Computational Mechanics

Chapter 3 Approximation of Trial Solutions, Weight Functions and Gauss Quadrature for 1D Problems

5-Step Analysis in FEM

- Preprocessing: subdividing the target domain into finite elements by automatic mesh generators.

Formulation of 1D elements

- Element formulation: development of equations for elements.
- Assembly: obtaining equations for the whole system by gathering ones at the element-level.

Computerized integration for weak form

- Solving equations.
- Postprocessing: calculation results visualization and output.

Requirements for Approximation

- Approximation functions are constructed at the element level.

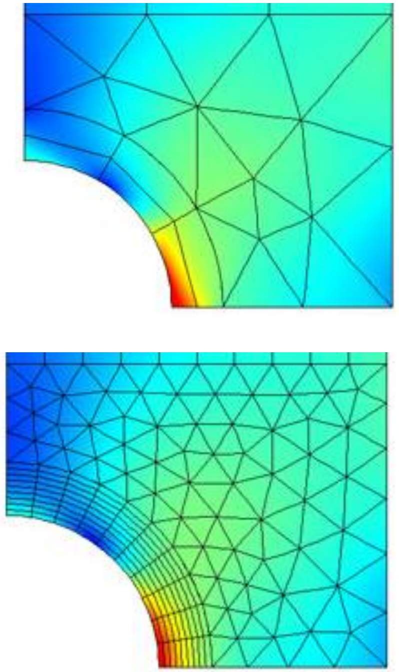
- Accuracy of finite element models needs to improve with mesh refinement.

- In practice: correct FEM models need to converge with mesh refinement.

- Necessary conditions for convergence:

  - Continuity: sufficiently smooth functions (H^1).

  - Completeness: approximation of solutions and 1st order derivatives converge to arbitrary constants.



Mesh examples[1]

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1. https://www.comsol.com/multiphysics/mesh-refinement

Notation and Nomenclature

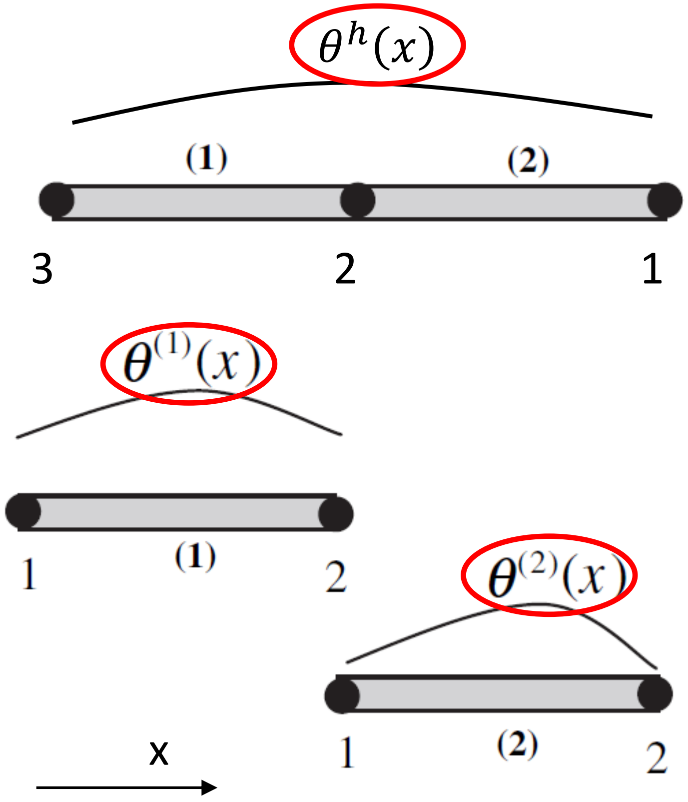
- Notation and nomenclature are defined for generalized 1D FEM problem:

$\theta(x)$: all functions, such as u and T.

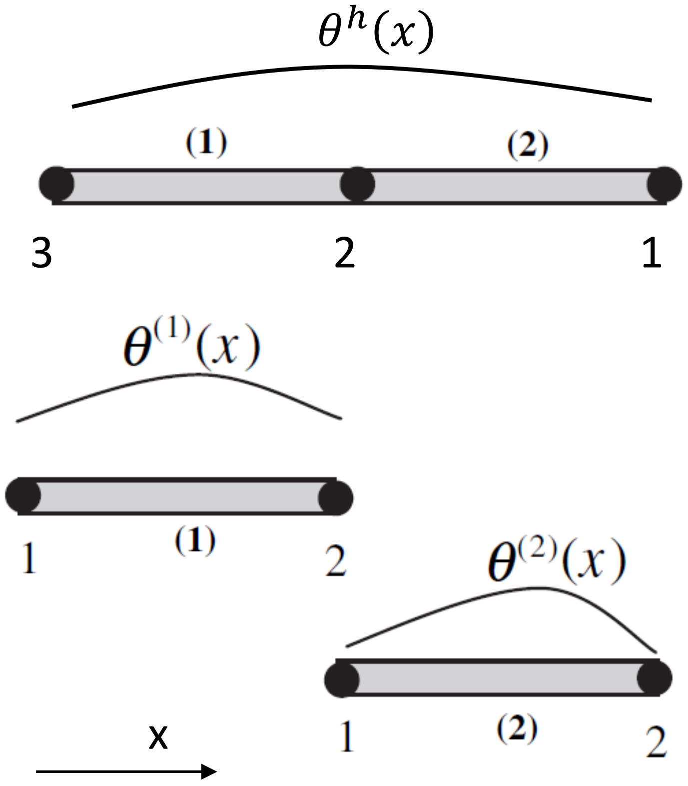
$\theta^h(x)$: global approximation.

