |\sigma - \alpha| - \sigma_Y(\bar{\varepsilon}) = 0$$

- Consistency condition:

$$\dot{f} = 0 \Rightarrow \dot{\bar{\varepsilon}} = \dot{\lambda} = \frac{E\dot{\varepsilon}sign(\sigma - \alpha)}{E + H + \kappa}$$

- Tangent modulus:

$$E^{tan} = \frac{E(H + \kappa)}{E + H + \kappa}$$

Multiaxial Hypoelastic-Plastic Materials (1/2)

- Elastic strains are small compared to plastic ones, causing negligible nonconservative energy error
- Decomposition of rate-of-deformation tensor:

$$\boldsymbol{D} = \boldsymbol{D}^e + \boldsymbol{D}^p$$

Objective

- Hypoelastic response of a specific model with Jaumann rate Cauchy (true) stress [1]:

$$\boldsymbol{\sigma}^{\nabla J} = \boldsymbol{C}_{el}^{\sigma J} : \boldsymbol{D}^e = \boldsymbol{C}_{el}^{\sigma J} : (\boldsymbol{D} - \boldsymbol{D}^p)$$

Constitutive law and objective stress measurement should be changed based on actual needs

- Rate of plastic flow:

$$\boldsymbol{D}^p = \dot{\lambda}\boldsymbol{r}(\boldsymbol{\sigma}, \boldsymbol{q}), \qquad D_{ij}^P = \dot{\lambda} r_{ij}(\boldsymbol{\sigma}, \boldsymbol{q})$$

Plastic flow direction   Scalar plastic flow rate

- Utilization of plastic flow potential:

$$\boldsymbol{r}(\boldsymbol{\sigma}, \boldsymbol{q}) = \frac{\partial \psi}{\partial \boldsymbol{\sigma}}$$

- Typical evolution of internal variable matrix $\boldsymbol{q}$ (containing scalars and tensors, such as effective plastic strain and backstress):

$$\dot{\boldsymbol{q}} = \dot{\lambda}\boldsymbol{h}(\boldsymbol{\sigma}, \boldsymbol{q}), \qquad \dot{q}_\alpha = \dot{\lambda} h_\alpha(\boldsymbol{\sigma}, \boldsymbol{q})$$

- Yield condition:

$$f(\boldsymbol{\sigma}, \boldsymbol{q}) = 0$$

$$\dot{\lambda} \geq 0, \qquad f \leq 0, \qquad \dot{\lambda} f = 0$$

- Consistency condition on yield:

$$0 = \dot{f} = \frac{\partial f}{\partial \sigma_{ij}} \dot{\sigma}_{ij} + \frac{\partial f}{\partial q_\alpha} \dot{q}_\alpha \ or \ 0 = f_{\boldsymbol{\sigma}} : \dot{\boldsymbol{\sigma}} + f_{\boldsymbol{q}} \cdot \dot{\boldsymbol{q}}$$

[1] https://abaqus-docs.mit.edu/2017/English/SIMACAETHERefMap/simathe-c-stressrates.htm

Multiaxial Hypoelastic-Plastic Materials (2/2)

$$0 = f_{\boldsymbol{\sigma}} \colon \dot{\boldsymbol{\sigma}} + f_{\boldsymbol{q}} \cdot \dot{\boldsymbol{q}}$$

Normal to the yield surface

- Associative flow rule:

$$\boldsymbol{r} = C f_{\boldsymbol{\sigma}}$$

$$\boldsymbol{r} = \frac{\partial \psi}{\partial \boldsymbol{\sigma}} \Rightarrow \psi = Cf$$

- Typical associative flow rule:

$$\psi = f, \qquad \boldsymbol{r} = f_{\boldsymbol{\sigma}}$$

- Property of Jaumann rate:

$$f_{\boldsymbol{\sigma}} \colon \dot{\boldsymbol{\sigma}} = f_{\boldsymbol{\sigma}} \colon \boldsymbol{\sigma}^{\nabla J}$$

