Computational Mechanics

Chapter 5 Continuum Mechanics

Introduction to Continuum Mechanics

- Building block for multidimensional and nonlinear finite element analysis
- Mathematical tools to capture multidimensional values – vectors and tensors
- Components of body movement – deformation and rigid body rotation
- Universal measurement in mechanics analysis – stress and strain
- Coordinate system rotation – polar decomposition and frame-invariant rates of stress

Introduction to 2nd Order Tensors

- 2nd order tensors are commonly used in mechanics
- 2nd order tensors can be express similarly to 2D matrices
- All tensors can be regarded as vector – vector function:

$$\boldsymbol{v} = \boldsymbol{F}(\boldsymbol{u})$$

- Fundamental properties of 2nd order
  - Homogenous:

$$\boldsymbol{F}(\alpha\boldsymbol{u}) = \alpha\boldsymbol{F}(\boldsymbol{u})$$

  - Linear:

$$\boldsymbol{F}(\boldsymbol{u}_1 + \boldsymbol{u}_2) = \boldsymbol{F}(\boldsymbol{u}_1) + \boldsymbol{F}(\boldsymbol{u}_2)$$

$$\Rightarrow \boldsymbol{v} = \boldsymbol{F}(\boldsymbol{u}) = \boldsymbol{F}\cdot\boldsymbol{u}$$

- Transpose of a tensor:

$$\boldsymbol{F}\cdot\boldsymbol{u} = \boldsymbol{u}\cdot\boldsymbol{F}^T$$

  - Same as matrices
  - Tensors are physical entities

- Inverse of a tensor:

$$\boldsymbol{v} = \boldsymbol{F}\cdot\boldsymbol{u} \leftrightarrow \boldsymbol{u} = \boldsymbol{F}^{-1}\cdot\boldsymbol{v}$$

- Orthogonal tensor – pure rotation:

$$|\boldsymbol{v}| = |\boldsymbol{A}\cdot\boldsymbol{u}| = |\boldsymbol{u}|$$

$$\Rightarrow \boldsymbol{u}\cdot\boldsymbol{u} = \boldsymbol{v}\cdot\boldsymbol{v} = \boldsymbol{u}\cdot\boldsymbol{A}^T\cdot\boldsymbol{A}\cdot\boldsymbol{u}$$

$$\Rightarrow \boldsymbol{A}^T\cdot\boldsymbol{A} = \boldsymbol{I}, \qquad \boldsymbol{A}^T = \boldsymbol{A}^{-1}$$

- Principal values and directions – linear scale:

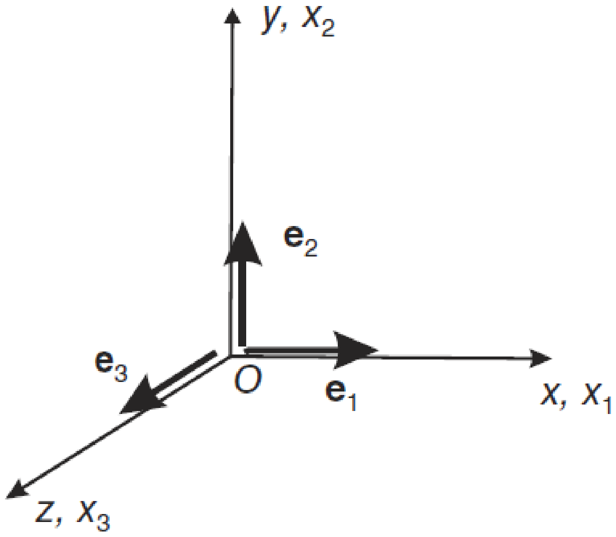
$$\boldsymbol{F}\cdot\boldsymbol{u} = \alpha\boldsymbol{u}$$