$\theta^e(x)$: approximation for a single element.

Nonzero only within the element



Polynomial Approximation



- Take the two-element model as an example:

$$\theta^e(x) = \alpha_0^e + \alpha_1^e x + \alpha_2^e x^2 + \alpha_3^e x^3 + \cdots$$

- Continuity – $\theta^h(x)$ should be C^0:

$$\theta^{(1)}\left(x_2^{(1)}\right) = \theta^{(2)}\left(x_1^{(2)}\right)$$

- Completeness – θ^e should contains the linear term:

$$\theta^e(\underline{l^e} \to 0) \to \alpha_0^e + \alpha_1^e \underline{x^e} + \cdots$$

Element length

Element starting position (fixed after meshing)

$$\frac{d\theta^e}{dx}(l^e \to 0) \to \alpha_1^e + \cdots$$

Linearity Requirement for Trial Solutions

- Incomplete polynomials with linear term:

$$\theta^e(x) = \alpha_0^e + \alpha_1^e x + \alpha_3^e x^3$$

$$\theta^e(l^e \to 0) \to \alpha_0^e + \alpha_1^e x^e + \alpha_3^e x^{e3}, \qquad \frac{d\theta^e}{dx}(l^e \to 0) \to \alpha_1^e + 3\alpha_3^e x^{e2}$$

Convergence rate comparable to pure linear approximation.

- Incomplete polynomials without linear term:

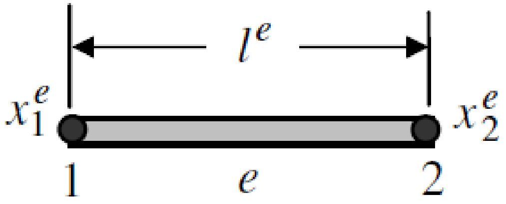
$$\theta^e(x) = \alpha_0^e + \alpha_3^e x^3$$

Nonarbitrary value when $x^{e2} = 0$

$$\theta^e(l^e \to 0) \to \alpha_0^e + \alpha_3^e x^{e3}, \qquad \frac{d\theta^e}{dx}(x_{local}^e \to 0) \to \boxed{3\alpha_3^e x^{e2}}$$

Cannot converge to target values, not suitable for FEM

Approximation for Two-node Linear Element



2 nodes are placed at the ends to ensure global continuity

- Simplest function that meets completeness requirement – linear function:

$$\underline{\theta^e(x) = \alpha_0^e + \alpha_1^e x}$$

2 parameters uniquely determined by 2 nodal displacements.

- Matrix form of the trial solution for computation convenience:

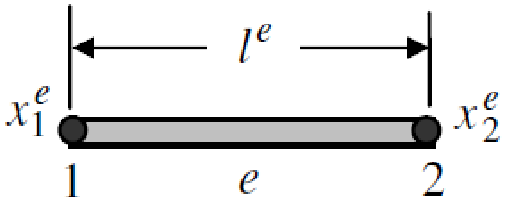
$$\theta^e(x) = \underbrace{[1 \quad x]}_{\boldsymbol{p}(x)} \underbrace{\begin{bmatrix} \alpha_0^e \\ \alpha_1^e \end{bmatrix}}_{\boldsymbol{\alpha}^e} = \boldsymbol{p}(x)\boldsymbol{\alpha}^e$$

- Nodal displacement conditions:

$$\begin{aligned} \theta_1^e &\equiv \theta^e(x_1^e) = \alpha_0^e + \alpha_1^e x_1^e \\ \theta_2^e &\equiv \theta^e(x_2^e) = \alpha_0^e + \alpha_1^e x_2^e \end{aligned} \Rightarrow \underbrace{\begin{bmatrix} \theta_1^e \\ \theta_2^e \end{bmatrix}}_{\boldsymbol{d}^e} = \underbrace{\begin{bmatrix} 1 & x_1^e \\ 1 & x_2^e \end{bmatrix}}_{\boldsymbol{M}^e} \begin{bmatrix} \alpha_0^e \\ \alpha_1^e \end{bmatrix}$$

$$\Rightarrow \boldsymbol{d}^e = \boldsymbol{M}^e \boldsymbol{\alpha}^e$$

Shape Function for Two-node Linear Element (1/2)



$$\theta^e(x) = \boldsymbol{p}(x)\boldsymbol{\alpha}^e, \qquad \boldsymbol{d}^e = \boldsymbol{M}^e\boldsymbol{\alpha}^e$$

$$\theta^e(x) = \boldsymbol{p}(x)\boldsymbol{\alpha}^e = \underline{\boldsymbol{p}(x)(\boldsymbol{M}^e)^{-1}}\boldsymbol{d}^e = \boldsymbol{N}^e(x)\boldsymbol{d}^e$$

Element shape function matrix $\boldsymbol{N}^e(x)$

- Components in $\boldsymbol{N}^e(x)$:

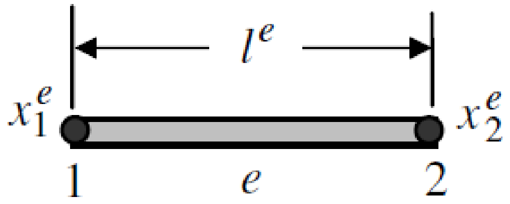
$$\boldsymbol{N}^e(x) = \boldsymbol{p}(x)(\boldsymbol{M}^e)^{-1} = [1 \quad x]\begin{bmatrix} 1 & x_1^e \\ 1 & x_2^e \end{bmatrix}^{-1}$$

