Computational Mechanics

Chapter 2 Strong and Weak Forms for 1D Problems

5-Step Analysis in FEM

- Preprocessing: subdividing the target domain into finite elements by automatic mesh generators.

Fundamental mechanics analysis related to FEM equation formulation

- Element formulation: development of equations for elements.

- Assembly: obtaining equations for the whole system by gathering ones at the element-level.

- Solving equations.

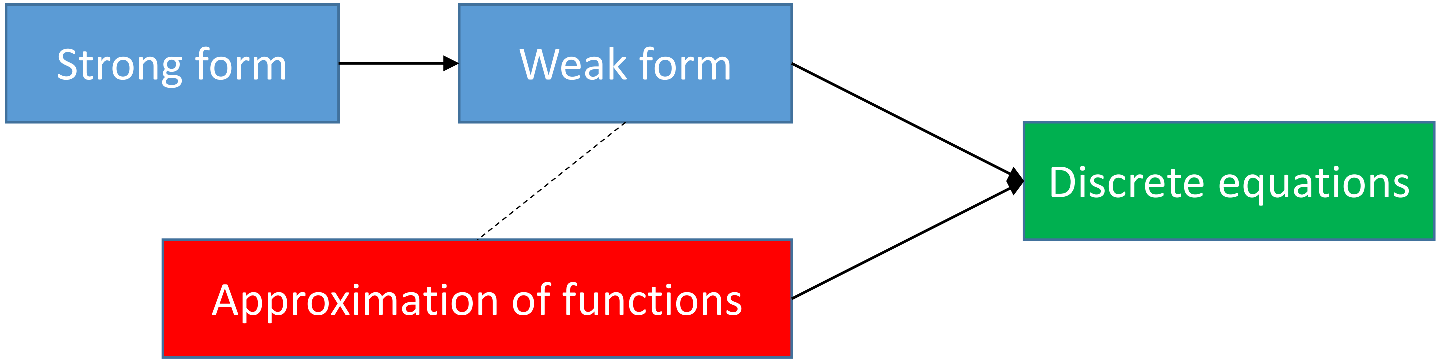
- Postprocessing: calculation results visualization and output.

Introduction to Strong and Weak Forms

- This section focuses on 1D analysis for various physical problems from the viewpoint of FEM.
- Strong form: governing equations (partial differential ones) and boundary conditions from physical analysis
- Weak form: an integral form of the strong form – to reduce requirement for the trial solution.

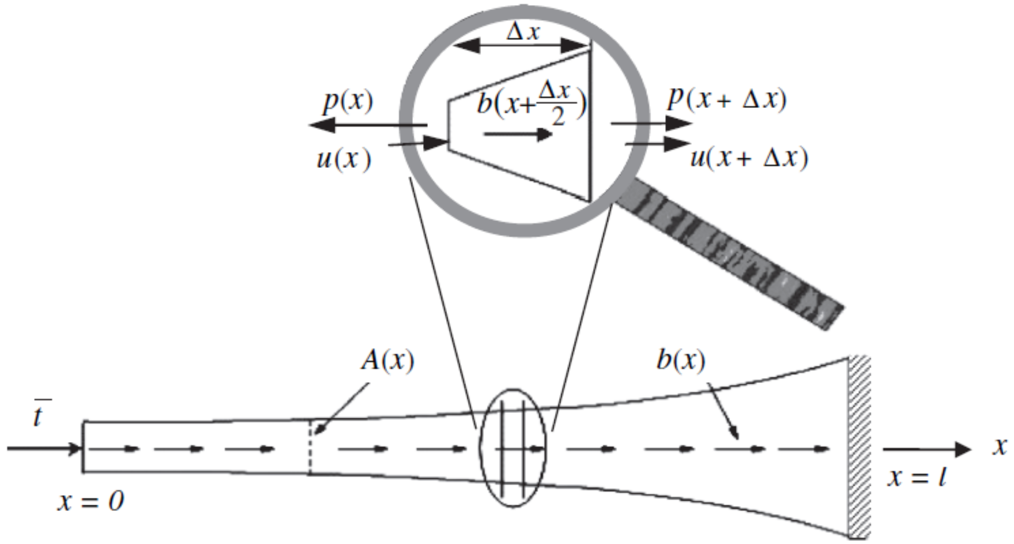
Strong form:	Weak form:
✓ Easy to build	✗ Complex to build
✗ Difficult to solve	✓ Easy to solve
Finite difference method with simple regions	Finite element method