- Expansion to nth order tensors:

$$\boldsymbol{F}^{(n)}\cdot\boldsymbol{u} = \boldsymbol{G}^{(n-1)}$$

Cartesian Coordinates

- Introduction to rectangular Cartesian system and base vectors to describe vectors and tensors:



$$\boldsymbol{e}_i \cdot \boldsymbol{e}_j = \delta_{ij}$$

- Vector components:

$$\boldsymbol{u} = u_1\boldsymbol{e}_1 + u_2\boldsymbol{e}_2 + u_3\boldsymbol{e}_3 = \underline{u_i\boldsymbol{e}_i}$$

Summation convention – two and only two same indices (i)

- Tensor Components:

$$\boldsymbol{v} = \boldsymbol{F} \cdot \boldsymbol{u}$$

$$\Rightarrow v_k = \boldsymbol{e}_k \cdot \boldsymbol{v} = \boldsymbol{e}_k \cdot \boldsymbol{F} \cdot \boldsymbol{u} = \boldsymbol{e}_k \cdot \boldsymbol{F} \cdot u_l\boldsymbol{e}_l$$

$$\Rightarrow v_k = \underbrace{(\boldsymbol{e}_k \cdot \boldsymbol{F} \cdot \boldsymbol{e}_l)}_{F_{kl}}u_l$$

$$\Rightarrow \boldsymbol{F} = F_{kl}\boldsymbol{e}_k\boldsymbol{e}_l$$

Vector Product in Cartesian System

- Calculation of vector product in component format (not the right-hand rule):

$$\boldsymbol{w} = \boldsymbol{u} \times \boldsymbol{v} = (u_i \boldsymbol{e}_i) \times (v_j \boldsymbol{e}_j) = u_i v_j \boldsymbol{e}_i \times \boldsymbol{e}_j = u_i v_j \epsilon_{ijk} \boldsymbol{e}_k$$

$$\epsilon_{ijk} = \begin{cases} 0, & if\ 2\ indices\ are\ the\ same \\ 1, & ijk = 123, 312, or\ 231 \\ -1, & ijk = 213, 321, or\ 132 \end{cases}$$

- Express vector product in matrix format for convenience:

$$\boldsymbol{w} = \begin{vmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

Determinant

Products of Tensors

- Tensor product:

$$\boldsymbol{F}\cdot\boldsymbol{G}=(F_{kl}\boldsymbol{e}_k\boldsymbol{e}_l)\cdot\left(G_{ij}\boldsymbol{e}_i\boldsymbol{e}_j\right)=\boldsymbol{e}_kF_{kl}\boldsymbol{e}_l\cdot\boldsymbol{e}_iG_{ij}\boldsymbol{e}_j=\boldsymbol{e}_kF_{kl}G_{lj}\boldsymbol{e}_j$$

$$\boldsymbol{H}=\boldsymbol{F}\cdot\boldsymbol{G}\Rightarrow H_{ij}=F_{il}G_{lj}$$

- Scalar product – type 1:

$$\boldsymbol{F}\cdot\cdot\,\boldsymbol{G}=(F_{kl}\boldsymbol{e}_k\boldsymbol{e}_l)\cdot\cdot\left(G_{ij}\boldsymbol{e}_i\boldsymbol{e}_j\right)=F_{kl}G_{ij}\left(\boldsymbol{e}_k\cdot\boldsymbol{e}_j\right)(\boldsymbol{e}_l\cdot\boldsymbol{e}_i)=F_{kl}G_{lk}$$

- Scalar product – type 2:

$$\boldsymbol{F}:\boldsymbol{G}=(F_{kl}\boldsymbol{e}_k\boldsymbol{e}_l):\left(G_{ij}\boldsymbol{e}_i\boldsymbol{e}_j\right)=F_{kl}G_{ij}(\boldsymbol{e}_k\cdot\boldsymbol{e}_i)\left(\boldsymbol{e}_l\cdot\boldsymbol{e}_j\right)=F_{kl}G_{kl}$$