Inverse of 2nd order matrix:

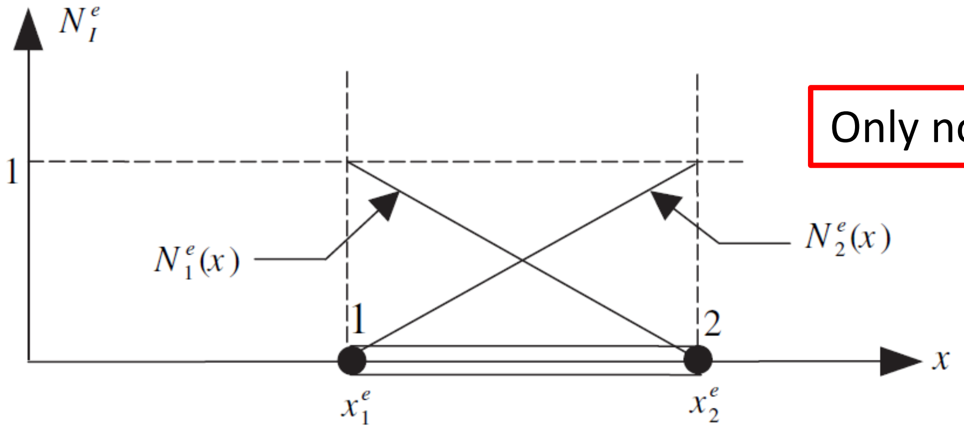
$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\Rightarrow \boldsymbol{N}^e(x) = \frac{1}{x_2^e - x_1^e}[1 \quad x]\begin{bmatrix} x_2^e & -x_1^e \\ -1 & 1 \end{bmatrix} = \frac{1}{l^e}[x_2^e - x \quad x - x_1^e] = \left[\underbrace{\frac{x_2^e - x}{l^e}}_{N_1^e(x)} \quad \underbrace{\frac{x - x_1^e}{l^e}}_{N_2^e(x)}\right]$$

Shape Function for Two-node Linear Element (2/2)



$$\boldsymbol{N}^e(x) = \left[\frac{x_2^e - x}{l^e} \quad \frac{x - x_1^e}{l^e}\right] = [N_1^e(x) \quad N_2^e(x)]$$



Only nonzero within the element

- Kronecker delta expression:

$$N_I^e(x_J^e) = \delta_{IJ} = \begin{cases} 1, & I = J \\ 0, & I \neq J \end{cases}$$

Properties of Shape Functions

- Interpolation properties – the approximation goes through all data points:

$$\theta^e(x) = \boldsymbol{N}^e(x)\boldsymbol{d}^e = [N_1^e(x) \quad N_2^e(x)]\begin{bmatrix}\theta_1^e \\ \theta_2^e\end{bmatrix} = \sum_I^{n=2} \theta_I^e N_I^e(x)$$

$$\Rightarrow \theta^e\left(x_J^e\right) = \sum_I^{n=2} \theta_I^e N_I^e\left(x_J^e\right) = \sum_I^{n=2} \theta_I^e\, \delta_{IJ} = \theta_J^e$$

Can be expand to multiple dimensions

- 1st order derivative of $\theta^e(x)$:

$$\frac{d\theta^e}{dx} = \left[\frac{dN_1^e(x)}{dx} \quad \frac{dN_2^e(x)}{dx}\right]\begin{bmatrix}\theta_1^e \\ \theta_2^e\end{bmatrix} = \boldsymbol{B}^e\boldsymbol{d}^e$$

$$\boldsymbol{B}^e = \left[\frac{dN_1^e(x)}{dx} \quad \frac{dN_2^e(x)}{dx}\right] = \left[d\left(\frac{x_2^e - x}{l^e}\right)/dx \quad d\left(\frac{x - x_1^e}{l^e}\right)/dx\right] = \frac{1}{l^e}[-1 \quad +1]$$

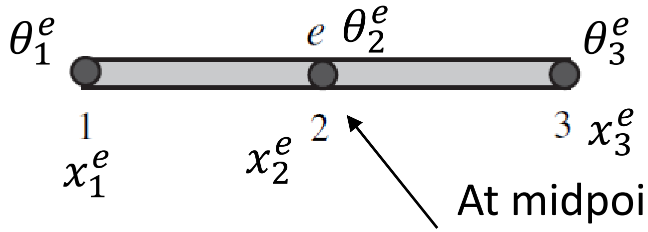
Only depends on element geometry

Quadratic 1D Element

$$\theta^e(x) = \alpha_0^e + \alpha_1^e x + \alpha_2^e x^2 = \begin{bmatrix} 1 & x & x^2 \end{bmatrix} \begin{bmatrix} \alpha_0^e \\ \alpha_1^e \\ \alpha_2^e \end{bmatrix} = \boldsymbol{p}(x)_{1\times 3}\boldsymbol{\alpha}_{3\times 1}^e$$

3 nodes are essential to uniquely determine 3 parameters.

Smoother transition within elements



At midpoint for convenience and performance.