Also works for other objective rates except the Truesdell rate

$$\Rightarrow 0 = f_{\boldsymbol{\sigma}} \colon \boldsymbol{\sigma}^{\nabla J} + f_{\boldsymbol{q}} \cdot \dot{\boldsymbol{q}}$$

- Calculation of scalar plastic flow rate:

$$0 = f_{\boldsymbol{\sigma}} \colon \boldsymbol{C}_{el}^{\sigma J} \colon (\boldsymbol{D} - \boldsymbol{D}^p) + f_{\boldsymbol{q}} \cdot \dot{\boldsymbol{q}} = f_{\boldsymbol{\sigma}} \colon \boldsymbol{C}_{el}^{\sigma J} \colon \left(\boldsymbol{D} - \dot{\lambda}\boldsymbol{r}\right) + f_{\boldsymbol{q}} \cdot \dot{\lambda}\boldsymbol{h}$$

$$\Rightarrow \dot{\lambda} = \frac{f_{\boldsymbol{\sigma}} \colon \boldsymbol{C}_{el}^{\sigma J} \colon \boldsymbol{D}}{-f_{\boldsymbol{q}} \cdot \boldsymbol{h} + f_{\boldsymbol{\sigma}} \colon \boldsymbol{C}_{el}^{\sigma J} \colon \boldsymbol{r}}$$

- Definition of continuum elasto-plastic tangent modulus:

$$\boldsymbol{\sigma}^{\nabla J} = \boldsymbol{C}_{el}^{\sigma J} \colon \left(\boldsymbol{D} - \dot{\lambda}\boldsymbol{r}\right)$$

$$= \boldsymbol{C}_{el}^{\sigma J} \colon \left(\boldsymbol{D} - \frac{f_{\boldsymbol{\sigma}} \colon \boldsymbol{C}_{el}^{\sigma J} \colon \boldsymbol{D}}{-f_{\boldsymbol{q}} \cdot \boldsymbol{h} + f_{\boldsymbol{\sigma}} \colon \boldsymbol{C}_{el}^{\sigma J} \colon \boldsymbol{r}} \boldsymbol{r}\right) = \boldsymbol{C}^{\sigma J} \colon \boldsymbol{D}$$

Minor symmetry due to symmetry of $\boldsymbol{\sigma}^{\nabla J}$ and $\boldsymbol{D}$

Calculation of Tangent Modulus

$$\boldsymbol{C}_{el}^{\sigma J}:\left(\boldsymbol{D}-\frac{f_{\boldsymbol{\sigma}}:\boldsymbol{C}_{el}^{\sigma J}:\boldsymbol{D}}{-f_{\boldsymbol{q}}\cdot\boldsymbol{h}+f_{\boldsymbol{\sigma}}:\boldsymbol{C}_{el}^{\sigma J}:\boldsymbol{r}}\boldsymbol{r}\right)=\boldsymbol{C}^{\sigma J}:\boldsymbol{D}$$

$$\Rightarrow C_{ijkl}^{\sigma J}D_{kl}=\left(C_{el}^{\sigma J}\right)_{ijkl}D_{kl}-\frac{(f_{\boldsymbol{\sigma}})_{mn}\left(C_{el}^{\sigma J}\right)_{mnkl}D_{kl}}{-\left(f_{\boldsymbol{q}}\right)_{\alpha}h_{\alpha}+(f_{\boldsymbol{\sigma}})_{rs}\left(C_{el}^{\sigma J}\right)_{rstu}r_{tu}}\left(C_{el}^{\sigma J}\right)_{ijpq}r_{pq}$$

$$\Rightarrow C_{ijkl}^{\sigma J}D_{kl}=\left[\left(C_{el}^{\sigma J}\right)_{ijkl}-\frac{(f_{\boldsymbol{\sigma}})_{mn}\left(C_{el}^{\sigma J}\right)_{mnkl}\left(C_{el}^{\sigma J}\right)_{ijpq}r_{pq}}{-\left(f_{\boldsymbol{q}}\right)_{\alpha}h_{\alpha}+(f_{\boldsymbol{\sigma}})_{rs}\left(C_{el}^{\sigma J}\right)_{rstu}r_{tu}}\right]D_{kl}$$