Determinant of 2nd Order Tensors

- Determinant calculation in component format:

$$\det(\boldsymbol{M}) = \begin{vmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{vmatrix} = \epsilon_{ijk} M_{1i} M_{2j} M_{3k}$$

- Transpose property: Write all terms out to prove

$$\det(\boldsymbol{M}) = \epsilon_{ijk} M_{1i} M_{2j} M_{3k} = \epsilon_{ijk} M_{i1} M_{j2} M_{k3} = \det(\boldsymbol{M}^T)$$

- Cofactor in determinant:

$$\det(\boldsymbol{M}) = \epsilon_{ijk} M_{1i} M_{2j} M_{3k} = M_{1i} c_{1i}$$

$$\Rightarrow \det(\boldsymbol{M}) = M_{pi} c_{pi} = M_{ip} c_{ip} \ (no\ sum\ on\ p)$$

$$c_{1i} = \frac{1}{2} \epsilon_{123} \epsilon_{ijk} M_{2j} M_{3k} + \frac{1}{2} \epsilon_{123} \epsilon_{ijk} M_{2j} M_{3k}$$

$$\Rightarrow c_{1i} = \frac{1}{2} \epsilon_{123} \epsilon_{ijk} M_{2j} M_{3k} + \frac{1}{2} \epsilon_{132} \epsilon_{ijk} M_{3j} M_{2k}$$

$$\Rightarrow c_{li} = \frac{1}{2} \epsilon_{lmn} \epsilon_{ijk} M_{mj} M_{nk}$$

Adjugate of $\boldsymbol{M}$

$$M_{li}^* = (c_{li})^T = \frac{1}{2} \epsilon_{imn} \epsilon_{ljk} M_{mj} M_{nk}$$

$$M_{pl} M_{li}^* = \frac{1}{2} \epsilon_{imn} \epsilon_{ljk} M_{pl} M_{mj} M_{nk} = \frac{1}{2} \epsilon_{imn} \epsilon_{pmn} \det(\boldsymbol{M})$$

Inverse of $\boldsymbol{M}$

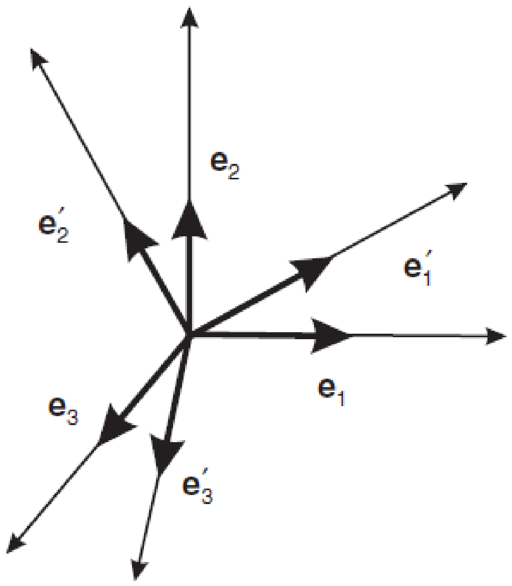
$$\Rightarrow M_{pl} M_{li}^* = \delta_{pi} \det(\boldsymbol{M}) \Rightarrow \boldsymbol{M}^{-1} = \frac{\boldsymbol{M}^*}{\det(\boldsymbol{M})}$$

Identity tensor $\boldsymbol{I}$

Transformation Matrix for Coordinate *Rotation*

Orthogonal transformation, no axis scaling!!!