- Nodal value conditions:

$$\begin{aligned} \theta_1^e &\equiv \theta^e(x_1^e) = \alpha_0^e + \alpha_1^e x_1^e + \alpha_2^e x_1^{e2} \\ \theta_2^e &\equiv \theta^e(x_2^e) = \alpha_0^e + \alpha_1^e x_2^e + +\alpha_2^e x_2^{e2} \\ \theta_3^e &\equiv \theta^e(x_3^e) = \alpha_0^e + \alpha_1^e x_3^e + +\alpha_2^e x_3^{e2} \end{aligned} \Rightarrow \underbrace{\begin{bmatrix} \theta_1^e \\ \theta_2^e \\ \theta_3^e \end{bmatrix}}_{\boldsymbol{d}^e} = \underbrace{\begin{bmatrix} 1 & x_1^e & x_1^{e2} \\ 1 & x_2^e & x_2^{e2} \\ 1 & x_3^e & x_3^{e2} \end{bmatrix}}_{\boldsymbol{M}^e} \begin{bmatrix} \alpha_0^e \\ \alpha_1^e \\ \alpha_2^e \end{bmatrix}$$

Components of Quadratic Shape Functions

$$\theta^e(x) = \begin{bmatrix}1 & x & x^2\end{bmatrix}\begin{bmatrix}\alpha_0^e\\ \alpha_1^e\\ \alpha_2^e\end{bmatrix}, \qquad \begin{bmatrix}\theta_1^e\\ \theta_2^e\\ \theta_3^e\end{bmatrix} = \begin{bmatrix}1 & x_1^e & x_1^{e2}\\ 1 & x_2^e & x_2^{e2}\\ 1 & x_3^e & x_3^{e2}\end{bmatrix}\begin{bmatrix}\alpha_0^e\\ \alpha_1^e\\ \alpha_2^e\end{bmatrix}$$

$$\Rightarrow \theta^e(x) = \underbrace{\begin{bmatrix}1 & x & x^2\end{bmatrix}\begin{bmatrix}1 & x_1^e & x_1^{e2}\\ 1 & x_2^e & x_2^{e2}\\ 1 & x_3^e & x_3^{e2}\end{bmatrix}^{-1}}_{\boldsymbol{N}^e(x)}\begin{bmatrix}\theta_1^e\\ \theta_2^e\\ \theta_3^e\end{bmatrix}$$

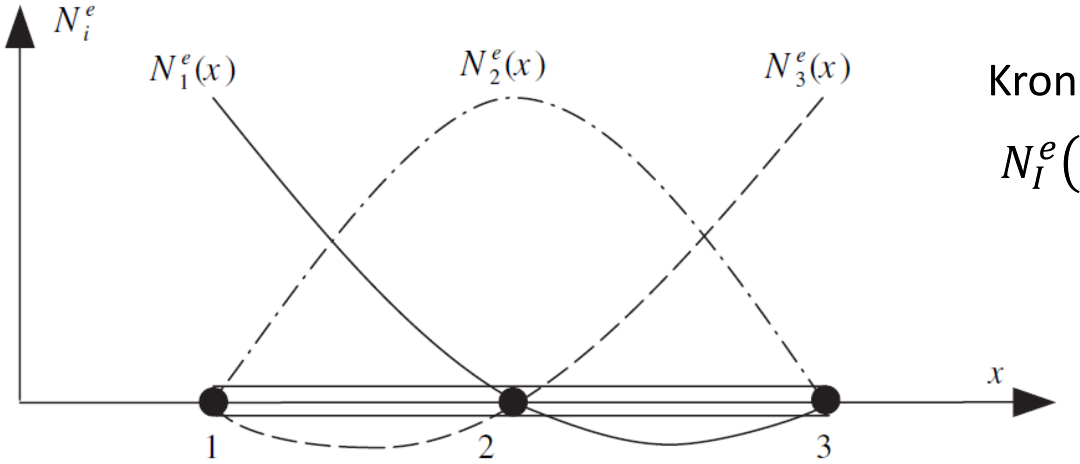
- Inverse of high order (3rd order here) matrices:

$$\left[\begin{array}{c|c}\underbrace{\begin{matrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{matrix}}_{\boldsymbol{a}} & \begin{matrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{matrix}\end{array}\right] \xrightarrow{\text{row reduction}} \left[\begin{array}{ccc|c}1 & 0 & 0 & \\ 0 & 1 & 0 & \boldsymbol{a}^{-1}\\ 0 & 0 & 1 & \end{array}\right]$$

Calculate $\boldsymbol{N}^e(x)$? ……

Interpolation of Quadratic Shape Functions

$$\boldsymbol{N}^e(x)_{1\times 3} = \frac{2}{l^{e2}}\left[(x - x_2^e)(x - x_3^e) \quad -2(x - x_1^e)(x - x_3^e) \quad (x - x_1^e)(x - x_2^e)\right]$$



Kronecker delta property:

$$N_I^e(x_J^e) = \delta_{IJ} = \begin{cases} 1, & I = J \\ 0, & I \neq J \end{cases}$$

$$\theta^e(x) = \boldsymbol{N}^e(x)\boldsymbol{d}^e = \sum_{I}^{n=3} \theta_I^e N_I^e(x)$$

Interpolation of nodal values

Direct Construction of 1D Shape Functions (1/2)

- Developed 1D shape functions are Lagrange interpolations.

- Derivation of high order (quadratic and cubic) shape functions by inversing $\boldsymbol{M}^e$ is time consuming.