Plastic flow contribution

$$\Rightarrow C_{ijkl}^{\sigma J}=\left(C_{el}^{\sigma J}\right)_{ijkl}-\frac{(f_{\boldsymbol{\sigma}})_{mn}\left(C_{el}^{\sigma J}\right)_{mnkl}\left(C_{el}^{\sigma J}\right)_{ijpq}r_{pq}}{-\left(f_{\boldsymbol{q}}\right)_{\alpha}h_{\alpha}+(f_{\boldsymbol{\sigma}})_{rs}\left(C_{el}^{\sigma J}\right)_{rstu}r_{tu}},\qquad \boldsymbol{C}^{\sigma J}=\boldsymbol{C}_{el}^{\sigma J}-\frac{\left(\boldsymbol{C}_{el}^{\sigma J}:\boldsymbol{r}\right)\left(f_{\boldsymbol{\sigma}}:\boldsymbol{C}_{el}^{\sigma J}\right)}{-f_{\boldsymbol{q}}\cdot\boldsymbol{h}+f_{\boldsymbol{\sigma}}:\boldsymbol{C}_{el}^{\sigma J}:\boldsymbol{r}}$$

Dyad (also can be denoted as ⊗) of two 2^{nd} order tensors for a 4^{th} order tensor

$\boldsymbol{C}^{\sigma J}$ has major symmetry for associative flow due to symmetry of $\boldsymbol{C}_{el}^{\sigma J}$ and $\boldsymbol{r}=f_{\boldsymbol{\sigma}}$

Summary of Hypoelastic-Plastic Constitutive Law

- Deformation rate tensor (objective):

$$\boldsymbol{D} = \boldsymbol{D}^e + \boldsymbol{D}^p$$

- Cauchy stress with objective Jaumann rate:

$$\boldsymbol{\sigma}^{\nabla J} = \boldsymbol{C}_{el}^{\sigma J} : \boldsymbol{D}^e = \boldsymbol{C}_{el}^{\sigma J} : (\boldsymbol{D} - \boldsymbol{D}^p)$$

Can be other stress rates with adjustment

- Plastic flow rule and internal variables evolution:

$$\boldsymbol{D}^p = \dot{\lambda}\boldsymbol{r}(\boldsymbol{\sigma}, \boldsymbol{q}), \qquad \dot{\boldsymbol{q}} = \dot{\lambda}\boldsymbol{h}(\boldsymbol{\sigma}, \boldsymbol{q})$$

- Yield condition:

$$f(\boldsymbol{\sigma}, \boldsymbol{q}) = 0$$

- Loading-unloading conditions:

$$\dot{\lambda} \geq 0, \qquad f \leq 0, \qquad \dot{\lambda} f = 0$$

- Scalar plastic rate parameter:

$$\dot{\lambda} = \frac{f_{\boldsymbol{\sigma}} : \boldsymbol{C}_{el}^{\sigma J} : \boldsymbol{D}}{-f_{\boldsymbol{q}} \cdot \boldsymbol{h} + f_{\boldsymbol{\sigma}} : \boldsymbol{C}_{el}^{\sigma J} : \boldsymbol{r}}$$

- Stress-rate and deformation rate relation:

$$\boldsymbol{\sigma}^{\nabla J} = \boldsymbol{C}^{\sigma J} : \boldsymbol{D}$$

$$\boldsymbol{C}^{\sigma J} = \boldsymbol{C}_{el}^{\sigma J} \ for\ elastic$$

$$\boldsymbol{C}^{\sigma J} = \boldsymbol{C}_{el}^{\sigma J} - \frac{\left(\boldsymbol{C}_{el}^{\sigma J} : \boldsymbol{r}\right)\left(f_{\boldsymbol{\sigma}} : \boldsymbol{C}_{el}^{\sigma J}\right)}{-f_{\boldsymbol{q}} \cdot \boldsymbol{h} + f_{\boldsymbol{\sigma}} : \boldsymbol{C}_{el}^{\sigma J} : \boldsymbol{r}} \ for\ plastic$$