- Two coordinate systems with relative rotation:



$$\Rightarrow \boldsymbol{A} = \boldsymbol{e}'_k \boldsymbol{e}_k$$

Dyadic product

- Transpose property of $\boldsymbol{A}$:

$$\boldsymbol{e}_m = \boldsymbol{A}^{-1} \cdot \boldsymbol{e}'_m = \boldsymbol{A}^T \cdot \boldsymbol{e}'_m$$

$$\Rightarrow \boldsymbol{e}'_n \cdot \boldsymbol{e}_m = \boldsymbol{e}'_n \cdot \boldsymbol{A}^T \cdot \boldsymbol{e}'_m = A^T_{nm} = \cos(m, n') = A_{mn}$$

$$\boldsymbol{I} = \boldsymbol{A} \cdot \boldsymbol{A}^T \Rightarrow \delta_{ij} = A_{ik} A^T_{kj} = A_{ik} A_{jk}$$

$$\boldsymbol{I} = \boldsymbol{A}^T \cdot \boldsymbol{A} \Rightarrow \delta_{ij} = A_{ki} A^T_{jk} = A_{ki} A_{kj} = A_{ik} A_{jk}$$

$$\Rightarrow \sum \cos(i, k') \cos(j, k') = \sum \cos(k, i') \cos(k, j') = \delta_{ij}$$

- Orthogonal tensor for rotation:

$$\boldsymbol{e}'_j = \boldsymbol{A} \cdot \boldsymbol{e}_j$$

$$\Rightarrow \boldsymbol{e}_i \cdot \boldsymbol{e}'_j = \cos(i, j') = \boldsymbol{e}_i \cdot \boldsymbol{A} \cdot \boldsymbol{e}_j = A_{ij}$$

Vector Components after Coordinate Rotation

- Consider one vector $\boldsymbol{v}$:
  - In system e1-e2-e3:

$$\boldsymbol{v} = v_i \boldsymbol{e}_i$$

  - In system e'1-e'2-e'3:

$$\boldsymbol{v} = v'_j \boldsymbol{e}'_j$$

$$\boldsymbol{v} = v_i \boldsymbol{e}_i = v'_j \boldsymbol{e}'_j$$

$$\Rightarrow v'_k = \boldsymbol{e}'_k \cdot \boldsymbol{v} = \boldsymbol{e}'_k \cdot (v_i \boldsymbol{e}_i) = v_i \boldsymbol{e}'_k \cdot \boldsymbol{e}_i = v_i A_{ik}$$

$$\Rightarrow [v'_1 \quad v'_2 \quad v'_3] = [v_1 \quad v_2 \quad v_3] \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

- Similarly:

$$v_k = \boldsymbol{e}_k \cdot \boldsymbol{v} = \boldsymbol{e}_k \cdot \left(v'_j \boldsymbol{e}'_j\right) = v'_j \boldsymbol{e}_k \cdot \boldsymbol{e}'_j = A_{kj} v'_j$$

$$\Rightarrow [v_1 \quad v_2 \quad v_3] = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} v'_1 \\ v'_2 \\ v'_3 \end{bmatrix}$$

Note: avoid format $\boldsymbol{u} = \boldsymbol{A} \cdot \boldsymbol{v}$ as we are working on the same vector $\boldsymbol{v}$!!!

Tensor Components after Coordinate Rotation

- Express two vectors and their linking tensor in e1-e2-e3 system:

$$v_k = F_{kl} u_l$$

The same vector $\boldsymbol{v}$ and $\boldsymbol{u}$ in e'1-e'2-e'3

$$\Rightarrow v'_n = A_{kn} v_k = A_{kn} F_{kl} u_l = A_{kn} F_{kl} A_{lj} u'_j = F'_{nj} u'_j$$

