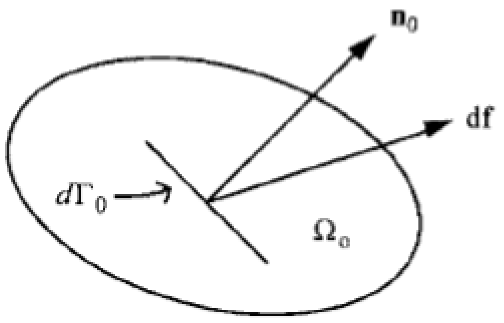
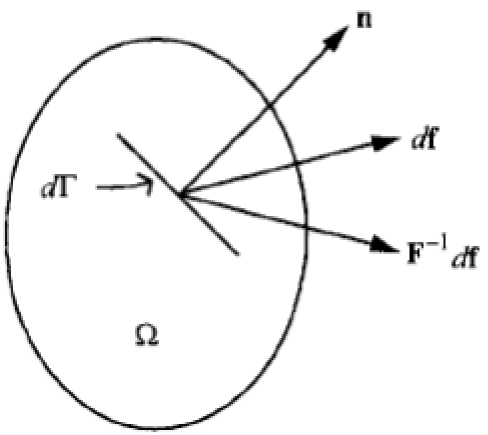
Computational Mechanics

Chapter 13 Supplementary Information for Constitutive Laws

Review – Stress in Different Configurations



Reference configuration



Current configuration

- Cauchy stress (current configuration):

$$\boldsymbol{n}\cdot\boldsymbol{\sigma}d\Gamma = d\boldsymbol{f} = \boldsymbol{t}d\Gamma, \qquad \boldsymbol{\sigma} = \boldsymbol{\sigma}^T$$

- Nominal stress (initial configuration):

$$\boldsymbol{n}_0\cdot\boldsymbol{P}d\Gamma_0 = d\boldsymbol{f} = \boldsymbol{t}_0 d\Gamma_0, \qquad \boldsymbol{P} \neq \boldsymbol{P}^T$$

- PK2 stress:

$$\boldsymbol{n}_0\cdot\boldsymbol{S}d\Gamma_0 = \underline{\boldsymbol{F}^{-1}}\cdot\boldsymbol{t}_0 d\Gamma_0$$

  - Make $\boldsymbol{S}$ symmetric
  - Make $\boldsymbol{S}$ conjugate to $\boldsymbol{E}^G$ in power

Review – Transformation of Stress Measures

- Stress measures can be transformed using deformation functions, refer to Box 3.2 of the textbook

Box 3.2 Transformations of stresses

Cauchy stress $\boldsymbol{\sigma}$	Nominal stress $\mathbf{P}$	2nd Piola–Kirchhoff stress $\mathbf{S}$	Corotational Cauchy stress $\hat{\boldsymbol{\sigma}}$
$\boldsymbol{\sigma} =$	$J^{-1}\mathbf{F}\cdot\mathbf{P}$	$J^{-1}\mathbf{F}\cdot\mathbf{S}\cdot\mathbf{F}^T$	$\mathbf{R}\cdot\hat{\boldsymbol{\sigma}}\cdot\mathbf{R}^T$
$\mathbf{P} = J\mathbf{F}^{-1}\cdot\boldsymbol{\sigma}$		$\mathbf{S}\cdot\mathbf{F}^T$	$J\mathbf{U}^{-1}\cdot\hat{\boldsymbol{\sigma}}\cdot\mathbf{R}^T$
$\mathbf{S} = J\mathbf{F}^{-1}\cdot\boldsymbol{\sigma}\cdot\mathbf{F}^{-T}$	$\mathbf{P}\cdot\mathbf{F}^{-T}$		$J\mathbf{U}^{-1}\cdot\hat{\boldsymbol{\sigma}}\cdot\mathbf{U}^{-1}$
$\hat{\boldsymbol{\sigma}} = \mathbf{R}^T\cdot\boldsymbol{\sigma}\cdot\mathbf{R}$	$J^{-1}\mathbf{U}\cdot\mathbf{P}\cdot\mathbf{R}$	$J^{-1}\mathbf{U}\cdot\mathbf{S}\cdot\mathbf{U}$	
$\boldsymbol{\tau} = J\boldsymbol{\sigma}$	$\mathbf{F}\cdot\mathbf{P}$	$\mathbf{F}\cdot\mathbf{S}\cdot\mathbf{F}^T$	$J\mathbf{R}\cdot\hat{\boldsymbol{\sigma}}\cdot\mathbf{R}^T$