*This hypoelastic-plastic law needs *isotropic* elastic moduli and yield function, but enough for most common engineering materials – metals

J2 Flow Theory (1/3)

- J2 flow model based on von Mises yield surface is suitable for metal plasticity

- Assume metal plastic flow is unaffected by pressure – supported by experiments

- Stress decomposition:

$$\boldsymbol{\sigma} = p\boldsymbol{I} + \boldsymbol{\sigma}^{dev}$$

$$p = \frac{1}{3} trace(\boldsymbol{\sigma}) = \frac{1}{3}\sigma_{ii}$$

- Von Mises effective stress $\bar{\sigma}$:

$$\bar{\sigma} = \sqrt{\frac{3}{2}\boldsymbol{\sigma}^{dev}:\boldsymbol{\sigma}^{dev}} = \sqrt{3J_2}$$

$$J_2 = -I_2(\boldsymbol{\sigma}^{dev}) = -\frac{1}{2}\left[tr(\boldsymbol{\sigma}^{dev})^2 - tr(\boldsymbol{\sigma}^{dev}\cdot\boldsymbol{\sigma}^{dev})\right] = \frac{1}{2}\boldsymbol{\sigma}^{dev}:\boldsymbol{\sigma}^{dev}$$

- Plastic flow rule:

$$\boldsymbol{D}^p = \dot{\lambda}\boldsymbol{r}(\boldsymbol{\sigma}, \boldsymbol{q})$$

- Internal variable – only *one* effective plastic strain:

$$\boldsymbol{q} = q_1 = \bar{\varepsilon} = \int \dot{\bar{\varepsilon}}\, dt$$

- Materials are usually characterized using the uniaxial tension tests, when along 1 direction:

$$D_{22}^p = D_{33}^p = -\frac{1}{2}D_{11}^p$$

Volume conservation in plastic deformation

$$D_{ij} = 0, \qquad i \neq j$$

J2 Flow Theory (2/3)

$$D_{22}^{p} = D_{33}^{p} = -\frac{1}{2}D_{11}^{p}$$

$$D_{ij} = 0, \qquad i \neq j$$

- Physical meaning of $\boldsymbol{D}$ is demonstrated in [1]
- In uniaxial tension, D_{11}^{p} is the same as the rate of effective plastic strain:

$$D_{11}^{p} = \dot{\bar{\varepsilon}}$$

Can be also obtained through:

$$\bar{\sigma}\dot{\bar{\varepsilon}} = \boldsymbol{\sigma}:\boldsymbol{D}^{p}$$

General uniaxial loading condition

$$\Rightarrow \dot{\bar{\varepsilon}} = \sqrt{\frac{2}{3}\boldsymbol{D}^{p}:\boldsymbol{D}^{p}}$$

- Yield condition:

$$f(\boldsymbol{\sigma}, \boldsymbol{q}) = \bar{\sigma} - \sigma_Y(\bar{\varepsilon}) = 0$$

Associative flow rule

$$\Rightarrow f_{\boldsymbol{\sigma}} = \frac{\partial \bar{\sigma}}{\partial \boldsymbol{\sigma}} = \frac{3}{2\bar{\sigma}}\boldsymbol{\sigma}^{dev} = \boldsymbol{r}$$

$$\Rightarrow \boldsymbol{D}^{p}:\boldsymbol{D}^{p} = \dot{\lambda}^{2}\boldsymbol{r}:\boldsymbol{r} = \dot{\lambda}^{2}\frac{9}{4\bar{\sigma}^{2}}\boldsymbol{\sigma}^{dev}:\boldsymbol{\sigma}^{dev} = \frac{3}{2}\dot{\lambda}^{2}$$

$$\Rightarrow \dot{\bar{\varepsilon}} = \dot{\lambda}$$

$$\dot{q}_1 = \dot{\lambda}h_1 = \dot{\bar{\varepsilon}}$$

$$\Rightarrow h_1 = 1$$

[1] Rudnicki JW. Fundamentals of continuum mechanics. John Wiley & Sons; 2014 Sep 22.