Roadmap of FEM



- Advantage of weak form – complicated geometry for real engineering problems

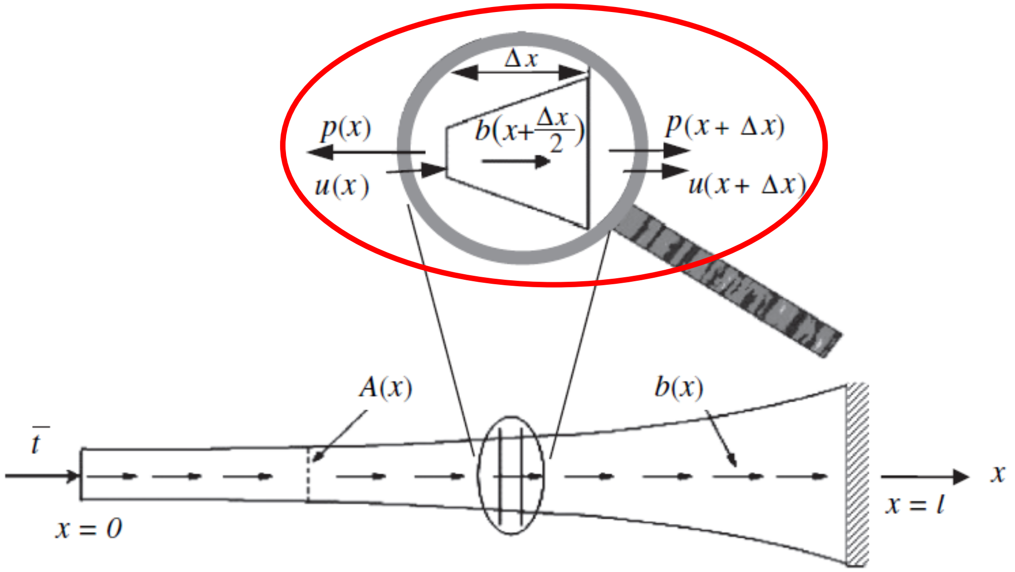
Analysis for An Axially Loaded Elastic Bar



- Static, elastic, linear and infinitesimal deformation:

1. Equilibrium constraint
2. Hooke's law of the material
3. Compatible displacement field
4. Strain-displacement equation

Force Equilibrium Conditions



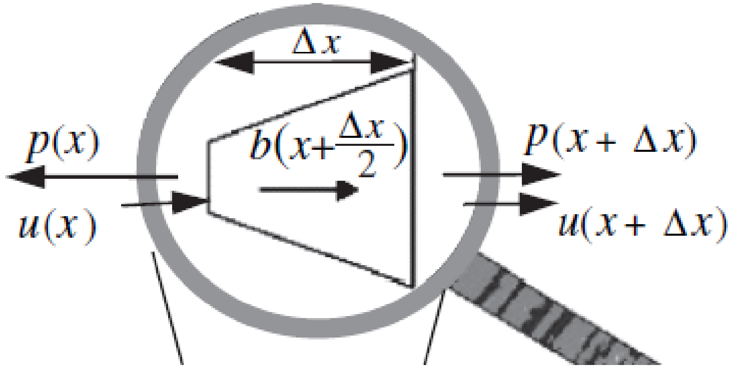
- Equilibrium of a small segment in a complex object:

$$-p(x) + b\left(x + \frac{\Delta x}{2}\right) \cdot \Delta x + p(x + \Delta x) = 0$$

$$\Rightarrow \frac{p(x + \Delta x) - p(x)}{\Delta x} + b\left(x + \frac{\Delta x}{2}\right) = 0$$

$$\Delta x \to 0 \Rightarrow \frac{dp(x)}{dx} + b(x) = 0$$

Calculation of the Governing Equation



- Hooke's Law:

$$\sigma(x) = E(x)\varepsilon(x)$$

$$\frac{dp(x)}{dx} + b(x) = 0 \Rightarrow \frac{d}{dx}\left(AE\frac{du}{dx}\right) + b = 0$$

2[nd] order ODE

- Stress:

$$\sigma(x) = \frac{p(x)}{A(x)} \Rightarrow p(x) = A(x)\sigma(x)$$

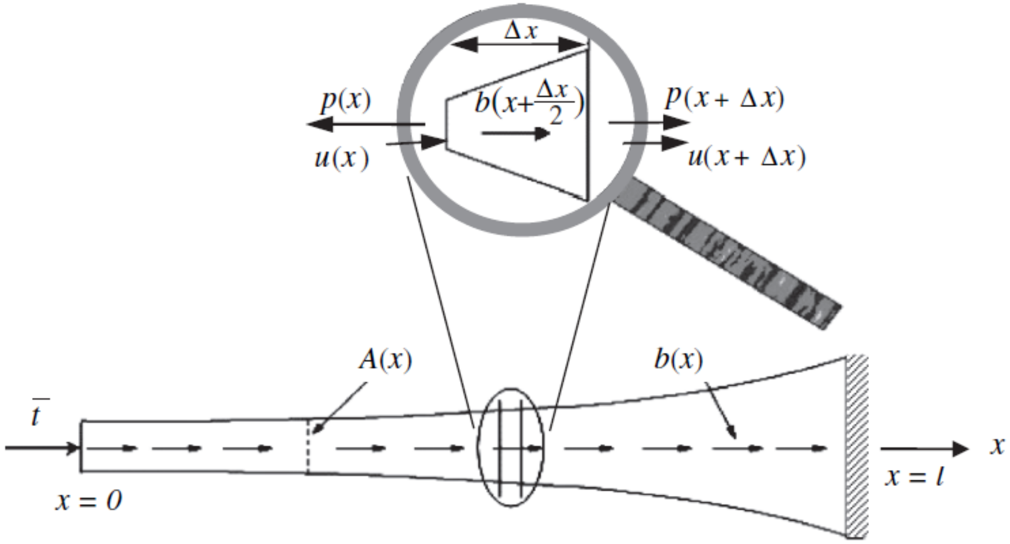
- Strain:

$$\varepsilon(x) = \frac{u(x+\Delta x) - u(x)}{\Delta x}$$

$$\Delta x \to 0 \Rightarrow \varepsilon(x) = \frac{du}{dx}$$

Definition based on infinitesimal deformation

Strong Form for an Axially Loaded Elastic Bar



- Governing equation:

$$\frac{d}{dx}\left(AE\frac{du}{dx}\right)+b=0, \qquad 0<x<l$$

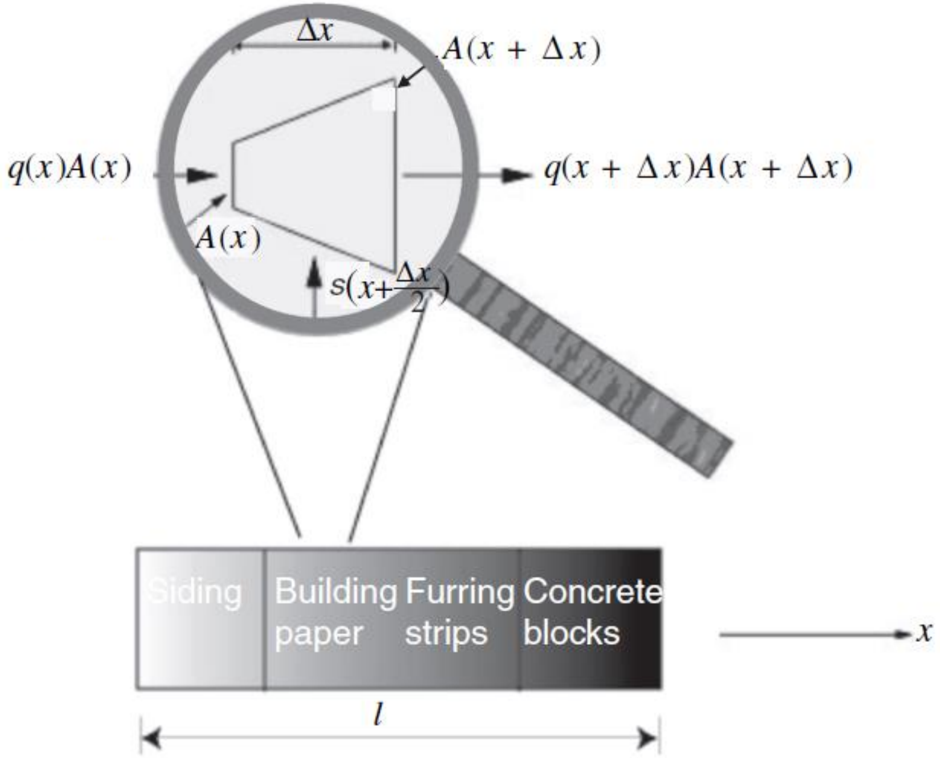
- Traction boundary condition:

$$\sigma(x=0)=E\varepsilon=E\frac{du}{dx}=-\bar{t}$$

- Displacement boundary condition:

$$u(x=l)=\bar{u}$$

Strong Form for 1D Heat Conduction



- Stable heat conduction analysis for a small segment:

$$\underset{P}{\underline{\underset{\sigma}{q(x)}A(x)}} + \underset{b}{s}\left(x + \frac{\Delta x}{2}\right) \cdot \Delta x - q(x + \Delta x)A(x + \Delta x) = 0$$

$$\Delta x \to 0 \Rightarrow \frac{d(qA)}{dx} - s = 0$$

- Fourier's law:

$$q = -k\frac{\overset{u}{dT}}{dx} \Rightarrow \frac{d}{dx}\left(Ak\frac{dT}{dx}\right) + s = 0$$

- Boundary conditions:

$$-q(x = 0) = k\frac{dT(x = 0)}{x} = \bar{q}, \qquad T(x = l) = \bar{T}$$

Similar to mechanical analysis, can be solved using one method

Introduction of Weight Function

- Stress analysis as an example:
  - Governing equation:

$$\frac{d}{dx}\left(AE\frac{du}{dx}\right)+b=0, 0<x<l \Rightarrow \int_0^l w\left[\frac{d}{dx}\left(AE\frac{du}{dx}\right)+b\right]dx=0, \forall w$$

Arbitrary smooth weight function

  - Traction boundary condition:

$$\left(E\frac{du}{dx}=-\bar{t}\right)_{x=0} \Rightarrow \left(wA\left(E\frac{du}{dx}+\bar{t}\right)\right)_{x=0}=0, \forall w$$