- Direct construction – example for quadratic shape functions:

$$N_1^e(x) = \frac{(x-a)(x-b)}{c}$$ - general quadratic function

Kronecker delta property:

$$N_I^e\left(x_J^e\right) = \delta_{IJ} = \begin{cases} 1, & I = J \\ 0, & I \neq J \end{cases}$$

$$N_1^e(x) = \frac{(x - x_2^e)(x - x_3^e)}{c}$$

$$N_1^e(x) = \frac{(x - x_2^e)(x - x_3^e)}{(x_1^e - x_2^e)(x_1^e - x_3^e)}$$

$$N_2^e(x) = \frac{(x - x_1^e)(x - x_3^e)}{(x_2^e - x_1^e)(x_2^e - x_3^e)}$$

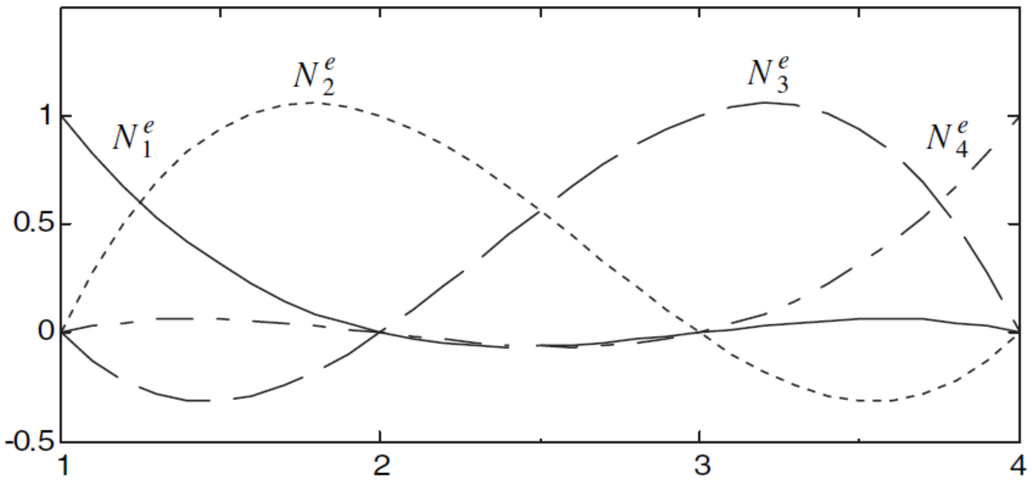
$$N_3^e(x) = \frac{(x - x_1^e)(x - x_2^e)}{(x_3^e - x_1^e)(x_3^e - x_2^e)}$$

Direct construction is the same as mathematical derivation as shape functions are uniquely determined by nodal positions and values.

Direct Construction of 1D Shape Functions (2/2)

- Using direct construction, 1D cubic shape functions can be also easily written out:

4-node element



$$N_1^e(x) = \frac{(x - x_2^e)(x - x_3^e)(x - x_4^e)}{(x_1^e - x_2^e)(x_1^e - x_3^e)(x_1^e - x_4^e)}, \qquad N_2^e(x) = \frac{(x - x_1^e)(x - x_3^e)(x - x_4^e)}{(x_2^e - x_1^e)(x_2^e - x_3^e)(x_2^e - x_4^e)}$$

$$N_3^e(x) = \frac{(x - x_1^e)(x - x_2^e)(x - x_4^e)}{(x_3^e - x_1^e)(x_3^e - x_2^e)(x_3^e - x_4^e)}, \qquad N_4^e(x) = \frac{(x - x_1^e)(x - x_2^e)(x - x_3^e)}{(x_4^e - x_1^e)(x_4^e - x_2^e)(x_4^e - x_1^e)}$$

Approximation of Weight Functions

- Arbitrary weight functions are the keys to solve weak form equations in FEM

- Common practice – utilize the same interpolations for trial solutions – Galerkin FEM:

$$w^e(x) = \boldsymbol{N}^e(x)\boldsymbol{w}^e = \begin{bmatrix} N_1^e(x) & \ldots \end{bmatrix}_{1\times n} \begin{bmatrix} w_1^e \\ \ldots \end{bmatrix}_{n\times 1}$$

$$\frac{dw^e}{dx} = \boldsymbol{B}^e\boldsymbol{w}^e = \begin{bmatrix} \frac{dN_1^e(x)}{dx} & \ldots \end{bmatrix}_{1\times n} \begin{bmatrix} w_1^e \\ \ldots \end{bmatrix}_{n\times 1}$$

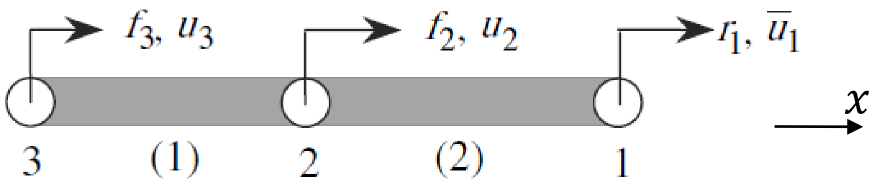
n is element node number and also shape function parameter number.

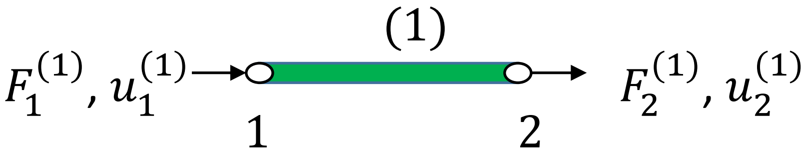
w_i^e are arbitrary.

