Computational Mechanics

Chapter 4 Finite Element Formulation for General 1D Problems

5-Step Analysis in FEM

Trivial for 1D problems

- Preprocessing: subdividing the target domain into finite elements by automatic mesh generators.

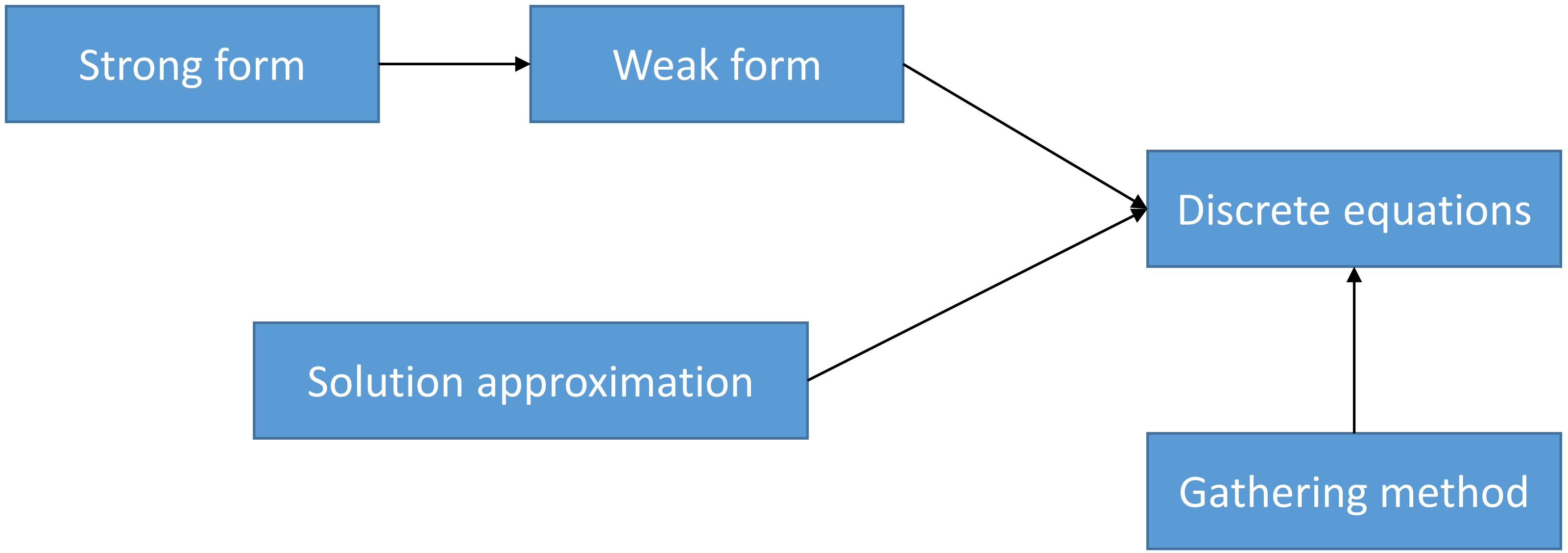
Focuses on solving 1D problems using FEM

- Element formulation: development of equations for elements.
- Assembly: obtaining equations for the whole system by gathering ones at the element-level.
- Solving equations.

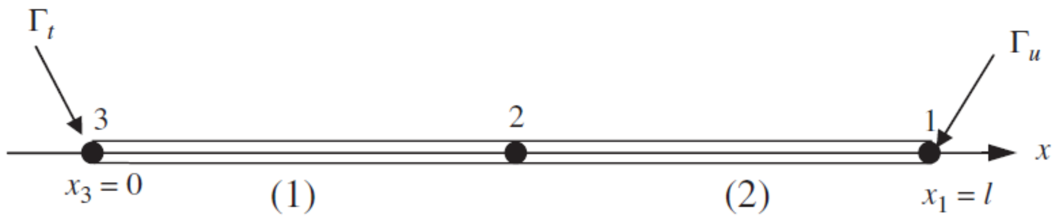
- Postprocessing: calculation results visualization and output.

Plot calculated curves

Components to Formulate FEM Equations



A 2-Element Example with Linear Approximation



- Review of the weak form with specific boundary:

  Find smooth $u(x)$ that satisfies $u(l) = \bar{u}_l$ so that

$$\int_0^l \frac{dw}{dx} AE \frac{du}{dx} dx - \int_0^l wbdx - (w\bar{t}A)\Big|_{x=0} = 0$$

$$\forall w \text{ with } w(l) = 0$$

- Review of approximation of weight functions and trial solutions:
  - Weight functions:

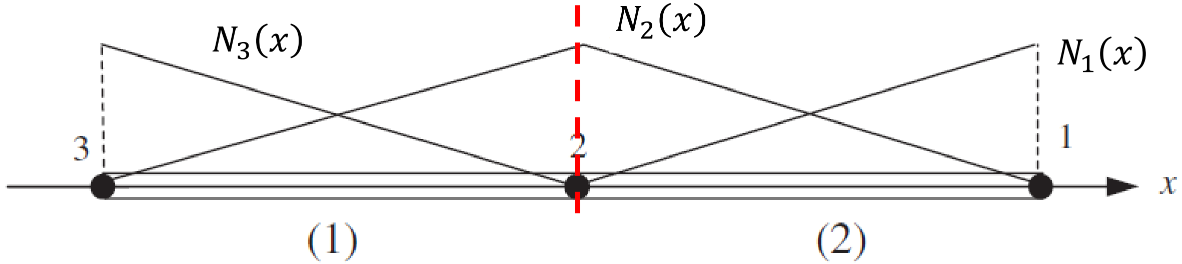
$$w(x) \approx w^h(x) = \boldsymbol{N}(x)\boldsymbol{w}$$

  - Trial solutions:

$$u(x) \approx u^h(x) = \boldsymbol{N}(x)\boldsymbol{d}$$

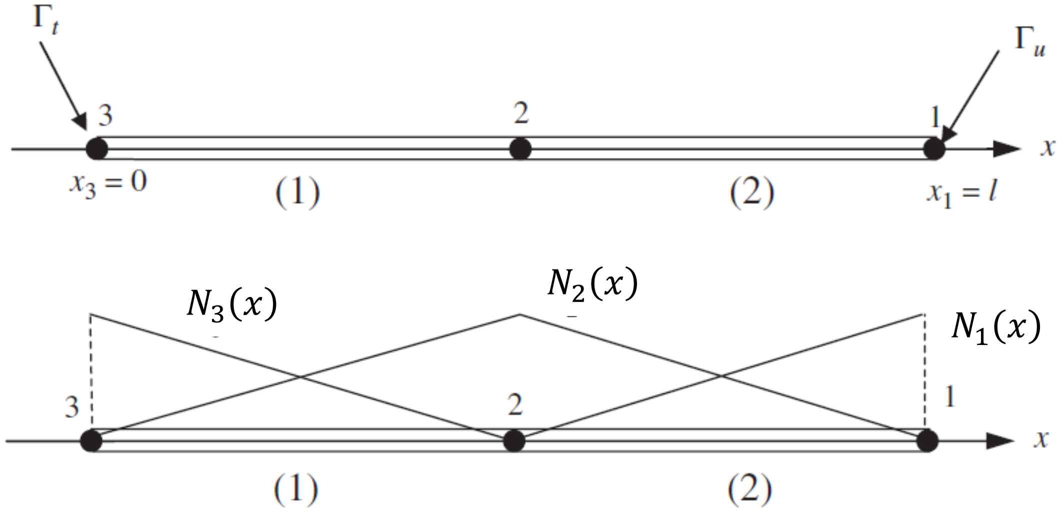
$$\boldsymbol{N}(x) = [N_1(x) \quad N_2(x) \quad N_3(x)]$$

$$\boldsymbol{w} = [w_1 \quad w_2 \quad w_3], \qquad \boldsymbol{d} = [d_1 \quad d_2 \quad d_3]$$



Approximation limited within elements

Nodal Value Conditions for the 2-Element Example



- Essential boundary conditions:

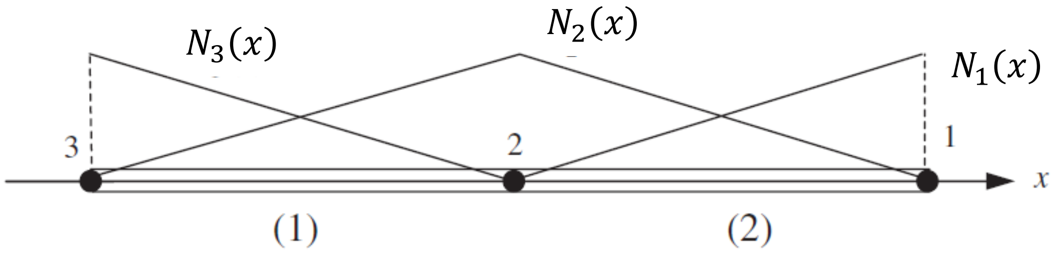
$$\bar{u}_1 = u(l) \approx u^h(l) = \boldsymbol{N}(x_1)\boldsymbol{d} = \sum_{I=1}^{n_{nd}} N_I(x_1)d_I$$

$$\xrightarrow{kronecker\ delta} \bar{u}_1 = 1 \times d_1 = u_1$$

- Weight functions conditions:

$$0 = w(l) \approx w^h(l) = \boldsymbol{N}(x_1)\boldsymbol{w} = 1 \times w_1 = w_1$$

Weak Form for the 2-Element Example



- $\boldsymbol{N}(x)$ and $d\boldsymbol{N}(x)/dx$ are discontinuous, so piecewise integration is essential:

$$0 = \int_0^l \frac{dw}{dx} AE \frac{du}{dx} dx - \int_0^l wbdx - (w\bar{t}A)\Big|_{x=0}$$

$$\Rightarrow$$

$$0 = \sum_{e=1}^{n_{el}} \left\{ \int_{x_1^e}^{x_2^e} \frac{dw^e}{dx} A^e E^e \frac{du^e}{dx} dx - \int_{x_1^e}^{x_2^e} w^e b^e dx \right\} - (w\bar{t}A)\Big|_{x=0}$$