Notes: $d\mathbf{x} = \mathbf{F}\cdot d\mathbf{X} = \mathbf{R}\cdot\mathbf{U}\cdot d\mathbf{X}$

$\mathbf{U}$ is the stretch tensor; see Section 3.7.1

$d\mathbf{x} = \mathbf{R}\cdot d\mathbf{X} = \mathbf{R}\cdot d\hat{\mathbf{x}}$ in rotation

$\boldsymbol{\tau}$ = Kirchhoff stress

How to derive?

Change between Current and Initial Configuration

- Nanson's formula from change in area enveloped by two Vectors (from Chapter 5):

$$\frac{d\Gamma}{d\Gamma_0} = \det(\boldsymbol{F})\,\boldsymbol{n}\cdot\boldsymbol{n}_0\cdot\boldsymbol{F}^{-1} \Rightarrow \underline{\boldsymbol{n}d\Gamma = \det(\boldsymbol{F})\,\boldsymbol{n}_0\cdot\boldsymbol{F}^{-1}d\Gamma_0}$$

Link between current and initial surfaces (direction + area)

- Example – transformation between Cauchy and nominal stress:

$$\boldsymbol{n}\cdot\boldsymbol{\sigma}d\Gamma = d\boldsymbol{f} = \boldsymbol{n}_0\cdot\boldsymbol{P}d\Gamma_0$$

Nanson's formula $\Rightarrow \det(\boldsymbol{F})\,\boldsymbol{n}_0\cdot\boldsymbol{F}^{-1}\cdot\boldsymbol{\sigma}d\Gamma_0 = J\boldsymbol{n}_0\cdot\boldsymbol{F}^{-1}\cdot\boldsymbol{\sigma}d\Gamma_0 = \boldsymbol{n}_0\cdot\boldsymbol{P}d\Gamma_0$

Arbitrary $\boldsymbol{n}_0$ and $d\Gamma_0$ $\Rightarrow J\boldsymbol{F}^{-1}\cdot\boldsymbol{\sigma} = \boldsymbol{P}$

- Expansion to PK2:

Transpose of a vector is the vector itself

$$\boldsymbol{F}\cdot(\boldsymbol{n}_0\cdot\boldsymbol{S})d\Gamma_0 = \boldsymbol{F}\cdot(\boldsymbol{S}^T\cdot\boldsymbol{n}_0)d\Gamma_0 = \boldsymbol{t}_0 d\Gamma_0 = \boldsymbol{n}_0\cdot\boldsymbol{P}d\Gamma_0 = \boldsymbol{P}^T\cdot\boldsymbol{n}_0 d\Gamma_0$$

Arbitrary $\boldsymbol{n}_0$ and $d\Gamma_0$ $\Rightarrow \boldsymbol{F}\cdot\boldsymbol{S}^T = \boldsymbol{P}^T \Rightarrow \boldsymbol{P} = \boldsymbol{S}\cdot\boldsymbol{F}^T = J\boldsymbol{F}^{-1}\cdot\boldsymbol{\sigma} \Rightarrow \boldsymbol{F}\cdot\boldsymbol{S}\cdot\boldsymbol{F}^T = \underline{J\boldsymbol{\sigma}}$

Kirchhoff stress $\boldsymbol{\tau}$

Lagrangian, Eulerian and Two-Point Tensors

- Lagrangian tensors – tensors defined in the intial/Lagragian configuration $X - Y - Z$:

$$\boldsymbol{C}\ (ds^2 = d\boldsymbol{X} \cdot \boldsymbol{C} \cdot d\boldsymbol{X} = C_{\boldsymbol{XX}}, \text{defined using initial } \boldsymbol{X}) \text{ and } \boldsymbol{E}^G = \frac{1}{2}(\boldsymbol{C} - \boldsymbol{I})$$