J2 Flow Theory (3/3)

Plastic modulus from uniaxial tension experiments

- Yield condition:

$$f(\boldsymbol{\sigma},\boldsymbol{q}) = \bar{\sigma} - \sigma_Y(\bar{\varepsilon}) = 0 \Rightarrow f_{\boldsymbol{q}} = f_{q_1} = -\frac{d\sigma_Y(\bar{\varepsilon})}{d\bar{\varepsilon}} = -H(\bar{\varepsilon})$$

- Plastic rate parameter:

$$\dot{\lambda} = \dot{\bar{\varepsilon}} = \frac{f_{\boldsymbol{\sigma}}:\boldsymbol{C}_{el}^{\sigma J}:\boldsymbol{D}}{-f_{\boldsymbol{q}}\cdot\boldsymbol{h} + f_{\boldsymbol{\sigma}}:\boldsymbol{C}_{el}^{\sigma J}:\boldsymbol{r}} = \frac{f_{\boldsymbol{\sigma}}:\boldsymbol{C}_{el}^{\sigma J}:\boldsymbol{D}}{-f_{q_1}h_1 + f_{\boldsymbol{\sigma}}:\boldsymbol{C}_{el}^{\sigma J}:\boldsymbol{r}} = \frac{\boldsymbol{r}:\boldsymbol{C}_{el}^{\sigma J}:\boldsymbol{D}}{H + \boldsymbol{r}:\boldsymbol{C}_{el}^{\sigma J}:\boldsymbol{r}}$$

- Tangent modulus:

$$\boldsymbol{C}^{\sigma J} = \boldsymbol{C}_{el}^{\sigma J} - \frac{\left(\boldsymbol{C}_{el}^{\sigma J}:\boldsymbol{r}\right)\left(f_{\boldsymbol{\sigma}}:\boldsymbol{C}_{el}^{\sigma J}\right)}{-f_{\boldsymbol{q}}\cdot\boldsymbol{h} + f_{\boldsymbol{\sigma}}:\boldsymbol{C}_{el}^{\sigma J}:\boldsymbol{r}} = \boldsymbol{C}_{el}^{\sigma J} - \frac{\left(\boldsymbol{C}_{el}^{\sigma J}:\boldsymbol{r}\right)\left(\boldsymbol{r}:\boldsymbol{C}_{el}^{\sigma J}\right)}{H + \boldsymbol{r}:\boldsymbol{C}_{el}^{\sigma J}:\boldsymbol{r}}$$

- Isotropic elastic law for the hypoelastic-plastic model:

$$\boldsymbol{C}_{el}^{\sigma J} = \lambda^e\delta_{ij}\delta_{kl} + \mu\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right) \Rightarrow \boldsymbol{C}_{el}^{\sigma J}:\boldsymbol{r} = 2\mu\boldsymbol{r}, \qquad \boldsymbol{r}:\boldsymbol{C}_{el}^{\sigma J}:\boldsymbol{r} = 3\mu$$

$\boldsymbol{r} = \frac{3}{2\bar{\sigma}}\boldsymbol{\sigma}^{dev}$ is deviatoric

$$\Rightarrow \boldsymbol{C}^{\sigma J} = \boldsymbol{C}_{el}^{\sigma J} - \frac{(2\mu\boldsymbol{r})(2\mu\boldsymbol{r})}{H + 3\mu} = \boldsymbol{C}_{el}^{\sigma J} - \frac{4\mu^2}{H + 3\mu}\boldsymbol{r}\boldsymbol{r}$$