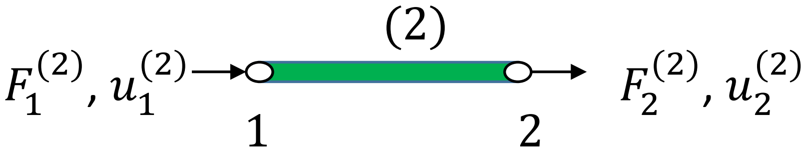
Weight functions are not completely arbitrary, they are constrained to polynomial with fixed order

Element Assembly – Review of Gather Matrices

- For the simplified 2-bar system:







- To enforce compatibility:

$$\boldsymbol{d}^{(1)} = \begin{bmatrix} u_1^{(1)} \\ u_2^{(1)} \end{bmatrix} = \begin{bmatrix} u_3 \\ u_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}}_{\boldsymbol{L}^{(1)}} \begin{bmatrix} \bar{u}_1 \\ u_2 \\ u_3 \end{bmatrix}$$

$$\boldsymbol{d}^{(2)} = \begin{bmatrix} u_1^{(2)} \\ u_2^{(2)} \end{bmatrix} = \begin{bmatrix} u_2 \\ u_1 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}}_{\boldsymbol{L}^{(2)}} \begin{bmatrix} \bar{u}_1 \\ u_2 \\ u_3 \end{bmatrix}$$

$$\Rightarrow \boldsymbol{d}^e = \boldsymbol{L}^e \boldsymbol{d}$$

Global Approximation and Continuity

$$\theta^e(x) = \boldsymbol{N}^e(x)\boldsymbol{d}^e, \qquad w^e(x) = \boldsymbol{N}^e(x)\boldsymbol{w}^e$$

- The element shape function matrices only works for specific elements, for global calculation:

$$\theta^h = \sum_{e=1}^{e=n_{el}} \boldsymbol{N}^e \boldsymbol{d}^e = \sum_{e=1}^{e=n_{el}} \boldsymbol{N}^e \boldsymbol{L}^e \boldsymbol{d} = \left(\sum_{e=1}^{e=n_{el}} \boldsymbol{N}^e \boldsymbol{L}^e \right) \boldsymbol{d}$$

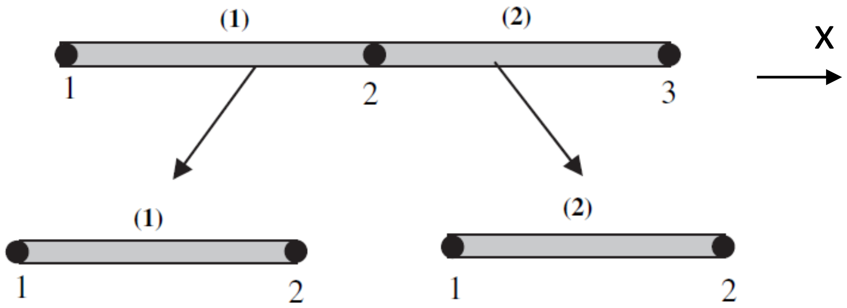
$$w^h = \sum_{e=1}^{e=n_{el}} \boldsymbol{N}^e \boldsymbol{w}^e = \sum_{e=1}^{e=n_{el}} \boldsymbol{N}^e \boldsymbol{L}^e \boldsymbol{w} = \underline{\left(\sum_{e=1}^{e=n_{el}} \boldsymbol{N}^e \boldsymbol{L}^e \right)} \boldsymbol{w}$$

Global shape function matrix $\boldsymbol{N}$

Summation for assembly – element shape functions vanish outside the element

- Enforced compatibility and continuity with global $\boldsymbol{d}$ and $\boldsymbol{w}$

One Example of Global Approximation



$$\boldsymbol{N} = \boldsymbol{N}^{(1)}\boldsymbol{L}^{(1)} + \boldsymbol{N}^{(2)}\boldsymbol{L}^{(2)} = \left[\underbrace{N_1^{(1)}}_{N_1} \quad \underbrace{N_2^{(1)} + N_1^{(2)}}_{N_2} \quad \underbrace{N_2^{(2)}}_{N_3}\right]$$

- $N_2^{(1)} + N_1^{(2)}$ are not direct summation
- $\boldsymbol{N}$ and $\boldsymbol{N}^e$ are the same on e

$$\boldsymbol{d}^{(1)} = \begin{bmatrix} \theta_1^{(1)} \\ \theta_2^{(1)} \end{bmatrix} = \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix} = \boldsymbol{L}^{(1)}\boldsymbol{d}$$

$$\Rightarrow$$

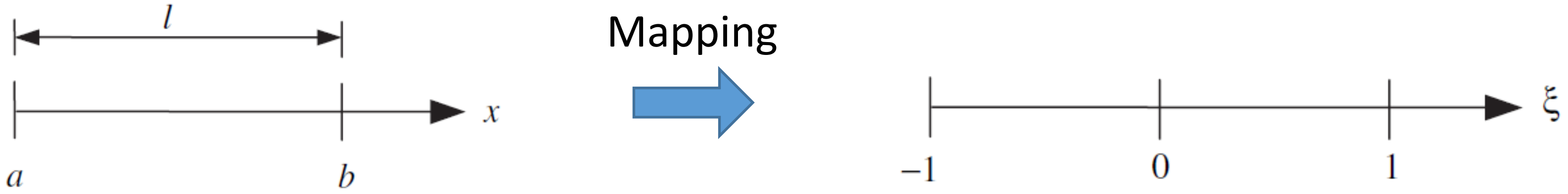
$$\theta^h = \boldsymbol{N}\boldsymbol{d} = [N_1 \quad N_2 \quad N_3] \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix} = \sum_{I=1}^{n_{np}} N_I d_I$$

$$\boldsymbol{d}^{(2)} = \begin{bmatrix} \theta_1^{(2)} \\ \theta_2^{(2)} \end{bmatrix} = \begin{bmatrix} \theta_2 \\ \theta_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix} = \boldsymbol{L}^{(2)}\boldsymbol{d}$$

$$w^h = \boldsymbol{N}\boldsymbol{w} = [N_1 \quad N_2 \quad N_3] \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \sum_{I=1}^{n_{np}} N_I w_I$$