- Eulerian tensors – tensors defined in the current/Eulerian configuration $x(t) - y(t) - z(t)$:

$$\boldsymbol{\sigma}\ (\text{defined with current } \boldsymbol{n}, d\Gamma \text{ and } d\boldsymbol{f}) \text{ and } \boldsymbol{\tau} = J\boldsymbol{\sigma}$$

$$\boldsymbol{L} = \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{x}} = \frac{\partial \dot{\boldsymbol{x}}}{\partial \boldsymbol{x}} \text{ and } \boldsymbol{D} = \frac{1}{2}(\boldsymbol{L} + \boldsymbol{L}^T)$$

- Two-point tensor – link initial and current configurations:

$$\boldsymbol{F} = \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{X}} = \frac{\partial x_k}{\partial X_m} \boldsymbol{e}_k \boldsymbol{e}_{0m} \text{ (Eulerian – Lagrangian)}$$

$$d\boldsymbol{f} = \boldsymbol{n}_0 \cdot \boldsymbol{P} d\Gamma_0 \Rightarrow \boldsymbol{P} = \boldsymbol{P}_{ij} \boldsymbol{e}_{0i} \boldsymbol{e}_j \text{ (Lagrangian - Eulerian)}$$

Pull-Back and Push-Forward Operations

- Pull-back – map current Eulerian entity back to its initial Lagrangian status:

$$d\boldsymbol{X} = \boldsymbol{F}^{-1} \cdot d\boldsymbol{x} = \varphi^{*} d\boldsymbol{x}$$

- Push-forward – map initial Lagrangian entity to its current Eulerian status:

$$d\boldsymbol{x} = \boldsymbol{F} \cdot d\boldsymbol{X} = \varphi_{*} d\boldsymbol{X}$$

- 2nd order tensors in solid mechanics:
  - Stress:

$$\begin{cases} \boldsymbol{\tau} = \boldsymbol{F} \cdot \boldsymbol{S} \cdot \boldsymbol{F}^{T} = \varphi_{*} \boldsymbol{S} \\ \boldsymbol{S} = \boldsymbol{F}^{-1} \cdot \boldsymbol{\tau} \cdot \boldsymbol{F}^{-T} = \varphi^{*} \boldsymbol{\tau} \end{cases}$$

  - Strain/deformation:

$$\begin{cases} \boldsymbol{D} = \boldsymbol{F}^{-T} \cdot \dot{\boldsymbol{E}}^{G} \cdot \boldsymbol{F}^{-1} = \varphi_{*} \dot{\boldsymbol{E}}^{G} \\ \dot{\boldsymbol{E}}^{G} = \boldsymbol{F}^{T} \cdot \boldsymbol{D} \cdot \boldsymbol{F} = \varphi^{*} \boldsymbol{D} \end{cases}$$

One special case:

$$\varphi_{*} \boldsymbol{C} = \boldsymbol{F}^{-T} \cdot \boldsymbol{C} \cdot \boldsymbol{F}^{-1} = \boldsymbol{I}$$

Material Frame Indifference

- Large deformation – constitutive laws should be independent from rigid body motion
- Constitutive laws should be the same for observers in relative motion – principle of material objectivity
  Translation and rotation

- Motion of one body described by two observers in relative movement (refer to Chapter 5):

$$\underline{\boldsymbol{x}^*(\boldsymbol{X},t)} = \boldsymbol{Q}(t)\cdot \underline{\boldsymbol{x}(\boldsymbol{X},t)} + \underline{\boldsymbol{c}(t)}, \qquad \boldsymbol{Q}^{-T} = \underline{\boldsymbol{Q}}, \qquad \begin{cases} \boldsymbol{Q}(0) = \boldsymbol{I} \\ \boldsymbol{c}(0) = 0 \end{cases}$$

Motion from observer #2 | Motion from observer #1 | Translation from frame #1 to #2 | Rotation from frame #1 to #2

Infinitesimal vector formed by two points

$$\Rightarrow d\boldsymbol{x}^* = \boldsymbol{Q}(t)\cdot d\boldsymbol{x}(\boldsymbol{X},t) = \boldsymbol{Q}\cdot \boldsymbol{F}\cdot d\boldsymbol{X} = \boldsymbol{F}^*\cdot d\boldsymbol{X}$$

$$\Rightarrow \underline{\boldsymbol{F}^* = \boldsymbol{Q}\cdot \boldsymbol{F}}$$