- Element functions are linked to global functions with gather matrices:

$$\boldsymbol{d}^e = \boldsymbol{L}^e \boldsymbol{d}, \qquad \boldsymbol{w}^e = \boldsymbol{L}^e \boldsymbol{w}$$

$$u^e(x) = \boldsymbol{N}^e(x)\boldsymbol{d}^e, \qquad \frac{du^e(x)}{dx} = \frac{d\boldsymbol{N}^e(x)}{dx}\boldsymbol{d}^e = \boldsymbol{B}^e(x)\boldsymbol{d}^e$$

$$w^e(x) = \boldsymbol{N}^e(x)\boldsymbol{w}^e, \qquad \frac{dw^e(x)}{dx} = \boldsymbol{B}^e(x)\boldsymbol{w}^e$$

$$\Rightarrow 0 = \sum_{e=1}^{n_{el}} \left\{ \begin{array}{c} \displaystyle\int_{x_1^e}^{x_2^e} \boldsymbol{B}^e(x)\boldsymbol{w}^e A^e E^e \boldsymbol{B}^e(x)\boldsymbol{d}^e dx \\ \displaystyle -\int_{x_1^e}^{x_2^e} \boldsymbol{N}^e(x)\boldsymbol{w}^e b^e dx - (\boldsymbol{N}^e(x)\boldsymbol{w}^e \bar{t}^e A^e)\Big|_{x=0} \end{array} \right\}$$

Re-organization of Weak form Integration

$$0 = \sum_{e=1}^{n_{el}} \left(\int_{x_1^e}^{x_2^e} \boldsymbol{B}^e(x)\boldsymbol{w}^e A^e E^e \boldsymbol{B}^e(x)\boldsymbol{d}^e dx - \int_{x_1^e}^{x_2^e} \boldsymbol{N}^e(x)\boldsymbol{w}^e b^e dx - \left(\boldsymbol{N}^e(x)\boldsymbol{w}^e \bar{t}^e A^e\right)\Big|_{x=0} \right)$$

Get *constants* out for generalized integration results

$$\Rightarrow 0 = \sum_{e=1}^{n_{el}} \left(\int_{x_1^e}^{x_2^e} (\boldsymbol{B}^e(x)\boldsymbol{w}^e)^T A^e E^e \boldsymbol{B}^e(x)\boldsymbol{d}^e dx - \int_{x_1^e}^{x_2^e} (\boldsymbol{N}^e(x)\boldsymbol{w}^e)^T b^e dx - \left((\boldsymbol{N}^e(x)\boldsymbol{w}^e)^T \bar{t}^e A^e\right)\Big|_{x=0} \right)$$

$$\Rightarrow 0 = \sum_{e=1}^{n_{el}} \left(\int_{x_1^e}^{x_2^e} \boldsymbol{w}^{eT}\boldsymbol{B}^{eT}(x) A^e E^e \boldsymbol{B}^e(x)\boldsymbol{d}^e dx - \int_{x_1^e}^{x_2^e} \boldsymbol{w}^{eT}\boldsymbol{N}^{eT}(x) b^e dx - \left(\boldsymbol{w}^{eT}\boldsymbol{N}^{eT}(x) \bar{t}^e A^e\right)\Big|_{x=0} \right)$$

Independent from x

$$\Rightarrow 0 = \sum_{e=1}^{n_{el}} \boldsymbol{w}^{eT} \left(\underline{\int_{x_1^e}^{x_2^e} \boldsymbol{B}^{eT}(x) A^e E^e \boldsymbol{B}^e(x) dx\, \boldsymbol{d}^e} - \underline{\int_{x_1^e}^{x_2^e} \boldsymbol{N}^{eT}(x) b^e dx} - \underline{\left(\boldsymbol{N}^{eT}(x) \bar{t}^e A^e\right)\Big|_{x=0}} \right)$$

$\boldsymbol{K}^e, x \in \Omega^e$ – element stiffness matrix

$\boldsymbol{f}_{\Omega^e}, x \in \Omega^e$

$\boldsymbol{f}_{\Gamma^e}, x \in \Gamma^e$

$\boldsymbol{f}_{\Omega^e} + \boldsymbol{f}_{\Gamma^e} = \boldsymbol{f}^e$ – element external force matrix

Weak Form Integration in Global Domain

$$0 = \sum_{e=1}^{n_{el}} \boldsymbol{w}^{eT}(\boldsymbol{K}^e\boldsymbol{d}^e - \boldsymbol{f}^e)$$

Independent from x

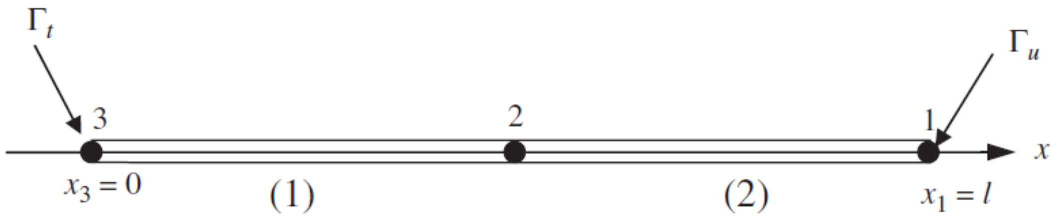
$$\boldsymbol{d}^e = \boldsymbol{L}^e\boldsymbol{d}, \qquad \boldsymbol{w}^e = \boldsymbol{L}^e\boldsymbol{w}$$

$$\Rightarrow 0 = \sum_{e=1}^{n_{el}} \boldsymbol{w}^T\boldsymbol{L}^{eT}(\boldsymbol{K}^e\boldsymbol{L}^e\boldsymbol{d} - \boldsymbol{f}^e)$$

$$\Rightarrow 0 = \boldsymbol{w}^T\left[\underbrace{\left(\sum_{e=1}^{n_{el}} \boldsymbol{L}^{eT}\boldsymbol{K}^e\boldsymbol{L}^e\right)}_{}\boldsymbol{d} - \underbrace{\sum_{e=1}^{n_{el}} \boldsymbol{L}^{eT}\boldsymbol{f}^e}_{}\right]$$

$\boldsymbol{K}$ – stiffness matrix $\qquad$ $\boldsymbol{f}$ – external force matrix

$$\Rightarrow 0 = \boldsymbol{w}^T[\boldsymbol{K}\boldsymbol{d} - \boldsymbol{f}], \forall \boldsymbol{w}\ with\ w_1 = w(l) = 0$$



- Utilization of residual $\boldsymbol{r}$:

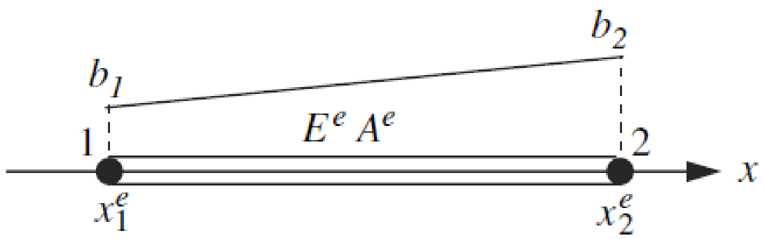
$$\boldsymbol{r} = \boldsymbol{K}\boldsymbol{d} - \boldsymbol{f}$$

$$\Rightarrow 0 = \boldsymbol{w}^T\boldsymbol{r} = [0 \quad w_2 \quad w_3]\begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} = w_2r_2 + w_3r_3$$

$$\Rightarrow 0 = r_2 = r_3$$

$$\boldsymbol{d} = [\bar{u}_1 \quad u_2 \quad u_3]$$

Example: Element Matrices for A Linear Element



- Linearly distributed body forces are applied to a linear 2-node element with constant cross-section area A^e and Young's modulus E^e.
- Review of linear shape functions:

$$\boldsymbol{N}^e = \frac{1}{l^e}\begin{bmatrix}(x_2^e - x) & (x - x_1^e)\end{bmatrix}$$

$$\boldsymbol{B}^e = \frac{d\boldsymbol{N}^e}{dx} = \frac{1}{l^e}\begin{bmatrix}-1 & 1\end{bmatrix}$$

- Calculate element stiffness matrices:

$$\boldsymbol{K}^e = \int_{x_1^e}^{x_2^e} \boldsymbol{B}^{eT}(x) A^e E^e \boldsymbol{B}^e(x)\, dx$$

$$\Rightarrow \boldsymbol{K}^e = \int_{x_1^e}^{x_2^e} \frac{1}{l^e}\begin{bmatrix}-1\\1\end{bmatrix} A^e E^e \frac{1}{l^e}\begin{bmatrix}-1 & 1\end{bmatrix} dx = \frac{A^e E^e}{l^e}\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix}$$

$$\boldsymbol{f}_{\Omega^e} = \int_{x_1^e}^{x_2^e} \boldsymbol{N}^{eT}(x) b^e\, dx = \int_{x_1^e}^{x_2^e} \boldsymbol{N}^{eT}(x)\boldsymbol{N}^e(x)\begin{bmatrix}b_1\\b_2\end{bmatrix} dx$$

$$\Rightarrow \boldsymbol{f}_{\Omega^e} = \frac{1}{l^{e2}}\int_{x_1^e}^{x_2^e}\begin{bmatrix}(x_2^e - x)\\(x - x_1^e)\end{bmatrix}\begin{bmatrix}(x_2^e - x) & (x - x_1^e)\end{bmatrix} dx \begin{bmatrix}b_1\\b_2\end{bmatrix}$$

$$\Rightarrow \boldsymbol{f}_{\Omega^e} = \frac{l^{e2}}{6}\begin{bmatrix}2 & 1\\1 & 2\end{bmatrix}\begin{bmatrix}b_1\\b_2\end{bmatrix}$$

Summation is total body force

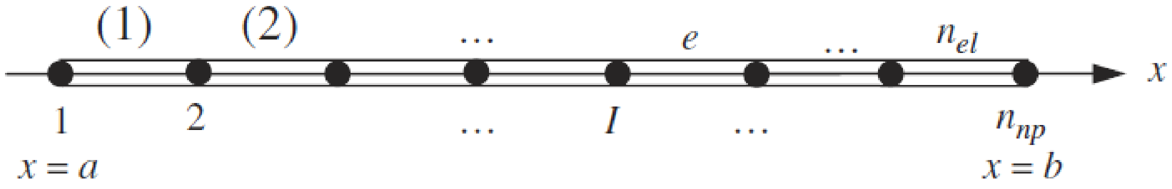
Note: Matrices similar to that of simple 2D bar element with constant properties, but the method can be directly applied to higher order elements and more complex problems.