Introduction to Gauss Quadrature

- Weak from FEM involves integral of trial and weight functions.
- General functions (not polynomial ones) do not have closed form integral.
- Gauss quadrature – most efficient numerical integration technique.
- Gauss quadrature formulas are constraint on parent domain $[-1, 1]$ – mapping process:



$$x = \frac{1}{2}(a + b) + \frac{1}{2}\xi(b - a), \text{ when } \xi \in [-1, 1], x \in [a, b]$$

Linear mapping

Gaussian Quadrature Procedure (1/2)

$$x = \frac{1}{2}(a+b) + \frac{1}{2}\xi(b-a) \in [a,b], \xi \in [-1,1]$$

$$I = \int_a^b f(x)dx = \int_{-1}^{1} f\big(x(\xi)\big)dx(\xi) = \underbrace{\frac{b-a}{2}}_{J} \underbrace{\int_{-1}^{1} g(\xi)d\xi}_{\hat{I}}$$

- Approximate $\hat{I}$ with Guass integration:

$$\hat{I} = W_1 g(\xi_1) + W_2 g(\xi_2) + \cdots = \underbrace{[W_1 \quad W_2 \quad \ldots]}_{\boldsymbol{W}^T} \underbrace{\begin{bmatrix} g(\xi_1) \\ g(\xi_2) \\ \ldots \end{bmatrix}}_{\boldsymbol{g}}$$

❖Change integral to multiplication and addition.

❖Select proper weights and integration points.

Polynomial functions have exact solution

- Approximate $g(\xi)$ by a polynomial:

$$g(\xi) = \alpha_1 + \alpha_2\xi + \cdots = \underbrace{[1 \quad \xi \quad \ldots]}_{\boldsymbol{p}(\xi)} \underbrace{\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \ldots \end{bmatrix}}_{\boldsymbol{\alpha}}$$

$$\Rightarrow g(\xi_i) = \alpha_1 + \alpha_2\xi_i + \cdots$$

$$\Rightarrow \begin{bmatrix} g(\xi_1) \\ g(\xi_2) \\ \ldots \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & \xi_1 & \ldots \\ 1 & \xi_2 & \ldots \\ \ldots & \ldots & \ldots \end{bmatrix}}_{\boldsymbol{M}} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \ldots \end{bmatrix}$$

$$\Rightarrow \hat{I} = \boldsymbol{W}^T\boldsymbol{g} = \boldsymbol{W}^T\boldsymbol{M}\boldsymbol{\alpha}$$

Gaussian Quadrature Procedure (2/2)

$$\hat{I} = \int_{-1}^{1} g(\xi)d\xi = \int_{-1}^{1}[1 \quad \xi \quad \dots]\begin{bmatrix}\alpha_1\\ \alpha_2\\ \dots\end{bmatrix}d\xi = \left[\xi \quad \frac{\xi^2}{2} \quad \dots\right]|_{-1}^{1}\boldsymbol{\alpha} = \underbrace{[2 \quad 0 \quad \dots]}_{\hat{\boldsymbol{P}}}\boldsymbol{\alpha}$$

$$\Rightarrow \hat{\boldsymbol{P}}\boldsymbol{\alpha} = \boldsymbol{W}^T\boldsymbol{M}\boldsymbol{\alpha}$$

$$\Rightarrow \hat{\boldsymbol{P}} = \boxed{\boldsymbol{W}^T\boldsymbol{M}}$$

$\boldsymbol{W}^T$: weight matrix
$\boldsymbol{M}$: integration point matrix

- Relationship between integration/Gauss point number n_{gp} and polymer order p for exact integration:

$$I = J\boldsymbol{W}^T\boldsymbol{g}$$

$$p + 1 \leq \underline{2n_{gp}}$$

n_{gp} weights + n_{gp} integration points

Number of controllable variables is not smaller than degree of freedom

An Example of Gauss Quadrature

- Calculate the integral:

$$I = \int_{2}^{5} (x^3 + x^2) dx$$

- Solution:
  - ➢ Step 1: determine n_{gp}

$$n_{gp} \geq \frac{p+1}{2} = 2$$

Select $n_{gp} = 2$ for convenience.

  - ➢ Step 2: Determine $\boldsymbol{W}^T$ and $\boldsymbol{M}$

$$[W_1 \quad W_2]\begin{bmatrix} 1 & \xi_1 & \xi_1^2 & \xi_1^3 \\ 1 & \xi_2 & \xi_2^2 & \xi_2^3 \end{bmatrix} = \underline{\begin{bmatrix} 2 & 0 & \frac{2}{3} & 0 \end{bmatrix}}$$