- Equation simplification at the essential boundary:

$$w(l)=0$$

Review of the *Integration by Parts*

$$\int_0^l w\left[\frac{d}{dx}\left(AE\frac{du}{dx}\right)+b\right]dx=0,\forall w$$

$$\Rightarrow \int_0^l w\frac{d}{dx}\left(AE\frac{du}{dx}\right)dx+\int_0^l wbdx=0,\forall w$$

- Derivation of *Integration by parts*:

$$\int_0^l w\frac{df}{dx}dx=(wf)_{x=l}-(wf)_{x=0}-\int_0^l f\frac{dw}{dx}dx$$

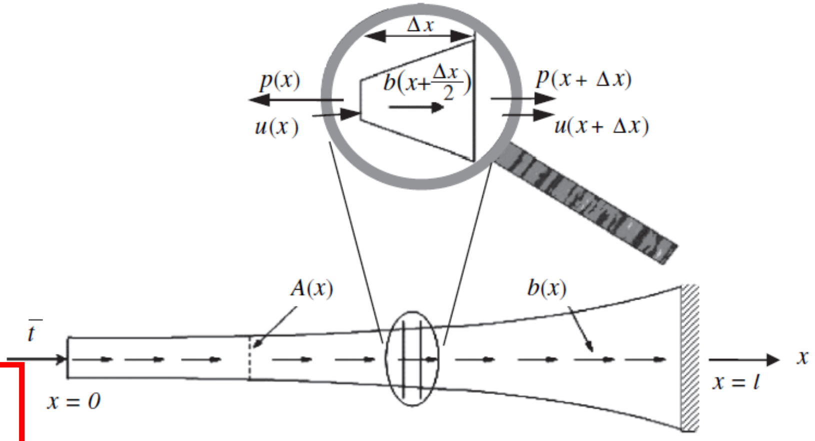
$$\int_0^l w\frac{d}{dx}\left(AE\frac{du}{dx}\right)dx$$
$$=\left(wAE\frac{du}{dx}\right)|_0^l-\int_0^l\left(AE\frac{du}{dx}\right)\frac{dw}{dx}dx$$

$$\Rightarrow\left(wAE\frac{du}{dx}\right)|_0^l-\int_0^l\left(AE\frac{du}{dx}\right)\frac{dw}{dx}dx+\int_0^l wbdx=0,\quad\forall w$$

$$w(l)=0\Rightarrow-\left(wAE\frac{du}{dx}\right)_{x=0}-\int_0^l\left(AE\frac{du}{dx}\right)\frac{dw}{dx}dx+\int_0^l wbdx=0,\ \forall w \text{ with } w(l)=0$$

1D Weak Form

$$-\left(wAE\underbrace{\frac{du}{dx}}_{\sigma}\right)_{x=0} - \int_0^l \left(AE\frac{du}{dx}\right)\frac{dw}{dx}dx + \int_0^l wbdx = 0,\ \forall w \text{ with } w(l) = 0$$



- Given the traction boundary condition:

$$wA\sigma(x=0) = -wA\bar{t}$$

$$\Rightarrow \int_0^l \frac{dw}{dx}AE\frac{du}{dx}dx = (wA\bar{t})_{x=0} + \int_0^l wbdx\,, \forall w \text{ with } w(l) = 0$$

1st order derivative

- Find solution $u(x)$ among the smooth functions that satisfy $u(l) = \bar{u}$ and the above integration

Weak form

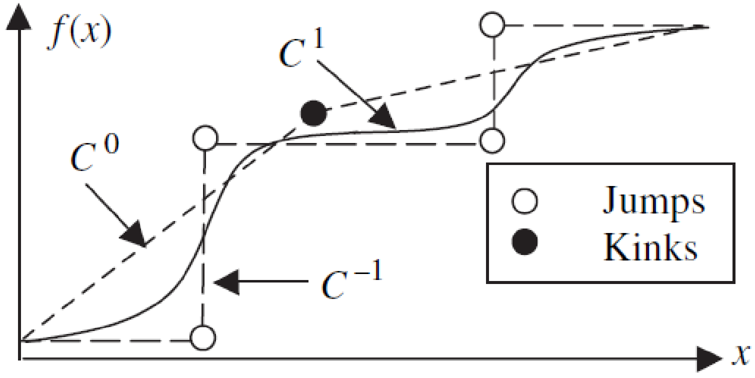
Strong form

$$\frac{d}{dx}\left(AE\frac{du}{dx}\right) + b = 0$$

2nd order derivative

Continuity of functions

- C^n function: its derivatives of order j for $0 \leq j \leq n$ exist and are continuous in the whole domain



Examples of C^{-1}, C^0 and C^1 functions.

Smoothness of functions.