Discrete Equations for *Arbitrary* Boundary Conditions

- Review of the generalized weak form:

  Find $u(x) \in U$ such that

$$w A \bar{t}\Big|_{\Gamma_t} + \int_{\Omega} wb dx = \int_{\Omega} \frac{dw}{dx} AE \frac{du}{dx} dx \,, \forall w \in U_0$$



Arbitrary element number and size for accuracy

- Piecewise integration through the whole domain:

$$0 = \sum_{e=1}^{n_{el}} \left\{ \int_{\Omega^e} \frac{dw^e}{dx} A^e E^e \frac{du^e}{dx} dx - \int_{\Omega^e} w^e b^e dx - (w\bar{t}A)\Big|_{\Gamma_t^e} \right\}$$

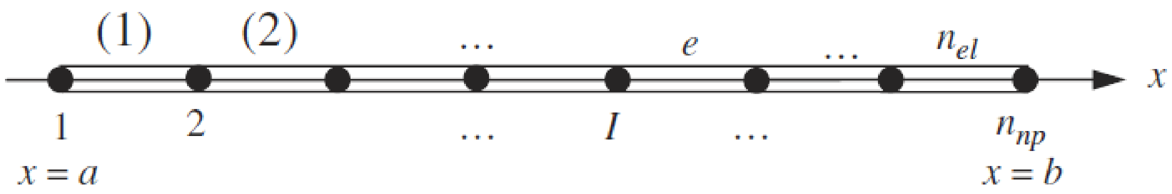
$$u^e(x) = \boldsymbol{N}^e(x)\boldsymbol{d}^e, \qquad \frac{du^e(x)}{dx} = \frac{d\boldsymbol{N}^e(x)}{dx}\boldsymbol{d}^e = \boldsymbol{B}^e(x)\boldsymbol{d}^e$$

$$w^e(x) = \boldsymbol{N}^e(x)\boldsymbol{w}^e, \qquad \frac{dw^e(x)}{dx} = \boldsymbol{B}^e(x)\boldsymbol{w}^e$$

$$\Rightarrow 0 = \sum_{e=1}^{n_{el}} \left\{ \begin{array}{l} \displaystyle\int_{x_1^e}^{x_2^e} \boldsymbol{B}^e(x)\boldsymbol{w}^e A^e E^e \boldsymbol{B}^e(x)\boldsymbol{d}^e dx \\ \displaystyle - \int_{x_1^e}^{x_2^e} \boldsymbol{N}^e(x)\boldsymbol{w}^e b^e dx - (\boldsymbol{N}^e(x)\boldsymbol{w}^e \bar{t}^e A^e)\Big|_{\Gamma_t^e} \end{array} \right\}$$

Same as the 2-element example

Partition of Global Matrices with Multiple Nodes



- Global matrices partition for ease of calculation:

$$\boldsymbol{d} = \begin{bmatrix} \bar{\boldsymbol{d}}_E \\ \boldsymbol{d}_F \end{bmatrix}, \qquad \boldsymbol{w} = \begin{bmatrix} \boldsymbol{w}_E \\ \boldsymbol{w}_F \end{bmatrix} = \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{w}_F \end{bmatrix}$$

$$\forall w \in U_0 \Rightarrow \forall \boldsymbol{w}_F$$

Manual calculation note: In the global system, number nodes with essential boundary conditions first

- Similar to the 2-element example:

$$\boldsymbol{r} = \boldsymbol{K}\boldsymbol{d} - \boldsymbol{f}, \qquad \boldsymbol{w}^T\boldsymbol{r} = \boldsymbol{0}, \forall \boldsymbol{w}_F$$

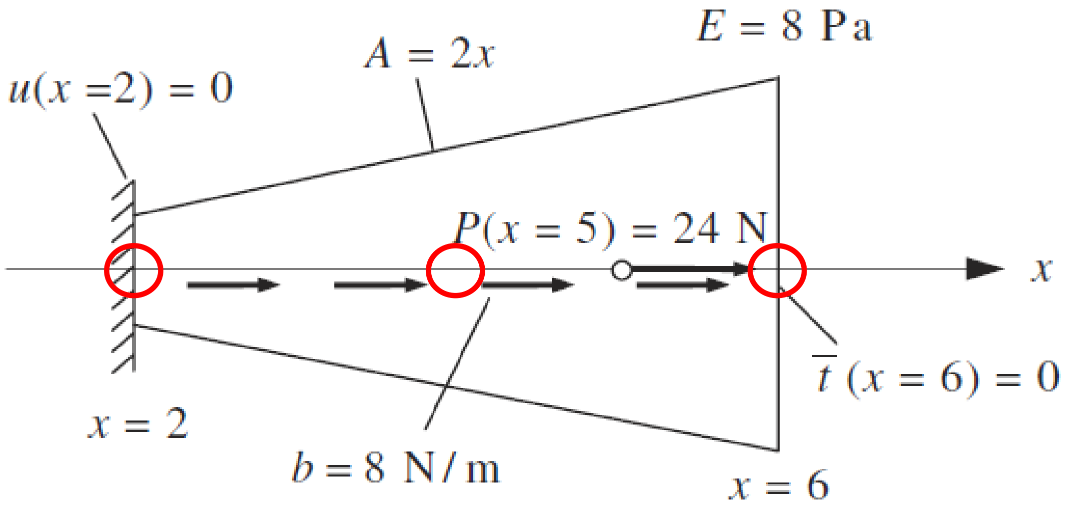
$$\Rightarrow [\boldsymbol{0} \quad \boldsymbol{w}_F^T] \begin{bmatrix} \boldsymbol{r}_E \\ \boldsymbol{r}_F \end{bmatrix} = \boldsymbol{w}_F^T \boldsymbol{r}_F = \boldsymbol{0}, \forall \boldsymbol{w}_F$$

$$\Rightarrow \boldsymbol{r}_F = \boldsymbol{0}$$

$$\Rightarrow \boldsymbol{r} = \begin{bmatrix} \boldsymbol{r}_E \\ \boldsymbol{0} \end{bmatrix} = \boldsymbol{K}\boldsymbol{d} - \boldsymbol{f} = \begin{bmatrix} \boldsymbol{K}_{EE} & \boldsymbol{K}_{EF} \\ \boldsymbol{K}_{EF}^T & \boldsymbol{K}_{FF} \end{bmatrix} \begin{bmatrix} \bar{\boldsymbol{d}}_E \\ \boldsymbol{d}_F \end{bmatrix} - \begin{bmatrix} \boldsymbol{f}_E \\ \boldsymbol{f}_F \end{bmatrix}$$

Solution similar to the discrete method

Stress Analysis Example: Tapered Elastic Bar



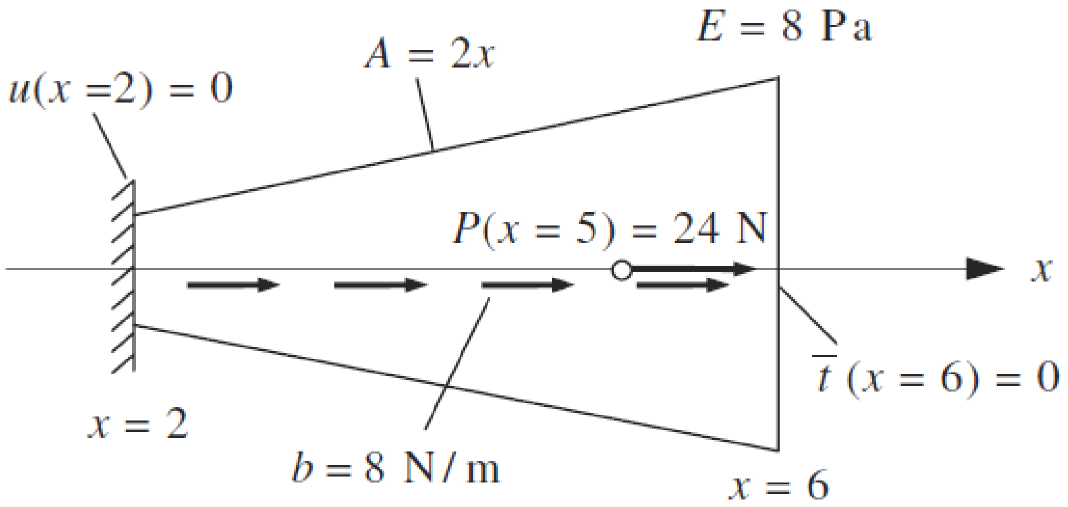
- Unit of coordinate is meter.
- Solution with 1 quadratic element?
- Solution with 2 linear elements?

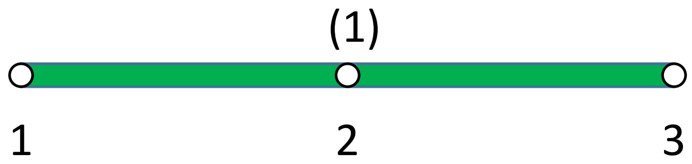
- Note: equally-spaced nodes are usually utilized for calculation convenience.
- General procedure:
  1. Meshing
  2. Determine shape functions
  3. Obtain global stiffness matrix;
  4. Obtain external force matrices
  5. Solve nodal values
  6. Post processing

  -> Displacement and stress fields

Gauss quadrature

Shape Functions of 1 *Quadratic* Element





- Shape functions:

$$N_1(x) = \frac{(x-4)(x-6)}{(2-4)(2-6)} = \frac{1}{8}(x-4)(x-6)$$

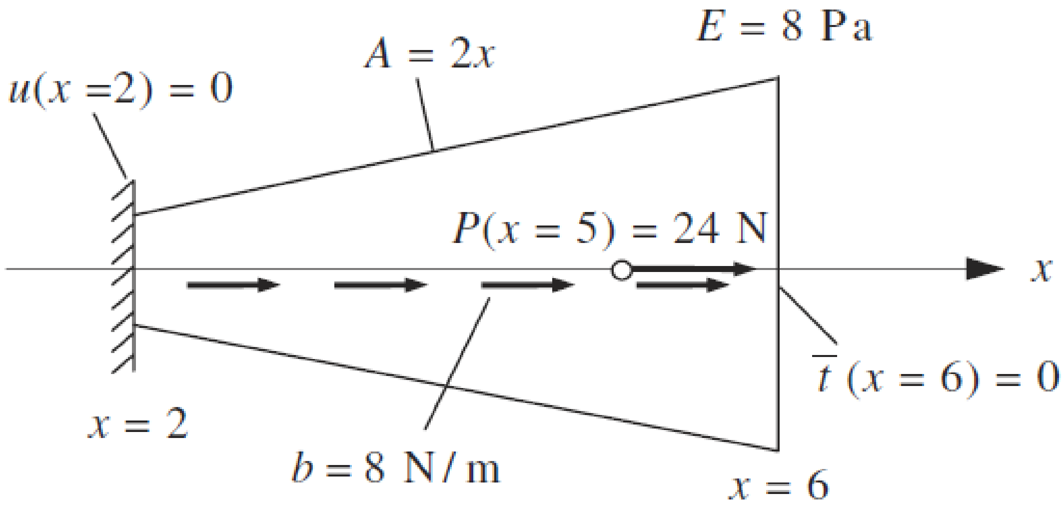
$$N_2(x) = \frac{(x-2)(x-6)}{(4-2)(4-6)} = -\frac{1}{4}(x-2)(x-6)$$