4 eqs for 4 unknowns

$$\Rightarrow W_1 = W_2 = 1, \qquad \xi_1 = -\xi_2 = \frac{1}{\sqrt{3}}$$

  - ➢ Step 3: calculate integration:

$$I = \int_{2}^{5} (x^3 + x^2) dx = \frac{5-2}{2} \boldsymbol{W}^T \boldsymbol{g}$$

$$x = \frac{1}{2}(5+2) + \frac{1}{2}\xi(5-2) = \frac{7+3\xi}{2}$$

$$\Rightarrow I = \frac{3}{2}[1 \quad 1]\begin{bmatrix} g(\xi_1) \\ g(\xi_1) \end{bmatrix} = \frac{3}{2} f\big(x(\xi_1)\big) + \frac{3}{2} f\big(x(\xi_2)\big)$$

$$\Rightarrow I = \frac{3}{2}\left[\left(\frac{7+3\xi_1}{2}\right)^3 + \left(\frac{7+3\xi_1}{2}\right)^2\right] + \frac{3}{2}\left[\left(\frac{7+3\xi_2}{2}\right)^3 + \left(\frac{7+3\xi_2}{2}\right)^2\right]$$

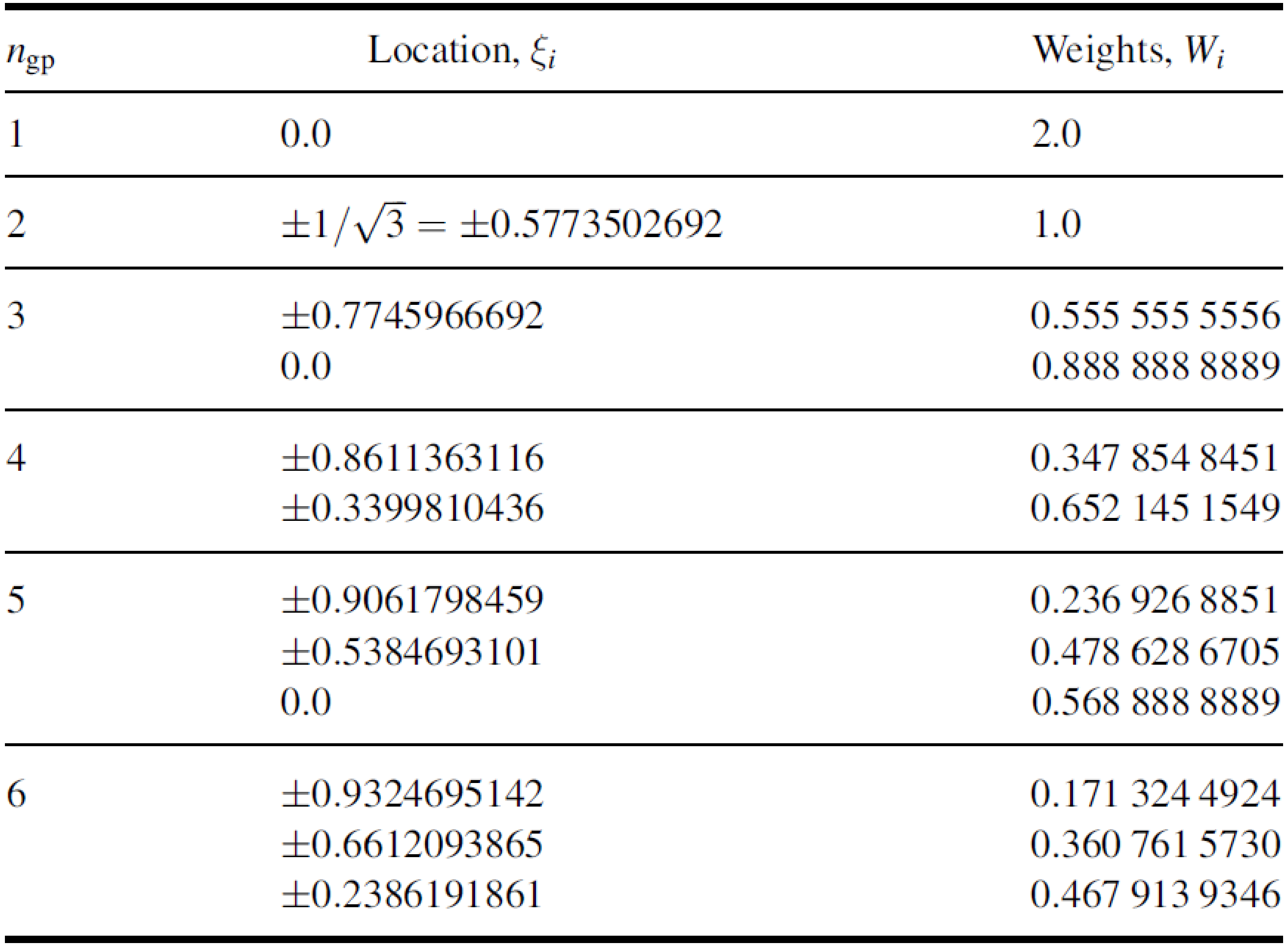
$$\Rightarrow I = 191.25 = I_{exact}$$

Polynomial functions get exact integral results

Weights and Integration Points Table

- Reduce calculation load

  *Integration points are fixed and can be obtained before hand.



n_{gp}	Location, ξ_i	Weights, W_i
1	0.0	2.0
2	$\pm 1/\sqrt{3} = \pm 0.5773502692$	1.0
3	±0.7745966692	0.555 555 5556
	0.0	0.888 888 8889
4	±0.8611363116	0.347 854 8451
	±0.3399810436	0.652 145 1549
5	±0.9061798459	0.236 926 8851
	±0.5384693101	0.478 628 6705
	0.0	0.568 888 8889
6	±0.9324695142	0.171 324 4924
	±0.6612093865	0.360 761 5730
	±0.2386191861	0.467 913 9346