Weak discontinuity

Strong discontinuity

Smoothness	Kinks	Jumps	Comments
C^{-1}	Yes	Yes	Piecewise continuous
C^0	Yes	No	Piecewise continuously differentiable
C^1	No	No	Continuously differentiable

- $\frac{d}{dx} C^n = C^{n-1}$

Weak form $\int_0^l \frac{dw}{dx} AE \frac{du}{dx} dx = (wA\bar{t})_{x=0} + \int_0^l wbdx$ ✓

Strong form $\frac{d}{dx}\left(AE \frac{du}{dx}\right) + b = 0$ ✗

- Smooth surface in CAD usually utilize C^1, while FEM employs C^0 approximation

Equivalence between Weak and Strong Forms (1/3)

- From weak form to strong form:

$$\int_0^l w \underline{\frac{d}{dx}\left(AE\frac{du}{dx}\right)} dx = \left(wAE\frac{du}{dx}\right)\Big|_0^l - \int_0^l \left(AE\frac{du}{dx}\right)\frac{dw}{dx} dx$$

part of strong form

$$\Rightarrow \int_0^l \frac{dw}{dx} AE \frac{du}{dx} dx = \left(wAE\frac{du}{dx}\right)\Big|_0^l - \int_0^l w \frac{d}{dx}\left(AE\frac{du}{dx}\right) dx$$

weak form

$$\underline{\int_0^l \frac{dw}{dx} AE \frac{du}{dx} dx = (wA\bar{t})_{x=0} + \int_0^l wb dx, \forall w \text{ with } w(l) = 0}$$

$$\left(wAE\frac{du}{dx}\right)\Big|_0^l - \int_0^l w \frac{d}{dx}\left(AE\frac{du}{dx}\right) dx = (wA\bar{t})_{x=0} + \int_0^l wb dx, \forall w \text{ with } w(l) = 0$$

$$\int_0^l w \left[\frac{d}{dx}\left(AE\frac{du}{dx}\right) + b\right] dx + wA(\bar{t} + \underline{\sigma})_{x=0} = 0, \forall w \text{ with } w(l) = 0$$

$$E\frac{du}{dx}$$

The Equivalence between Weak and Strong Forms (2/3)

$$\int_0^l w\left[\frac{d}{dx}\left(AE\frac{du}{dx}\right)+b\right]dx + wA(\bar{t}+\sigma)_{x=0} = 0, \forall w \text{ with } w(l)=0$$

- Utilize arbitrariness of $w(x)$:

$$w(x) = \psi(x)\left[\frac{d}{dx}\left(AE\frac{du}{dx}\right)+b\right]$$

$$\psi(x)\begin{cases} smooth\ function \\ > 0, \quad 0 < x < l \ \ (\psi(x) = x(l-x)) \\ = 0, \quad x = 0\ or\ l \end{cases}$$

$$\Rightarrow \int_0^l \psi(x)\left[\frac{d}{dx}\left(AE\frac{du}{dx}\right)+b\right]^2 dx = 0 \Rightarrow \frac{d}{dx}\left(AE\frac{du}{dx}\right)+b = 0, 0 < x < l$$

The Equivalence between Weak and Strong Forms (3/3)

$$\int_0^l w\underbrace{\left[\frac{d}{dx}\left(AE\frac{du}{dx}\right)+b\right]}_{0}dx + wA(\bar{t}+\sigma)_{x=0} = 0, \forall w \text{ with } w(l) = 0$$

$$\Rightarrow wA(\bar{t}+\sigma)_{x=0} = 0, \forall w \text{ with } w(l) = 0$$

- Utilize the arbitrariness of $w(x)$:

$$w(0) = 1 \; and \; w(l) = 0 \; (w = (l-x)/x)$$

$$\Rightarrow \underline{(\bar{t}+\sigma)_{x=0} = 0}$$

Traction boundary condition

Generalized Strong Form for 1D Stress Analysis

Exact positions of boundary conditions can be arbitrary

$$\frac{d}{dx}\left(AE\frac{du}{dx}\right) + b = 0, \qquad l_0 < x < l_1$$

Traction and stress sign difference:

$n = -1$ when $x = l_0$

$n = +1$ when $x = l_1$

$$\sigma n = En\frac{du}{dx} = \bar{t}, \qquad x \in \Gamma_t$$

- Traction boundary conditions
- Natural boundary conditions
- 1st order derivative boundaries

$$u = \bar{u}, \qquad x \in \Gamma_u$$

- Displacement boundary conditions
- Essential boundary conditions
- Function boundaries

- Complementary boundaries:

$$\Gamma_t \cup \Gamma_u = \Gamma, \qquad \Gamma_t \cap \Gamma_u = 0$$

Generalized Weak Form for 1D Stress Analysis

- Strong form:

$$\frac{d}{dx}\left(AE\frac{du}{dx}\right)+b=0, \qquad 0<x<l$$

$$\sigma n = En\frac{du}{dx}=\bar{t}, \qquad x\in\Gamma_t$$

$$u=\bar{u}, \qquad x\in\Gamma_u$$

- Derivation to weak form:

$$\int_\Omega w\left[\frac{d}{dx}\left(AE\frac{du}{dx}\right)+b\right]dx=0, \forall w$$

$$w(\bar{t}-\sigma n)=0, \qquad x\in\Gamma_t$$

$$\int_\Omega w\frac{d}{dx}\left(AE\frac{du}{dx}\right)dx$$

$$=wAE\underbrace{\frac{du}{dx}n}_{\bar{t}}\Big|_{\Gamma_u+\Gamma_t}-\int_\Omega \frac{dw}{dx}AE\frac{du}{dx}dx$$

$$\Rightarrow wA\bar{t}\Big|_{\Gamma_t}+\int_\Omega wbdx=\int_\Omega\frac{dw}{dx}AE\frac{du}{dx}dx$$