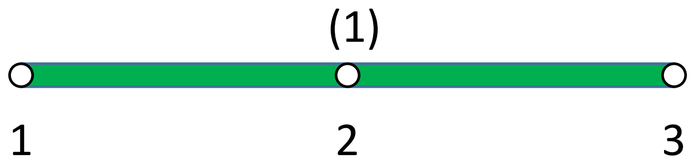
$$N_3(x) = \frac{(x-2)(x-4)}{(6-2)(6-4)} = \frac{1}{8}(x-2)(x-4)$$

$$\boldsymbol{B}(x) = \left[\frac{dN_1}{dx} \quad \frac{dN_3}{dx} \quad \frac{dN_3}{dx}\right]$$

$$\Rightarrow \boldsymbol{B}(x) = \frac{1}{4}[x-5 \quad 8-2x \quad x-3]$$

Stiffness Matrix of 1 *Quadratic* Element





- Global stiffness matrix:

$$\boldsymbol{K} = \boldsymbol{K}^{(1)} = \int_{x_1}^{x_3} \boldsymbol{B}^{eT} A^{(1)} E^{(1)} \boldsymbol{B}^{e} dx$$

$$\Rightarrow \boldsymbol{K} = \int_2^6 \frac{1}{4}\begin{bmatrix} x-5 \\ 8-2x \\ x-3 \end{bmatrix} 2x \cdot 8 \frac{1}{4}[x-5 \quad 8-2x \quad x-3]dx$$

- Utilize Gauss quadrature:

$$\boldsymbol{K} = \begin{bmatrix} 26.67 & -32 & 5.33 \\ -32 & 85.33 & -53.33 \\ 5.33 & -53.33 & 48 \end{bmatrix}$$

Review of Gauss Quadrature

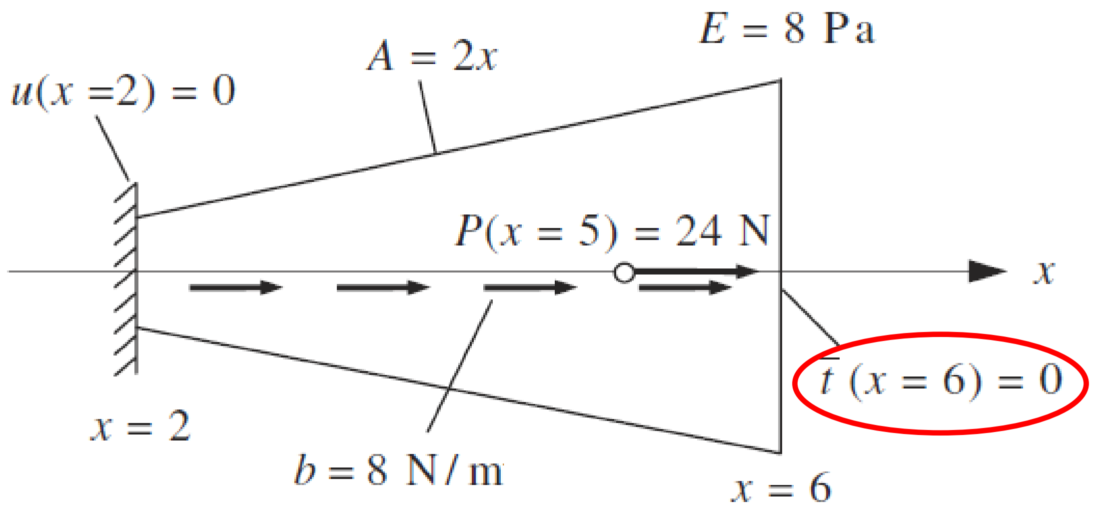
$$\boldsymbol{K} = \int_2^6 \frac{1}{4}\begin{bmatrix} x-5 \\ 8-2x \\ x-3 \end{bmatrix} 2x \cdot 8\frac{1}{4}[x-5 \quad 8-2x \quad x-3]dx = \int_2^6 x\begin{bmatrix} x-5 \\ 8-2x \\ x-3 \end{bmatrix}[x-5 \quad 8-2x \quad x-3]dx$$

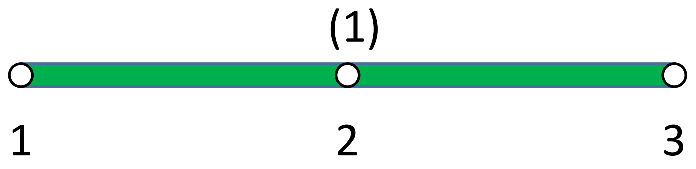
$$p = 3 \Rightarrow 2n_{gp} \geq 3+1 \Rightarrow n_{gp} = 2 \Rightarrow \xi_1 = -\xi_2 = \frac{1}{\sqrt{3}}, \qquad W_1 = W_2 = 1$$

$$x = \frac{1}{2}(2+6) + \frac{1}{2}(6-2)\xi = 4 - 2\xi$$

$$\Rightarrow \boldsymbol{K} = \frac{6-2}{2}\left[1 \cdot \left(x\begin{bmatrix} x-5 \\ 8-2x \\ x-3 \end{bmatrix}[x-5 \quad 8-2x \quad x-3]\right)\Big|_{\xi=\frac{1}{\sqrt{3}}} + 1 \cdot \left(x\begin{bmatrix} x-5 \\ 8-2x \\ x-3 \end{bmatrix}[x-5 \quad 8-2x \quad x-3]\right)\Big|_{\xi=-\frac{1}{\sqrt{3}}}\right]$$

Force Matrix of 1 *Quadratic* Element





- Global force matrix (0 traction):

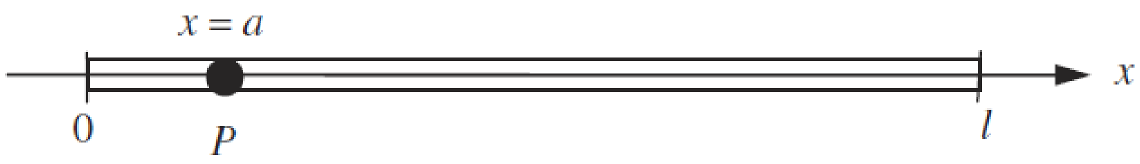
$$\boldsymbol{f}_{\Omega} = \boldsymbol{f}_{\Omega^{(1)}} = \int_{2}^{6} \boldsymbol{N}^{eT}(x) b^{e} dx$$

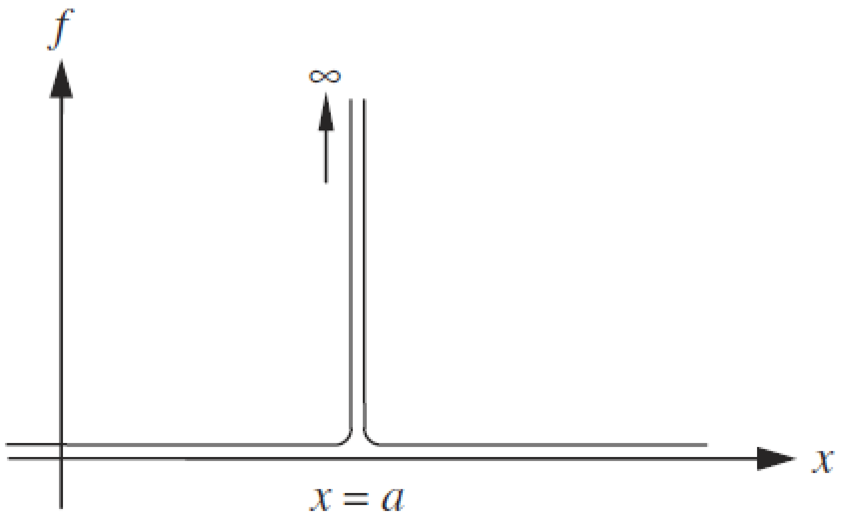
$$\Rightarrow \boldsymbol{f}_{\Omega} = \int_{2}^{6} \frac{1}{8} \begin{bmatrix} (x-4)(x-6) \\ -2(x-2)(x-6) \\ (x-2)(x-4) \end{bmatrix} (8 + 24\delta(x-5))\, dx$$

- Utilize Gauss quadrature:

$$\boldsymbol{f}_{\Omega} = \begin{bmatrix} 2.33 \\ 39.33 \\ 14.33 \end{bmatrix}$$

Formulation of Concentrated Body Force





- Regard concentrated body force as a distribution:

$$P = \int_0^l f(x)dx$$

- Concentrate body force is applied within a infinitesimal region:

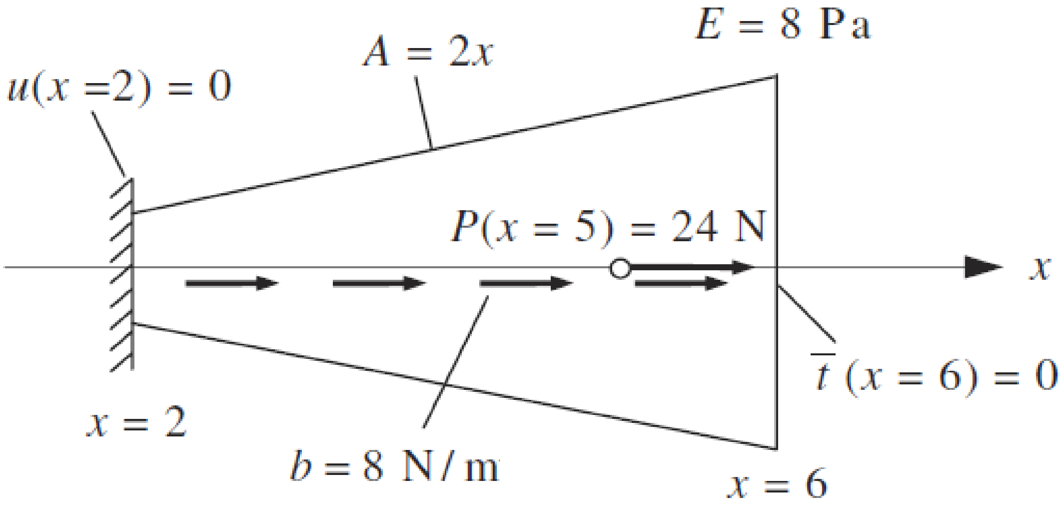
$$f = P\delta(x)$$

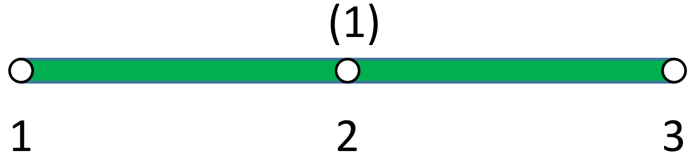
- Properties of Dirac delta function:

$$\int_{x_1}^{x_2} \delta(x-a)dx = 1 \; if \; x_1 < a < x_2$$

$$\int_{x_1}^{x_2} g(x)\delta(x-a)dx = g(a) \; if \; x_1 < a < x_2$$

Calculation of Nodal Values





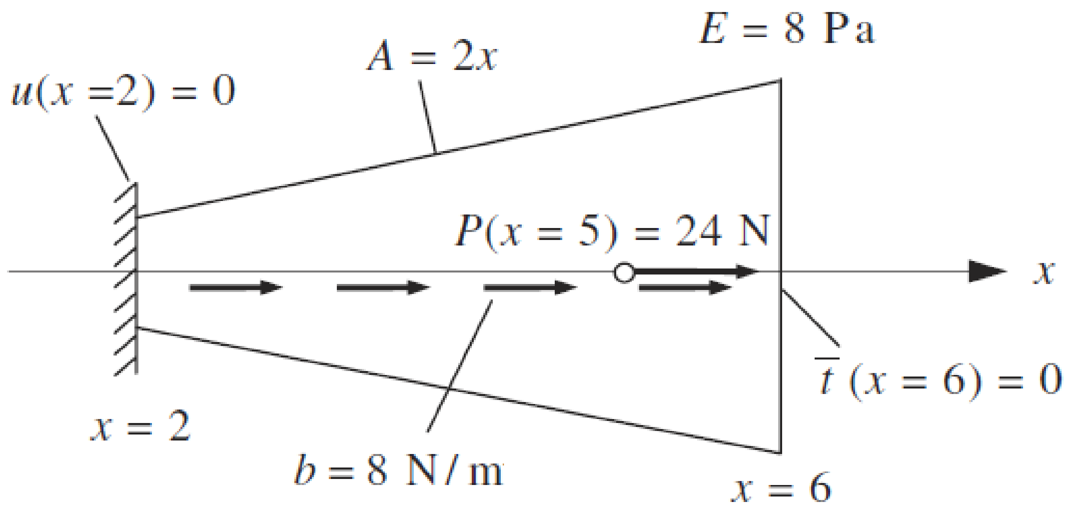
$$\boldsymbol{r} = \boldsymbol{K}\boldsymbol{d} - \boldsymbol{f}$$

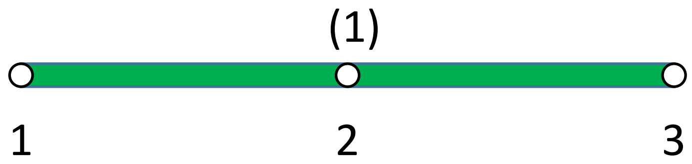
$$\Rightarrow$$

$$\begin{bmatrix} r_1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 26.67 & -32 & 5.33 \\ -32 & 85.33 & -53.33 \\ 5.33 & -53.33 & 48 \end{bmatrix} \begin{bmatrix} 0 \\ u_2 \\ u_3 \end{bmatrix} - \begin{bmatrix} 2.33 \\ 39.33 \\ 14.33 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 2.1193 \\ 2.6534 \end{bmatrix}$$

Postprocessing





$$\begin{bmatrix} u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 2.1193 \\ 2.6534 \end{bmatrix} \Rightarrow \boldsymbol{d} = \begin{bmatrix} \bar{u}_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 2.1193 \\ 2.6534 \end{bmatrix}$$

- Displacement field:

$$u = N_1 u_1 + N_2 u_2 + N_3 u_3$$
$$= -\frac{2.1193}{4}(x-2)(x-6) + \frac{2.6534}{8}(x-2)(x-4)$$

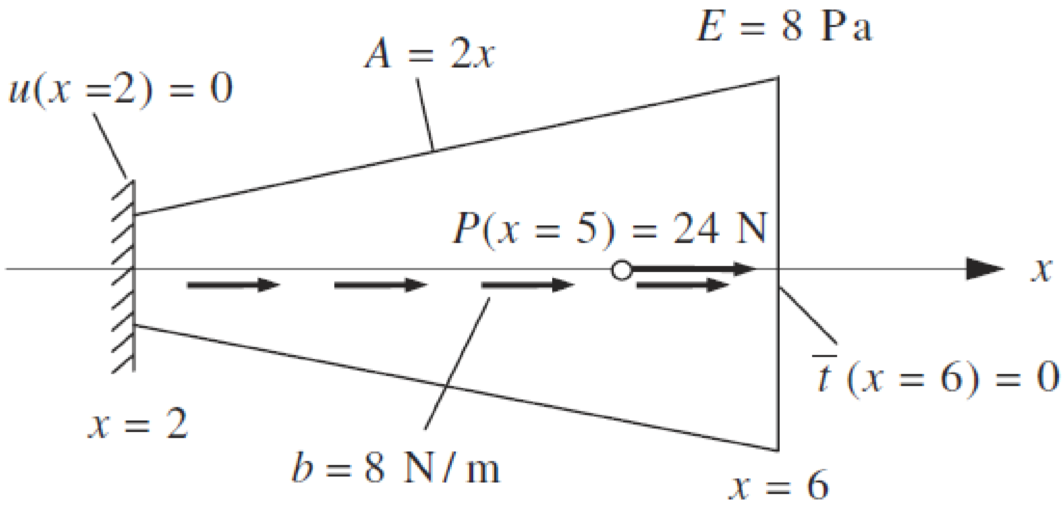
*Unit – *m*

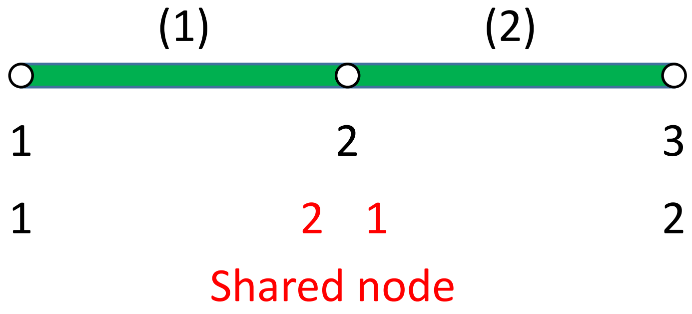
- Stress field:

$$\sigma = E\frac{du}{dx} = \boldsymbol{Bd}$$
$$= \frac{1}{4}\begin{bmatrix} x-5 & 8-2x & x-3 \end{bmatrix}\begin{bmatrix} 0 \\ 2.1193 \\ 2.6534 \end{bmatrix}$$
$$= -3.17x + 17.99$$

*Unit – *Pa*

Shape Functions of 2 *Linear* Elements





- Shape functions:

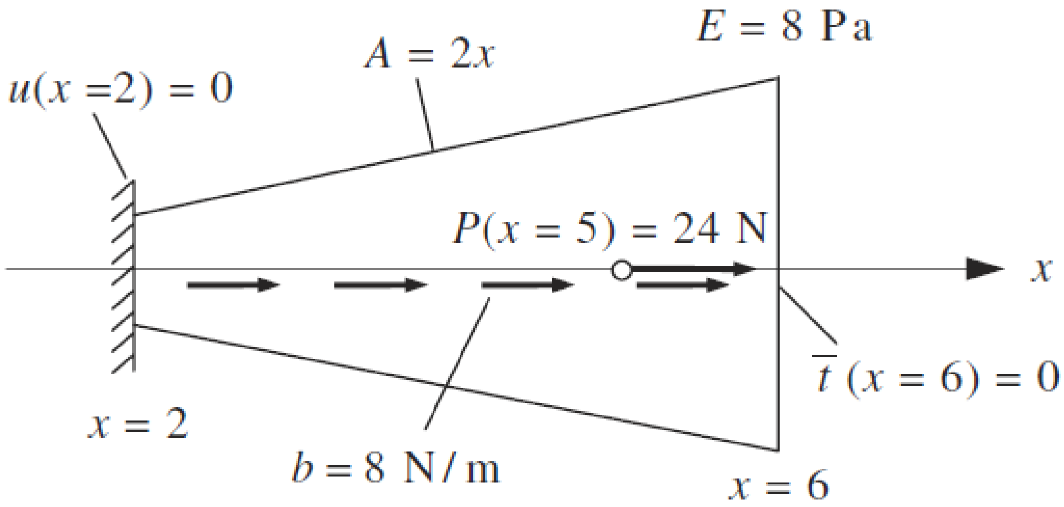
$$N_1^{(1)}(x) = \frac{x-4}{2-4}, \qquad N_2^{(1)}(x) = \frac{x-2}{4-2}$$

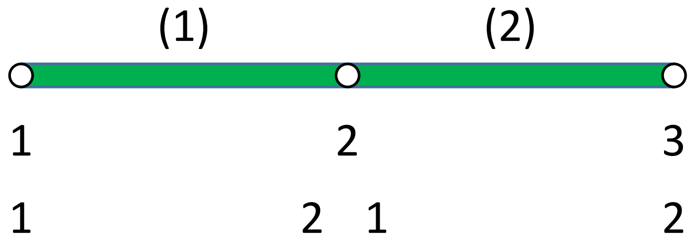
$$N_1^{(2)}(x) = \frac{x-6}{4-6}, \qquad N_2^{(2)}(x) = \frac{x-4}{6-4}$$

$$\boldsymbol{B}^{(1)}(x) = \begin{bmatrix} \frac{dN_1^{(1)}}{dx} & \frac{dN_2^{(1)}}{dx} \end{bmatrix} = \frac{1}{2}[-1 \quad 1]$$

$$\boldsymbol{B}^{(2)}(x) = \begin{bmatrix} \frac{dN_1^{(2)}}{dx} & \frac{dN_2^{(2)}}{dx} \end{bmatrix} = \frac{1}{2}[-1 \quad 1]$$