Transformation of Deformation gradient

Transformation of Basic Quantities

$$\boldsymbol{x}^*(\boldsymbol{X},t) = \boldsymbol{Q}(t)\cdot\boldsymbol{x}(\boldsymbol{X},t) + \boldsymbol{c}(t)$$

- Velocity ($\boldsymbol{v}^* = \dot{\boldsymbol{x}}^*$, and $\boldsymbol{v} = \dot{\boldsymbol{x}}$):

$$\boldsymbol{v}^* = \dot{\boldsymbol{Q}}\cdot\boldsymbol{x} + \boldsymbol{Q}\cdot\boldsymbol{v} + \dot{\boldsymbol{c}}$$

- Right Cauchy-Green deformation tensor:

$$\boldsymbol{F}^* = \boldsymbol{Q}\cdot\boldsymbol{F}$$

$$\Rightarrow \boldsymbol{C}^* = \boldsymbol{F}^{*T}\cdot\boldsymbol{F}^* = \boldsymbol{F}^T\cdot\boldsymbol{Q}^T\cdot\boldsymbol{Q}\cdot\boldsymbol{F} = \boldsymbol{F}^T\cdot\boldsymbol{F}$$

- Spatial velocity gradient:

$$\boldsymbol{L}^* = \dot{\boldsymbol{F}}^*\cdot(\boldsymbol{F}^*)^{-1} = (\boldsymbol{Q}\dot{\cdot}\boldsymbol{F})\cdot(\boldsymbol{Q}\cdot\boldsymbol{F})^{-1}$$

$$\Rightarrow \boldsymbol{L}^* = \dot{\boldsymbol{Q}}\cdot\boldsymbol{F}\cdot\boldsymbol{F}^{-1}\cdot\boldsymbol{Q}^T + \boldsymbol{Q}\cdot\dot{\boldsymbol{F}}\cdot\boldsymbol{F}^{-1}\cdot\boldsymbol{Q}^T$$

$$\Rightarrow \boldsymbol{L}^* = \dot{\boldsymbol{Q}}\cdot\boldsymbol{Q}^T + \boldsymbol{Q}\cdot\boldsymbol{L}\cdot\boldsymbol{Q}^T$$

- Rate of deformation:

$$\boldsymbol{D}^* = \frac{1}{2}(\boldsymbol{L}^* + \boldsymbol{L}^{T*}) = \frac{1}{2}\left[\underbrace{(\boldsymbol{Q}\dot{\cdot}\boldsymbol{Q}^T)}_{\dot{\boldsymbol{I}}\,=\,\boldsymbol{0}} + 2\boldsymbol{Q}\cdot\boldsymbol{D}\cdot\boldsymbol{Q}^T\right]$$

$$\Rightarrow \boldsymbol{D}^* = \boldsymbol{Q}\cdot\boldsymbol{D}\cdot\boldsymbol{Q}^T$$

Objectivity of Tensor Fields

- Base of a Cartesian system rotating with the observer #2 relative to observer #1:

$$d\boldsymbol{x}^* = \boldsymbol{Q}\cdot d\boldsymbol{x} \Rightarrow \boldsymbol{e}_i^* = \boldsymbol{Q}\cdot\boldsymbol{e}_i = \boldsymbol{e}_i\cdot\boldsymbol{Q}^T$$

$$\Rightarrow dx_i^* = \boldsymbol{e}_i^*\cdot d\boldsymbol{x}^* = \boldsymbol{e}_i\cdot\boldsymbol{Q}^T\cdot\boldsymbol{Q}\cdot d\boldsymbol{x} = dx_i$$

$$D_{ij}^* = \boldsymbol{e}_i^*\cdot\boldsymbol{D}^*\cdot\boldsymbol{e}_j^* = \boldsymbol{e}_i\cdot\boldsymbol{Q}^T\cdot\boldsymbol{Q}\cdot\boldsymbol{D}\cdot\boldsymbol{Q}^T\cdot\boldsymbol{Q}\cdot\boldsymbol{e}_j$$

$$\Rightarrow D_{ij}^* = D_{ij}$$

$d\boldsymbol{x}$ and $\boldsymbol{D}$ are objective *Eulerian* tensors, but $\boldsymbol{v}$ and $\boldsymbol{L}$ are not