$\forall w$ with $w|_{\Gamma_u}=0$

Weak form

- Define smooth function set $H^1 \in C^0$:
  - Trial solution set: $U=\{u|u\in H^1, u|_{\Gamma_u}=\bar{u}\}$
  - Weight function set: $U_0=\{w|w\in H^1, w|_{\Gamma_u}=0\}$

From Elastic to Heat Conduction Analysis

- Previous slides show elastic and heat conduction analysis share similar governing equations and boundary conditions in strong form:

Stress analysis:

$$\frac{d}{dx}\left(AE\frac{du}{dx}\right)+b=0, \qquad x\in\Omega$$

$$\sigma n=En\frac{du}{dx}=-\bar{t}, \qquad x\in\Gamma_t$$

$$u=\bar{u}, \qquad x\in\Gamma_u$$

Heat conduction analysis:

$$\frac{d}{dx}\left(Ak\frac{dT}{dx}\right)+b=0, \qquad x\in\Omega$$

$$qn=-kn\frac{dT}{dx}=\bar{q}, \qquad x\in\Gamma_q$$

$$T=\bar{T}, \qquad x\in\Gamma_T$$

- Weak form of elastic analysis can be transferred directly to weak form of heat conduction analysis:

Elasticity	Heat conduction
u	T
E	k
b	s
$\bar{t}$	$-\bar{q}$
$\bar{u}$	$\bar{T}$
Γ_t	Γ_q
Γ_u	Γ_T
k	h

This transferring method can also be applied to other similar systems, such as diffusion.

Approximation of Solutions to 1D Weak Form

- Obtain a solution to the weak form:

  Find $u(x) \in U = \{u(x)|u(x) \in H^1,\ u = 10^{-4} \text{ on } \Gamma_u\}$ such that

  $$\int_\Omega \frac{dw}{dx} AE \frac{du}{dx} dx = 10wA\Big|_{\Gamma_t} + \int_\Omega 10wAxdx\,, \forall w \in U_0 = \{w(x)|w(x) \in H^1, w = 0 \; on \; \Gamma_u\}$$

  where
  - Γ_u is $x = 0$, Ω is $0 < x < 2$ and Γ_t is $x = 2$
  - A is constant and $E = 10^5$
- Approximation of solutions – FEM uses C^0 functions:
  - Trial solutions: $u(x) = \alpha_0 + \alpha_1 x$
  - Weight functions: $w(x) = \beta_0 + \beta_1 x$

  - α_i are unknown parameters
  - β_i are arbitrary parameters

C^0 functions can be piecewise for complex geometry.

Determination of Parameters (1/2)

- Trial solutions: $u(x) = \alpha_0 + \alpha_1 x$
- Weight functions: $\mathrm{w}(x) = \beta_0 + \beta_1 x$

- $\mathrm{w}(x)$ vanishes at the essential boundary Γ_u:

$$\mathrm{w}(x = 0) = \beta_0 + \beta_1 \cdot 0 = \beta_0 = 0$$

$$\Rightarrow \mathrm{w}(x) = \beta_1 x$$

$$\Rightarrow \frac{dw}{dx} = \beta_1$$

- Essential boundary condition:

$$u(x = 0) = \alpha_0 + \alpha_1 \cdot 0 = \alpha_0 = 10^{-4}$$

$$\Rightarrow u(x) = 10^{-4} + \alpha_1 x$$

$$\Rightarrow \frac{du}{dx} = \alpha_1$$

- Integral in weak form:

$$A \int_0^2 \beta_1 E \alpha_1 dx = A \cdot 10\beta_1 x \Big|_{x=2} + A \int_0^2 10\beta_1 x^2 \, dx$$

Determination of Parameters (2/2)

- Integral in weak form:

$$A\int_0^2 \beta_1 E\alpha_1 dx = A\cdot 10\beta_1 x\Big|_{x=2} + A\int_0^2 10\beta_1 x\, dx$$

$$\Rightarrow 2\beta_1 E\alpha_1 = 20\beta_1 + \int_0^2 10\beta_1 x^2\, dx$$

$$\Rightarrow 2\beta_1 E\alpha_1 = 20\beta_1 + \frac{80}{3}\beta_1, \qquad \forall\beta_1$$

$$\Rightarrow \alpha_1 = \frac{70}{3E} = \frac{7}{3}\times 10^{-4}$$

- Linear trial solution:

$$u^{lin}(x) = 10^{-4} + \frac{7}{3}\times 10^{-4}x$$

$$\sigma^{lin} = nE\frac{du}{dx} = E\frac{du}{dx} = \frac{70}{3}$$

Only approximation because weight function type is constrained

- Quadratic solution?
  - Displacement: $u^{quad}(x) = \alpha_1 + \alpha_2 x + \alpha_3 x^2$
  - Weight function: $w^{quad}(x) = \beta_1 + \beta_2 x + \beta_3 x^2$