Stiffness Matrix of 2 Linear Elements





- Element stiffness matrices:

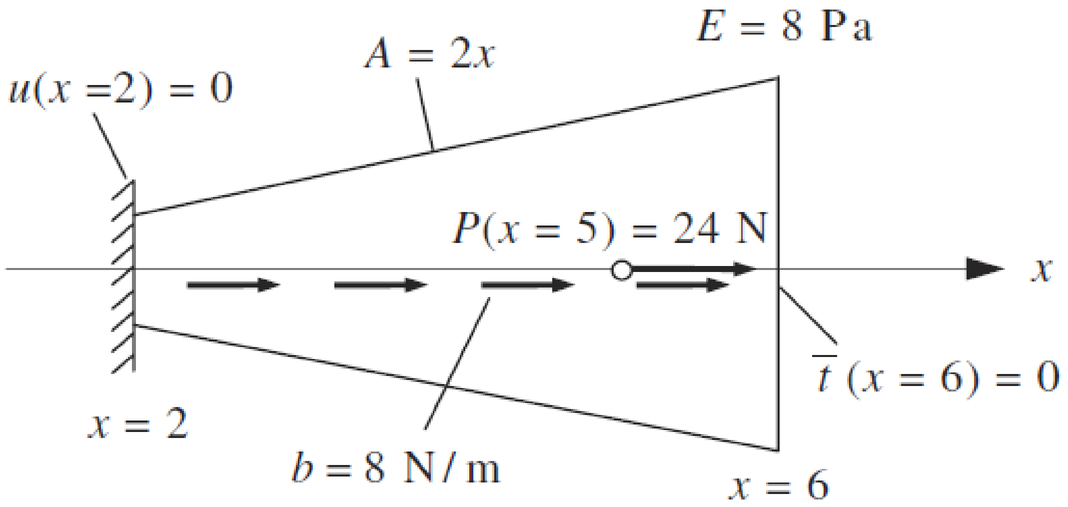
$$\boldsymbol{K}^{(1)} = \int_{x_1}^{x_2} \boldsymbol{B}^{(1)T} A^{(1)} E^{(1)} \boldsymbol{B}^{(1)} dx$$

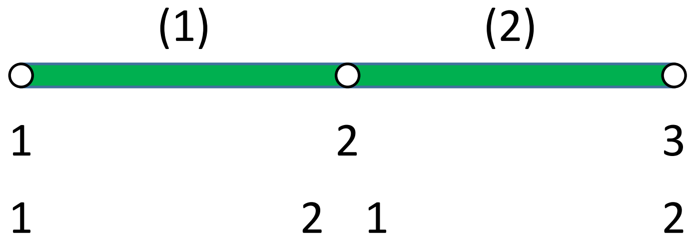
$$\Rightarrow \boldsymbol{K}^{(1)} = \int_{2}^{4} 4 \begin{bmatrix} -1 \\ 1 \end{bmatrix} x \begin{bmatrix} -1 & 1 \end{bmatrix} dx = 24 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\boldsymbol{K}^{(2)} = \int_{x_2}^{x_3} \boldsymbol{B}^{(1)T} A^{(1)} E^{(1)} \boldsymbol{B}^{(1)} dx$$

$$\Rightarrow \boldsymbol{K}^{(2)} = \int_{4}^{6} 4 \begin{bmatrix} -1 \\ 1 \end{bmatrix} x \begin{bmatrix} -1 & 1 \end{bmatrix} dx = 40 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Global Stiffness Matrix of the Bar



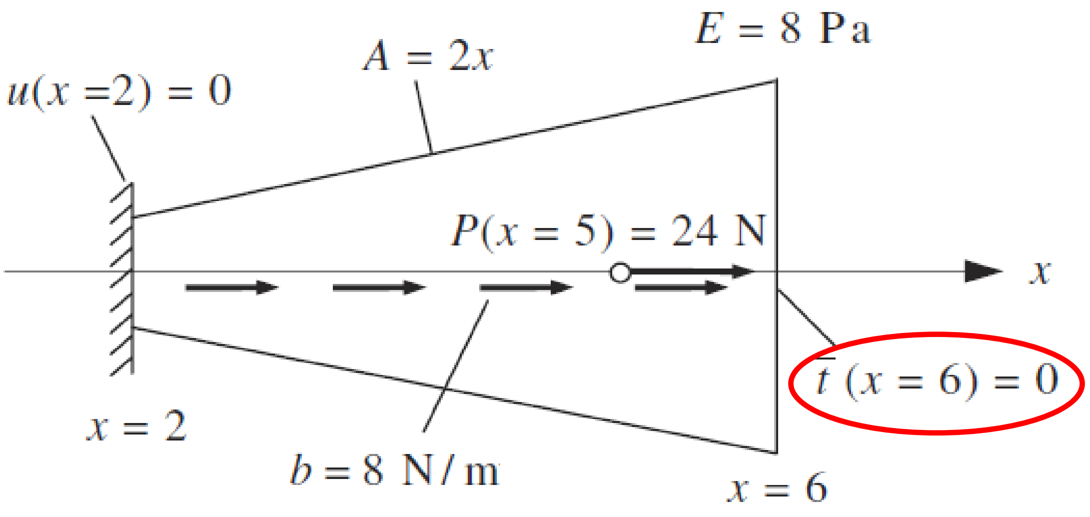


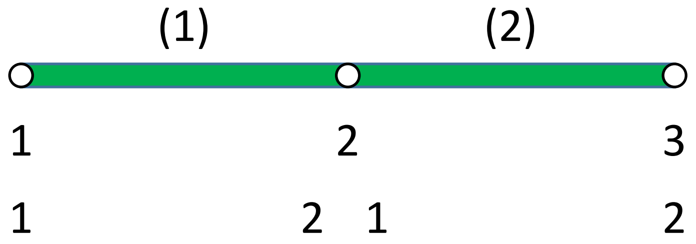
- Direct assembly:

$$K^{(1)} = 24 \begin{array}{cc} \begin{array}{cc} 1 & 2 \end{array} & \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} & \begin{array}{c} 1 \\ 2 \end{array} \end{array}, \quad K^{(2)} = 40 \begin{array}{cc} \begin{array}{cc} 2 & 3 \end{array} & \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} & \begin{array}{c} 2 \\ 3 \end{array} \end{array}$$

$$\Rightarrow K = \begin{bmatrix} 24 & -24 & 0 \\ -24 & 64 & -40 \\ 0 & -40 & 40 \end{bmatrix}$$

Force Matrices Calculation for Linear Elements





- Element force matrices (0 traction):

$$\boldsymbol{f}_{\Omega^{(1)}} = \int_{x_1}^{x_2} \boldsymbol{N}^{(1)T} b dx$$

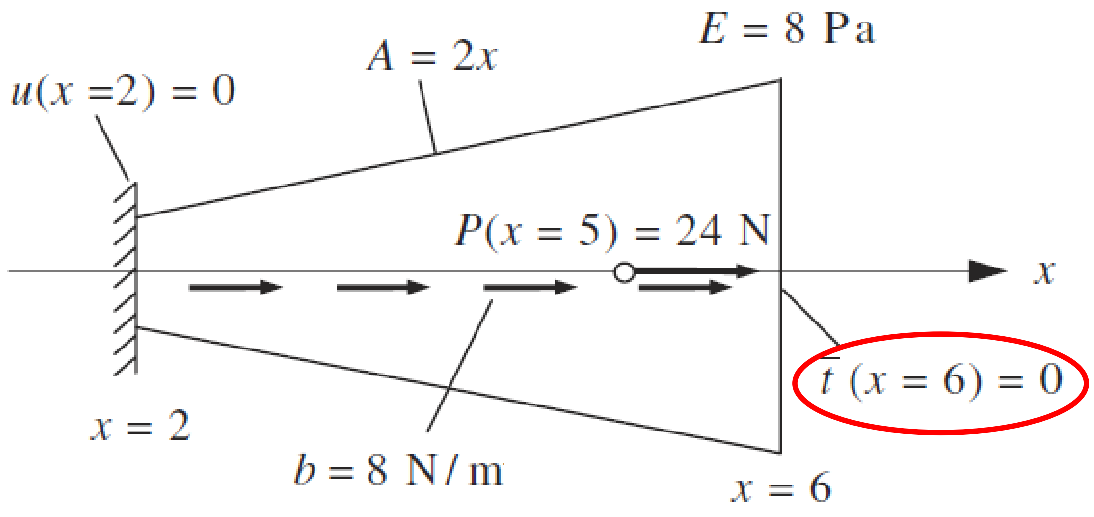
$$\Rightarrow \boldsymbol{f}_{\Omega^{(1)}} = \int_{2}^{4} \frac{1}{2} \begin{bmatrix} 4 - x \\ x - 2 \end{bmatrix} 8dx = 8 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

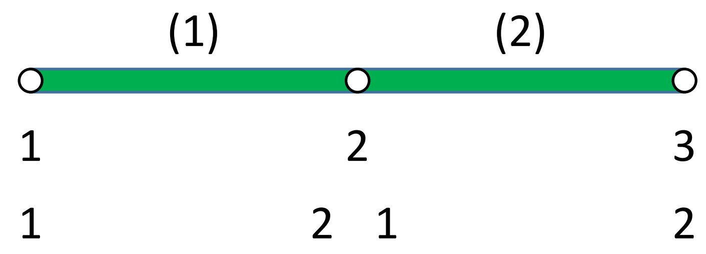
$$\boldsymbol{f}_{\Omega^{(2)}} = \int_{x_2}^{x_3} \boldsymbol{N}^{(2)T} b dx$$

$$\Rightarrow \boldsymbol{f}_{\Omega^{(2)}} = \int_{4}^{6} \frac{1}{2} \begin{bmatrix} 6 - x \\ x - 4 \end{bmatrix} \big(8 + 24\delta(x - 5)\big) dx$$

$$= 20 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Nodal Values Calculation





- Global force matrix (0 traction):

$$\boldsymbol{f}_{\Omega^{(1)}} = 8\begin{bmatrix}1\\1\end{bmatrix}, \qquad \boldsymbol{f}_{\Omega^{(2)}} = 20\begin{bmatrix}1\\1\end{bmatrix}$$