- Objective requirement on *Eulerian* tensors:

$$\begin{cases} \boldsymbol{a}^* = \boldsymbol{Q}\cdot\boldsymbol{a}, & 1st\ order \\ \boldsymbol{A}^* = \boldsymbol{Q}\cdot\boldsymbol{A}\cdot\boldsymbol{Q}^T, & 2nd\ order \end{cases}$$

- Objectivity of *Lagrangian* tensors ($t = 0$):

$$d\boldsymbol{x}^*(\boldsymbol{X},0) = \boldsymbol{Q}(0)\cdot d\boldsymbol{x}(\boldsymbol{X},0)$$

$$\Rightarrow d\boldsymbol{X}^* = \boldsymbol{I}\cdot d\boldsymbol{X} = d\boldsymbol{X}$$

- Objective requirement on *Lagrangian* tensors:

$$\begin{cases} \boldsymbol{a}_0^* = \boldsymbol{a}_0, & 1st\ order \\ \boldsymbol{A}_0^* = \boldsymbol{A}_0, & 2nd\ order \end{cases}$$

- Objective of *Eulerian-Lagrangian* tensors (for example $\boldsymbol{F}$):

$$B_{ij}^* = \boldsymbol{e}_i^*\cdot\boldsymbol{B}^*\cdot\boldsymbol{e}_j = B_{ij} = \boldsymbol{e}_i\cdot\boldsymbol{B}\cdot\boldsymbol{e}_j$$

or

$$\boldsymbol{B}^* = \boldsymbol{Q}\cdot\boldsymbol{B}$$

Objective Scalar Functions

- Tensor based constitutive equations frequently utilize scalar functions, such as the yield function $f(\boldsymbol{\sigma})$

- Scalar functions represent material properties and should be objective:

$$f^*(\boldsymbol{\sigma}^*) = f(\boldsymbol{\sigma}^*) = f(\boldsymbol{Q} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{Q}^T) = f(\boldsymbol{\sigma}), \qquad \forall \boldsymbol{Q}$$

$$\Rightarrow f(\boldsymbol{\sigma}) = g[I_1(\boldsymbol{\sigma}), I_2(\boldsymbol{\sigma}), I_3(\boldsymbol{\sigma})] \leftrightarrow f \ is\ an\ isotropic\ function\ of\, \boldsymbol{\sigma}$$

- To capture anisotropic yield behavior, objective stress measurement, such as PK2 $\boldsymbol{S}$, should be applied

Frame Invariance Restriction on Elastic Moduli

- Hypoelastic constitutive equation after transformation:

$$(\boldsymbol{\sigma}^{\nabla J})^* = \boldsymbol{C}_{el}^{\sigma J} : (\boldsymbol{D}^e)^*$$

Frame invariance of material properties requires constant $\boldsymbol{C}_{el}^{\sigma J}$

Objective stress and deformation rates

$$\Rightarrow Q_{im} Q_{jn} \sigma_{mn}^{\nabla J} = \left(\boldsymbol{C}_{el}^{\sigma J}\right)_{ijkl} (Q_{kr} Q_{ls} D_{rs}^e)$$

$$\Rightarrow \sigma_{ij}^{\nabla J} = \left[Q_{mi} Q_{nj} Q_{pk} Q_{ql} \left(\boldsymbol{C}_{el}^{\sigma J}\right)_{mnpq}\right] D_{kl}^e$$

$$\sigma_{ij}^{\nabla J} = \left(\boldsymbol{C}_{el}^{\sigma J}\right)_{ijkl} D_{kl}^e$$

$$\Rightarrow \left(\boldsymbol{C}_{el}^{\sigma J}\right)_{ijkl} = Q_{mi} Q_{nj} Q_{pk} Q_{ql} \left(\boldsymbol{C}_{el}^{\sigma J}\right)_{mnpq}, \qquad \forall \boldsymbol{Q}$$

$\boldsymbol{C}_{el}^{\sigma J}$ is the same regardless of the coordinate rotation – isotropic!