……

Minimum Potential Energy Approach

- The theorem of minimum potential energy:

  The solution of the strong form is the minimizer of

$$W\big(u(x)\big) \text{ for } \forall u(x) \in U \text{ where } W\big(u(x)\big) = \underbrace{\frac{1}{2}\int_{\Omega} AE\left(\frac{du}{dx}\right)^2 dx}_{W_{int}} - \underbrace{\left(\int_{\Omega} ubdx + (uA\bar{t})|_{\Gamma_t}\right)}_{W_{ext}}$$

- The theorem holds for any elastic system, as it only consider internal elastic potential and external work made by traction and body force

Deriving FEM Equations Using Energy Approach (1/2)

$$W\big(u(x)\big) = \frac{1}{2}\int_\Omega AE\left(\frac{du}{dx}\right)^2 dx - \left(\int_\Omega ubdx + (uA\bar{t})|_{\Gamma_t}\right)$$, find the minimizer $u(x) \in U$

- Introduce an infinitesimal change in function $u(x)$:

$$\delta u(x) = \varsigma w(x), \qquad 0 < \varsigma \ll 1, \qquad w(x) \in U_0$$

$$U = \{u | u \in H^1, u|_{\Gamma_u} = \bar{u}\}$$
$$U_0 = \{w | w \in H^1, w|_{\Gamma_u} = 0\}$$

- Internal elastic energy:

$$\delta W_{int} = \frac{1}{2}\int_\Omega AE\left(\frac{du}{dx} + \varsigma\frac{dw}{dx}\right)^2 dx - \frac{1}{2}\int_\Omega AE\left(\frac{du}{dx}\right)^2 dx = \varsigma\int_\Omega AE\frac{du}{dx}\frac{dw}{dx}dx \ \ (\varsigma^2 \to 0)$$

- External work:

$$\delta W_{ext}^{\Omega} = \int_\Omega (u + \varsigma w)bdx - \int_\Omega ubdx = \varsigma\int_\Omega wbdx$$

$$\delta W_{ext}^{\Gamma} = (u + \varsigma w)A\bar{t}\Big|_{\Gamma_t} - (uA\bar{t})\Big|_{\Gamma_t} = \varsigma(wA\bar{t})\Big|_{\Gamma_t}$$

$$\Rightarrow \delta W_{ext} = \varsigma\left(\int_\Omega wbdx + (wA\bar{t})\Big|_{\Gamma_t}\right)$$

Deriving FEM Equations Using Energy Approach (2/2)

$$W\big(u(x)\big) = W_{int} - W_{ext}$$

$$\delta W_{int} = \varsigma \int_{\Omega} AE \frac{du}{dx}\frac{dw}{dx} dx\,, \qquad \delta W_{ext} = \varsigma\left(\int_{\Omega} wb dx + (wA\bar{t})\Big|_{\Gamma_t}\right)$$

- The minimizer corresponds to the stationary point of W:

$u(x) \in U$

$$\delta W = \delta W_{int} - \delta W_{ext} = 0$$

Weak form

$$\Rightarrow \int_{\Omega} \frac{dw}{dx} AE \frac{du}{dx} dx = \int_{\Omega} wb dx + (wA\bar{t})\Big|_{\Gamma_t}, \qquad \forall w(x) \in U_0$$

Link to Principle of Virtual Work

- Weak form from the theorem of potential energy:

  Find $u(x) \in U$ such that

$$\int_{\Omega} \frac{dw}{dx} AE \frac{du}{dx} dx - \int_{\Omega} wbdx - (wA\bar{t})\Big|_{\Gamma_t} = 0, \qquad \forall w(x) \in U_0$$

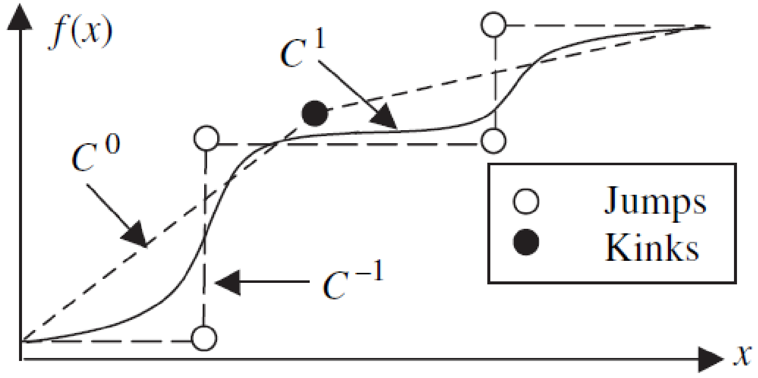
$$\Rightarrow \delta W = \int_{\Omega} AE \underbrace{\frac{du}{dx}}_{\varepsilon} \frac{d(\delta u)}{dx} dx - \int_{\Omega} \delta u \cdot bdx - (\delta u \cdot A\bar{t})\Big|_{\Gamma_t} = 0, \qquad \forall \delta u \in U_0$$

$$\delta W = \underbrace{\int_{\Omega} AE\varepsilon\delta\varepsilon \, dx}_{\delta W_{int}} - \underbrace{\int_{\Omega} \delta u \cdot bdx - (\delta u \cdot A\bar{t})\Big|_{\Gamma_t}}_{\delta W_{ext}} = 0$$