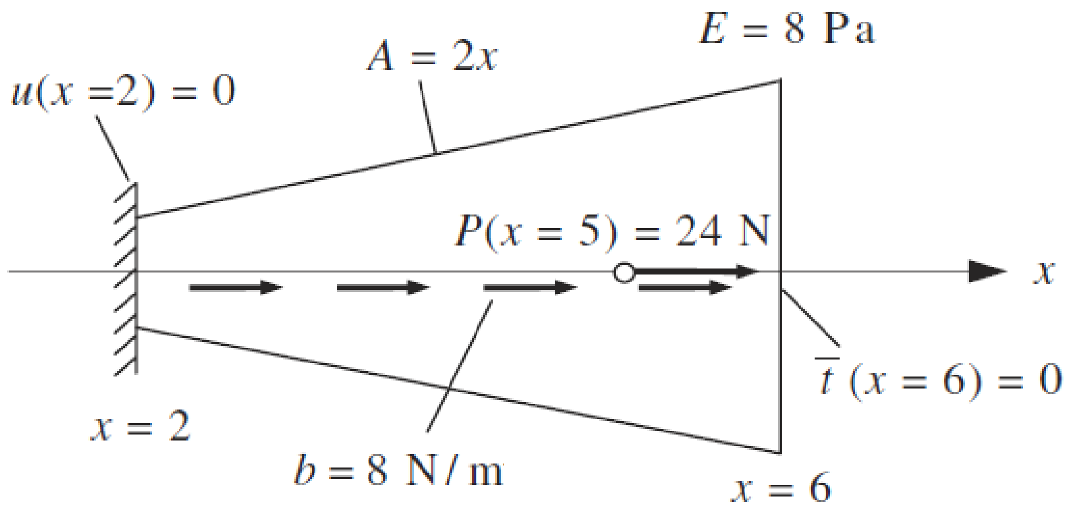
$$\Rightarrow \boldsymbol{f}_{\Omega} = \begin{bmatrix}8\\28\\20\end{bmatrix}$$

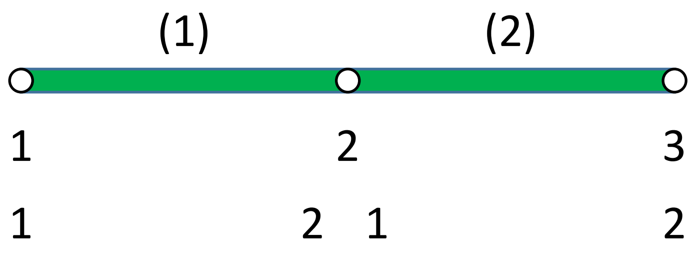
$$\boldsymbol{Kd} = \boldsymbol{f} + \boldsymbol{r}$$

$$\Rightarrow \begin{bmatrix}24 & -24 & 0\\ -24 & 64 & -40\\ 0 & -40 & 40\end{bmatrix}\begin{bmatrix}0\\u_2\\u_3\end{bmatrix} = \begin{bmatrix}8+r_1\\28\\20\end{bmatrix}$$

$$\Rightarrow u_2 = 2, u_3 = 2.5, r_1 = -56N$$

Postprocessing





- Displacement field:

$$u(x) = \boldsymbol{N}^{(1)}\boldsymbol{d}^{(1)} + \boldsymbol{N}^{(2)}\boldsymbol{d}^{(2)}$$

$$\Rightarrow u = \frac{1}{2}\begin{bmatrix}4-x\\ x-2\end{bmatrix}\begin{bmatrix}0 & 2\end{bmatrix}\Big|_{2<x<4} + \frac{1}{2}\begin{bmatrix}6-x\\ x-4\end{bmatrix}\begin{bmatrix}2 & 2.5\end{bmatrix}\Big|_{4<x<6}$$

$$\Rightarrow u = \begin{cases} x-2, & 2 \le x \le 4^- \\ 0.25x+1, & 4^+ \le x \le 6 \end{cases}$$

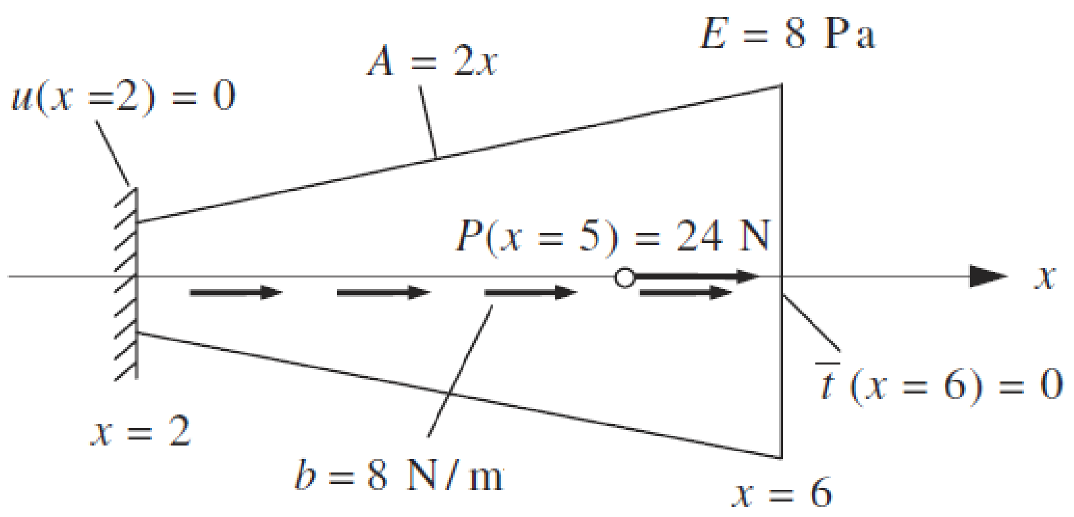
*Unit – *m*

- Stress field:

$$\sigma(x) = E\frac{du}{dx} = \begin{cases} 8, & 2 \le x \le 4^- \\ 2, & 4^+ \le x \le 6 \end{cases}$$ *Unit – *Pa*

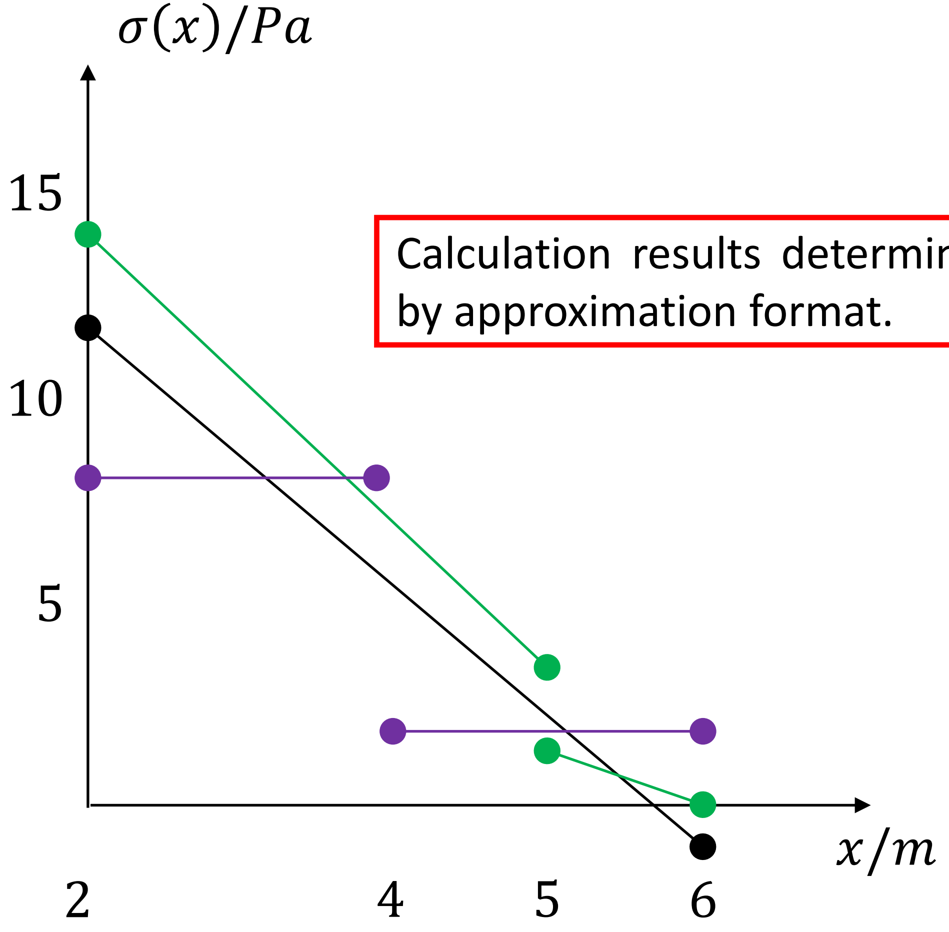
Note: Different elements have different approximation function.

Stress Comparison



- Stress comparison of:
  - Quadratic element model (black line)
  - Linear element model (purple lines)
  - Exact solution (green curves):

$$\sigma(x) = \frac{p(x)}{2x} = \begin{cases} \dfrac{8(6-x)+24}{2x} = \dfrac{36-4x}{x}, x \le 5 \\ \dfrac{8(6-x)}{2x} = \dfrac{24-4x}{x}, x > 5 \end{cases}$$



Measurement for Approximation Error in 1D

- L_2 error – utilization of function norm:

$$\bar{e}_{L_2} = \frac{\overbrace{\left|\left|u^{ex}(x) - u^h(x)\right|\right|_{L_2}}^{||e||_{L_2}}}{\underline{\left|\left|u^{ex}(x)\right|\right|_{L_2}}} = \frac{\left(\int_{x_1}^{x_2}\left(u^{ex}(x) - u^h(x)\right)^2 dx\right)^{\frac{1}{2}}}{\underline{\left(\int_{x_1}^{x_2}\left(u^{ex}(x)\right)^2 dx\right)^{\frac{1}{2}}}}$$

Similar to vector length

- Normalized error in energy – derivative of function:

Derivative of displacement

$$\bar{e}_{en} = \frac{\overbrace{\left|\left|u^{ex}(x) - u^h(x)\right|\right|_{en}}^{||e||_{en}}}{\underline{\left|\left|u^{ex}(x)\right|\right|_{en}}} = \frac{\left(\frac{1}{2}\int_{x_1}^{x_2} EA\left(\varepsilon^{ex}(x) - \varepsilon^h(x)\right)^2 dx\right)^{\frac{1}{2}}}{\underline{\left(\frac{1}{2}\int_{x_1}^{x_2} EA\left(\varepsilon^{ex}(x)\right)^2 dx\right)^{\frac{1}{2}}}}$$

Similar to elastic potential energy

High order Gauss quadrature formulas are usually essential to integrate exact solutions.