Multiaxial Kinematic Hardening (1/2)

- Introduction of multiaxial backstress $\boldsymbol{\alpha}$

- Derived from J2 theory with isotropic hardening

- Definition of overall stress tensor:

$$\boldsymbol{\Sigma} = \boldsymbol{\sigma} - \boldsymbol{\alpha}$$

- Linear hardening law for backstress evolution:

$$\boldsymbol{\alpha}^{\nabla J} = \kappa \boldsymbol{D}^p$$

Jaumann rate works for strain smaller than about 0.4

- Modified von Mises stress:

$$\bar{\sigma} = \sqrt{\frac{3}{2}\boldsymbol{\Sigma}^{dev} : \boldsymbol{\Sigma}^{dev}}$$

- Evolution of backstress as internal variable:

$$\boldsymbol{\alpha}^{\nabla J} = \kappa \boldsymbol{D}^p = \kappa \dot{\lambda} \boldsymbol{r}(\boldsymbol{\Sigma}, \boldsymbol{q})$$

- Yield condition:

$$f(\boldsymbol{\Sigma}, \boldsymbol{q}) = \bar{\sigma} - \sigma_Y(\bar{\varepsilon}) = 0$$

$$\Rightarrow f_{\Sigma} = \frac{3}{2\bar{\sigma}} \boldsymbol{\Sigma}^{dev} = \boldsymbol{r}$$

Associative plasticity

$$f_{q_1} = -\frac{d\sigma_Y(\bar{\varepsilon})}{d\bar{\varepsilon}} = -H(\bar{\varepsilon})$$

Multiaxial Kinematic Hardening (2/2)

$$f_{\boldsymbol{\Sigma}} = \frac{3}{2\bar{\sigma}}\boldsymbol{\Sigma}^{dev} = \boldsymbol{r}, \qquad f_{q_1} = -H(\bar{\varepsilon}), \qquad \boldsymbol{\alpha}^{\nabla J} = \kappa \boldsymbol{D}^p = \kappa\dot{\lambda}\boldsymbol{r}(\boldsymbol{\Sigma}, \boldsymbol{q})$$

- Scalar plastic rate parameter from consistency condition:

$$\dot{\lambda} = \dot{\bar{\varepsilon}} = \frac{f_{\boldsymbol{\Sigma}}: \boldsymbol{C}_{el}^{\sigma J}: \boldsymbol{D}}{-f_{q_1}h_1 - \kappa f_{\boldsymbol{\alpha}}: \boldsymbol{r} + f_{\boldsymbol{\Sigma}}: \boldsymbol{C}_{el}^{\sigma J}: \boldsymbol{r}} = \frac{\boldsymbol{r}: \boldsymbol{C}_{el}^{\sigma J}: \boldsymbol{D}}{H + \frac{3}{2}\kappa + \boldsymbol{r}: \boldsymbol{C}_{el}^{\sigma J}: \boldsymbol{r}}$$

$$f_{\boldsymbol{\alpha}} = f_{\boldsymbol{\sigma}-\boldsymbol{\Sigma}} = -f_{\boldsymbol{\Sigma}} = -\boldsymbol{r}$$

- Continuum elasto-plastic tangent modulus:

$$\boldsymbol{C}^{\sigma J} = \boldsymbol{C}_{el}^{\sigma J} - \frac{\left(\boldsymbol{C}_{el}^{\sigma J}: \boldsymbol{r}\right)\left(\boldsymbol{r}: \boldsymbol{C}_{el}^{\sigma J}\right)}{H + \frac{3}{2}\kappa + \boldsymbol{r}: \boldsymbol{C}_{el}^{\sigma J}: \boldsymbol{r}} = \boldsymbol{C}_{el}^{\sigma J} - \frac{4\mu^2}{H + \frac{3}{2}\kappa + 3\mu}\boldsymbol{r}\boldsymbol{r}$$

Isotropic for the hypoelastic-plastic model

The End