Principle of virtual work

Integrability

$$wA\bar{t}\Big|_{\Gamma_t} + \int_{\Omega} wbdx = \int_{\Omega} \frac{dw}{dx} AE \frac{du}{dx} dx$$



Examples of C^{-1}, C^0 and C^1 functions.

- C^1 (usually splines) are complicate to build.
- Derivatives of C^{-1} (Dirac delta functions) are integrable, but the integral of the product of two Dirac delta functions are meaningless.
- C^{-1} form of displacement field does not ensure compatibility.
- C^0 is the choice for weight functions and trial solutions.
- Derivatives of w and u should be square integrable. H^1

An Approximation Function Example (1/2)

$$u = \begin{cases} -\left(\frac{1}{2}\right)^{\lambda}\frac{x}{l}, 0 \le x \le \frac{l}{2} \\ \left(\frac{x}{l} - \frac{1}{2}\right)^{\lambda} - \left(\frac{1}{2}\right)^{\lambda}\frac{x}{l}, \frac{l}{2} < x \le l \end{cases}$$

$$\frac{du}{dx} = \begin{cases} -\left(\frac{1}{2}\right)^{\lambda}\frac{1}{l}, 0 \le x \le \frac{l}{2} \\ \frac{\lambda}{l}\left(\frac{x}{l} - \frac{1}{2}\right)^{\lambda-1} - \left(\frac{1}{2}\right)^{\lambda}\frac{1}{l}, \frac{l}{2} < x \le l \end{cases}$$

- Check C^0 continuity:

$$u\Big|_{x\to\frac{l^-}{2}} = -\left(\frac{1}{2}\right)^{\lambda}\frac{1}{2} = -\left(\frac{1}{2}\right)^{\lambda+1}$$

$$u\Big|_{x\to\frac{l^+}{2}} = \left(\frac{1}{2} - \frac{1}{2}\right)^{\lambda} - \left(\frac{1}{2}\right)^{\lambda}\frac{1}{2} = -\left(\frac{1}{2}\right)^{\lambda+1} = u\Big|_{x\to\frac{l^-}{2}}$$

$$\frac{du}{dx}\Big|_{x\to\frac{l^-}{2}} = -\left(\frac{1}{2}\right)^{\lambda}\frac{1}{l}$$

$$\frac{du}{dx}\Big|_{x\to\frac{l^+}{2}} = \frac{\lambda}{l}(0^+)^{\lambda-1} - \left(\frac{1}{2}\right)^{\lambda}\frac{1}{2} \neq \frac{du}{dx}\Big|_{x\to\frac{l^-}{2}}$$

When $0 < \lambda < 1$, u belongs to C^0

An Approximation Function Example (2/2)

$$\frac{du}{dx} = \begin{cases} -\left(\frac{1}{2}\right)^{\lambda}\frac{1}{l}, 0 \le x \le \frac{l}{2} \\ \frac{\lambda}{l}\left(\frac{x}{l}-\frac{1}{2}\right)^{\lambda-1} - \left(\frac{1}{2}\right)^{\lambda}\frac{1}{l}, \frac{l}{2} < x \le l \end{cases}$$

- Check square integrability H^1:

$$\int_0^l \left(\frac{du}{dx}\right)^2 dx = \int_0^{\frac{l}{2}} \left(\frac{1}{2}\right)^{2\lambda} \frac{1}{l^2} dx + \int_{\frac{l}{2}}^{l} \left[\frac{\lambda^2}{l^2}\left(\frac{x}{l}-\frac{1}{2}\right)^{2\lambda-2} - 2\frac{\lambda}{l^2}\left(\frac{x}{l}-\frac{1}{2}\right)^{\lambda-1}\left(\frac{1}{2}\right)^{\lambda} + \left(\frac{1}{2}\right)^{2\lambda}\frac{1}{l^2}\right] dx$$

$$\int_{\frac{l}{2}}^{l} \left(\frac{x}{l}-\frac{1}{2}\right)^{2\lambda-2} dx = \frac{l}{2\lambda-1}\left(\frac{x}{l}-\frac{1}{2}\right)^{2\lambda-1} \Big|_{\frac{l}{2}^+}^{l} = \frac{l}{2\lambda-1}\left(\frac{1}{2}\right)^{2\lambda-1} - \frac{l}{2\lambda-1}(0^+)^{2\lambda-1}$$

$\rightarrow -\infty$ when $0 < \lambda < \frac{1}{2}$

u does not belong to H^1

The End