Error from 1D Approximation

- Weak form for the exact solution:

  Find $u \in U$ so that

$$\int_{\Omega} \frac{dw}{dx} AE \frac{du}{dx} dx = \int_{\Omega} wbdx - (w\bar{t}A)\Big|_{\Gamma_t}, \forall w \in U_0$$

- Weak form for the approximation:

  Find $u^h \in U^h$ so that

$$\int_{\Omega} \frac{dw^h}{dx} AE \frac{du^h}{dx} dx = \int_{\Omega} w^h bdx - \left(w^h\bar{t}A\right)\Big|_{\Gamma_t}, \forall w^h \in U_0^h$$

- Approximation does not cover all possibilities:

$$U^h \subset U, \qquad U_0^h \subset U_0$$

1D Convergence Analysis

- For an arbitrary function $u^* \in U^h$:

$$||u-u^*||_{en}^2 = \Big|\Big|\underbrace{(u-u^h)}_{e} + \underbrace{(u^h-u^*)}_{w^h \in U_0^h}\Big|\Big|_{en}^2$$

$$\left|\left|e+w^h\right|\right|_{en}^2 = \frac{1}{2}\int_{x_1}^{x_2} EA\left(\frac{de}{dx}+\frac{dw^h}{dx}\right)^2 dx = \frac{1}{2}\int_{x_1}^{x_2} EA\left[\left(\frac{de}{dx}\right)^2 + 2\frac{de}{dx}\frac{dw^h}{dx} + \left(\frac{dw^h}{dx}\right)^2\right]dx$$

$$\Rightarrow ||u-u^*||_{en}^2 = \left|\left|e+w^h\right|\right|_{en}^2 = ||e||_{en}^2 + \left|\left|w^h\right|\right|_{en}^2 + \int_{x_1}^{x_2}\frac{dw^h}{dx}EA\frac{de}{dx}dx$$

- Subtracting the two weak forms with $w^h = w$:

Exact and approximated

$$\int_\Omega \frac{dw^h}{dx}AE\frac{de}{dx}dx = 0$$

$$\Rightarrow ||u-u^*||_{en}^2 \geq \underbrace{\left|\left|u-u^h\right|\right|_{en}^2 = ||e||_{en}^2}_{\text{Minimum energy norm}}$$

Quantitative Estimation for Linear $||e||_{en}$ in 1D (1/3)

$$||e||_{en} = \left|\left|u^{ex}(x) - u^h(x)\right|\right|_{en} = \left(\frac{1}{2}\int_{x_1}^{x_2} EA\left(\varepsilon^{ex}(x) - \varepsilon^h(x)\right)^2 dx\right)^{\frac{1}{2}}$$

- Define an auxiliary function for trial solutions:

$$\tilde{u} \in U^h \Rightarrow \tilde{e}^i = u - \tilde{u} \text{ for } (i-1)h \le x \le ih$$

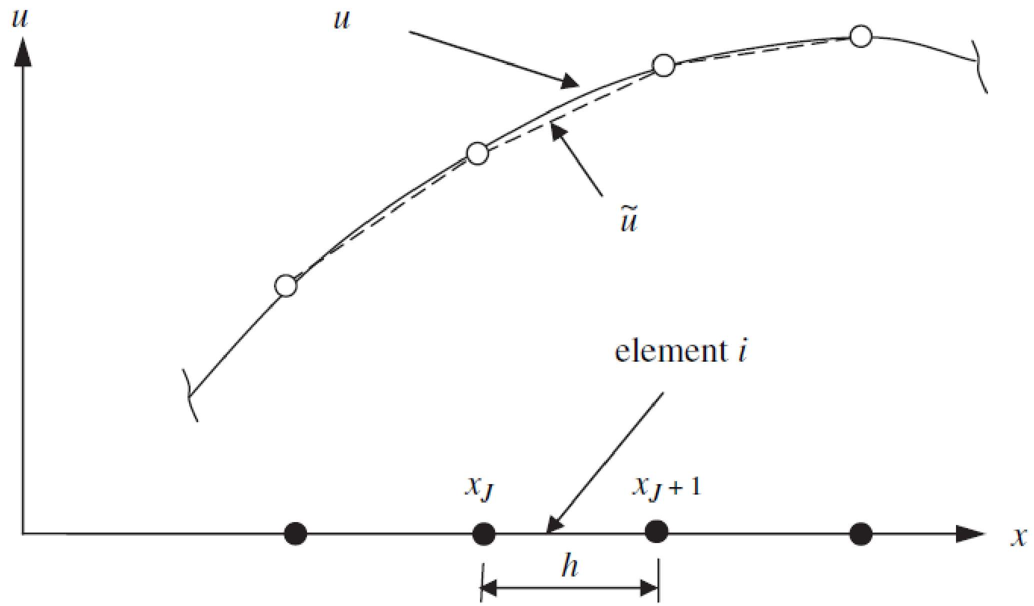
where i is the element number and $h = l/n$ is the length of n equal-size elements.

- Assume $\tilde{u}$ is a linear interpolation for nodal values:

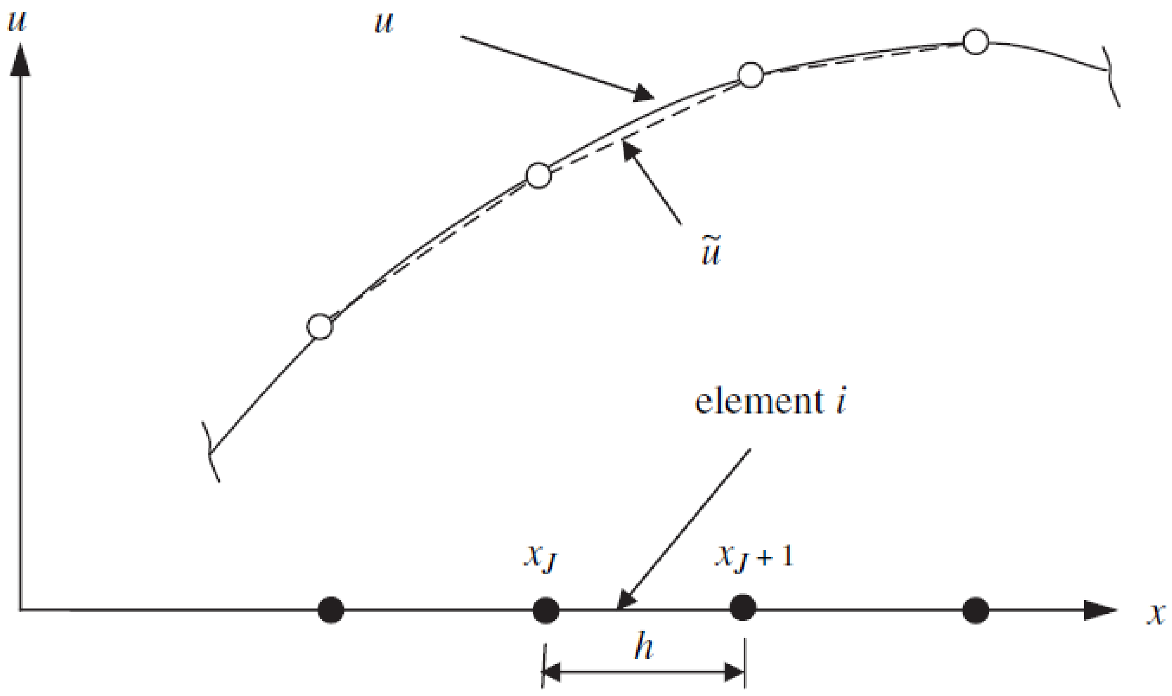
$$u(x_J) = \tilde{u}(x_J)$$

$$\frac{d\tilde{u}}{dx} = \frac{\tilde{u}(x_{J+1}) - \tilde{u}(x_J)}{x_{J+1} - x_J}$$

where $x_J = (i-1)h$



Quantitative Estimation for Linear $||e||_{en}$ in 1D (2/3)



$$\tilde{e}^i = u - \tilde{u} \text{ for } (i-1)h \le x \le ih$$

$$u(x_J) = \tilde{u}(x_J), \qquad \frac{d\tilde{u}}{dx} = \frac{\tilde{u}(x_{J+1}) - \tilde{u}(x_J)}{x_{J+1} - x_J}$$

- Since u is smooth and differentiable:

$$\frac{d\tilde{u}}{dx} = \frac{du(c)}{dx}, \qquad \exists c \in [x_J, x_{J+1}]$$

- Utilize Taylor series around c with remainder:

$$\frac{du(x)}{dx} = \frac{du(c)}{dx} + (x-c)\frac{d^2u(\delta)}{dx^2}$$

$$\Rightarrow \left|\frac{d\tilde{e}^i}{dx}\right| = \left|\frac{du(x)}{dx} - \frac{d\tilde{u}}{dx}\right| = \underbrace{\left|\frac{d^2u(\delta)}{dx^2}\right|}_{\le \alpha} \underbrace{|x-c|}_{\le h}$$

Quantitative Estimation for Linear $||e||_{en}$ in 1D (3/3)

$$\left|\frac{d\tilde{e}^i}{dx}\right| = \left|\frac{d^2u(\delta)}{dx^2}\right| |x-c| \le \alpha h$$

$$\Rightarrow ||\tilde{e}||_{en}^2 = \frac{1}{2}\int_\Omega EA\left(\frac{d\tilde{e}}{dx}\right)^2 dx = \frac{1}{2}\sum_i^n \int_{(i-1)h}^{ih} \underbrace{EA}_{E(x)A(x)\le K}\left(\frac{d\tilde{e}^i}{dx}\right)^2 dx \le \frac{1}{2}\underbrace{nh}_{l}K(\alpha h)^2$$

- Finite element approximation u^h lead to the least energy error in U^h:

$$||e||_{en}^2 \le ||u-u^*||_{en}^2$$

$$\Rightarrow ||e||_{en} \le ||\tilde{e}||_{en} \le \sqrt{\frac{1}{2}lK(\alpha h)^2} = Ch$$

Derivation for higher order elements is similar.

The End