$$\Rightarrow F'_{nj} = A_{kn} F_{kl} A_{lj} = A^T_{nk} F_{kl} A_{lj}$$

$$\Rightarrow [F'] = [A^T][F][A]$$

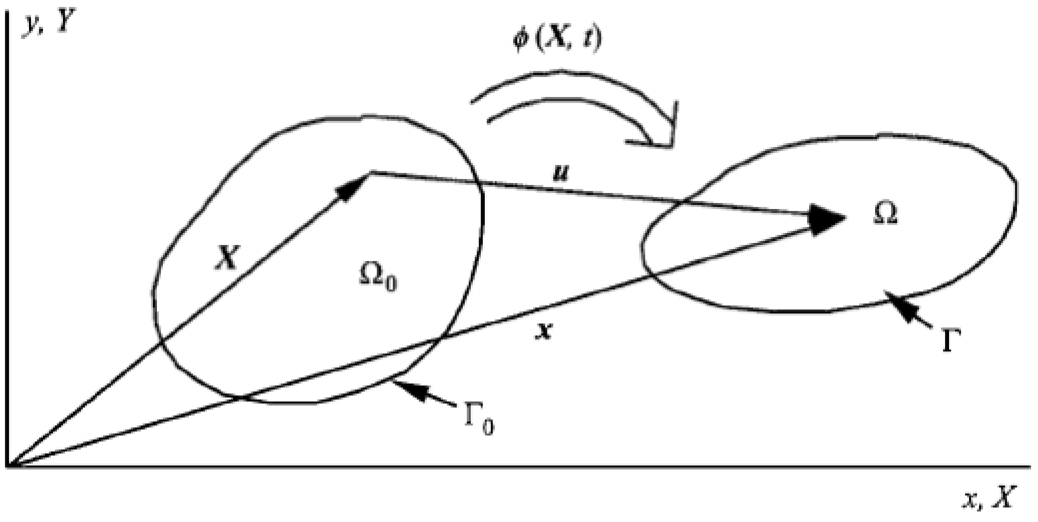
- Similarly:

$$[F] = [A][F'][A^T]$$

Note: $[F]$ and $[F']$are matrix formats of the same tensor $\boldsymbol{F}$ in different rotated coordinates!!!

Definition of Domains

- Continuum mechanics focuses on macroscopic behaviors and ignore discontinuity within materials
- Different multi-dimensional body domains induced by deformation:



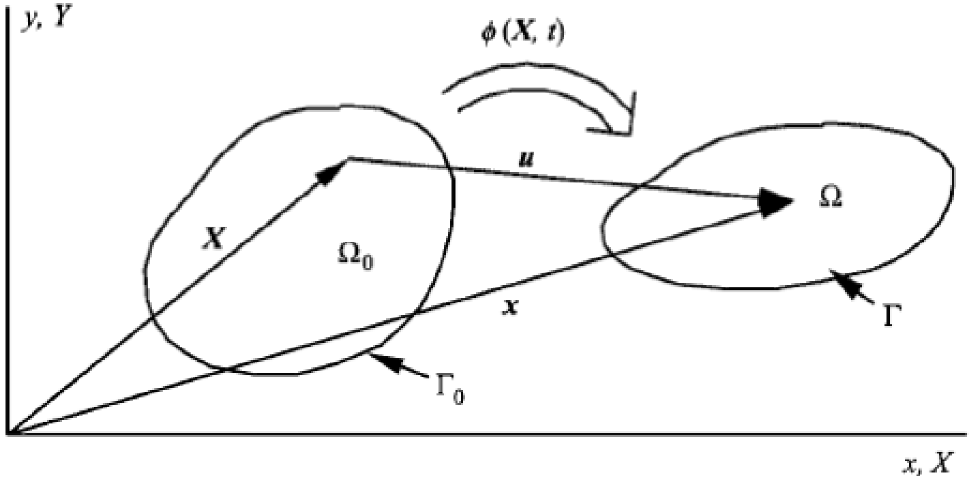
- Ω_0: initial/undeformed* configuration, usually also reference configuration to define motion

  Undeformed*: idealization at $t = 0$

- $\boldsymbol{\Phi}(\boldsymbol{X}, t)$: deformation descriptor

- Ω: current/deformed configuration

Lagrangian and Eulerian Description



- Lagrangian/material coordinates – position of one material point using initial configuration with no time effect:

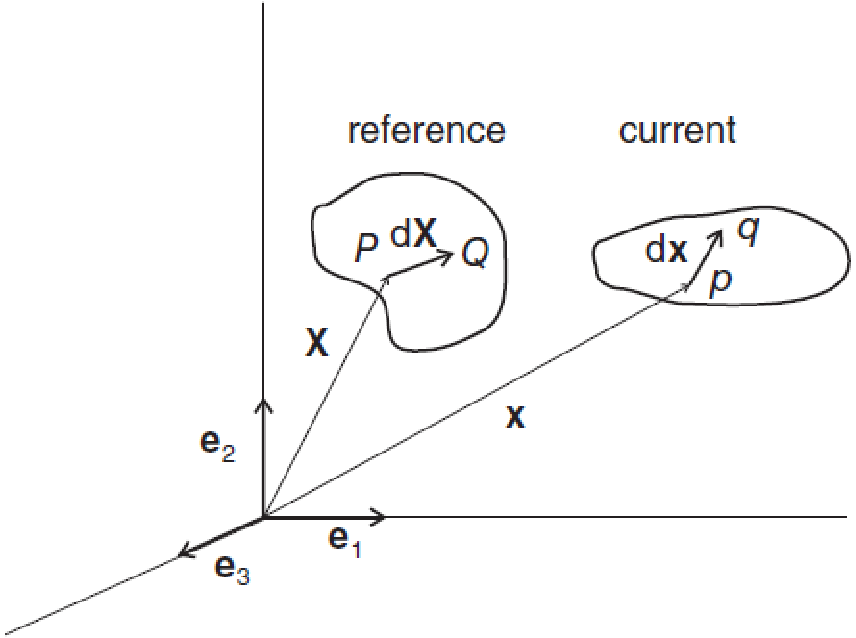
$$\boldsymbol{X}(point\ p) = X_i \mathbf{e}_i$$

- Eulerian/spatial coordinates – trace spatial motion of one material point using reference condition at time t:

$$x(point\ p, t) = \boldsymbol{\Phi}(\boldsymbol{X}(point\ p), t)$$

- Lagrangian description is preferred in solid mechanics to take deformation history into consideration

Deformation Gradient



- Mapping original vector $d\boldsymbol{X}$ to deformed vector $d\boldsymbol{x}$

$$dx_k = \frac{\partial x_k}{\partial X_m} dX_m$$

- Definition of deformation gradient tensor:

$$F_{km} = \frac{\partial x_k}{\partial X_m}(\boldsymbol{X})$$

- Coordinate free notation:

$$d\boldsymbol{x} = \boldsymbol{F} \cdot d\boldsymbol{X} = d\boldsymbol{X} \cdot \boldsymbol{F}^T$$

Change in Length of Lines

- Obtain length by scalar product of vectors:

$$(ds)^2 = d\boldsymbol{x} \cdot d\boldsymbol{x} = (d\boldsymbol{X} \cdot \boldsymbol{F}^T) \cdot (\boldsymbol{F} \cdot d\boldsymbol{X})$$

$$\Rightarrow (ds)^2 = d\boldsymbol{X} \cdot (\boldsymbol{F}^T \cdot \boldsymbol{F}) \cdot d\boldsymbol{X}$$

$$d\boldsymbol{X} = \boldsymbol{N} dS$$

$$\Rightarrow \left(\frac{ds}{dS}\right)^2 = \boldsymbol{N} \cdot (\boldsymbol{F}^T \cdot \boldsymbol{F}) \cdot \boldsymbol{N} = \boldsymbol{N} \cdot \underline{\boldsymbol{C}} \cdot \boldsymbol{N}$$

Green deformation tensor

- Calculate stretch ratio:

$$\Lambda = \frac{ds}{dS} = \sqrt{\boldsymbol{N} \cdot \boldsymbol{C} \cdot \boldsymbol{N}}$$

- Calculate reciprocal value of Λ:

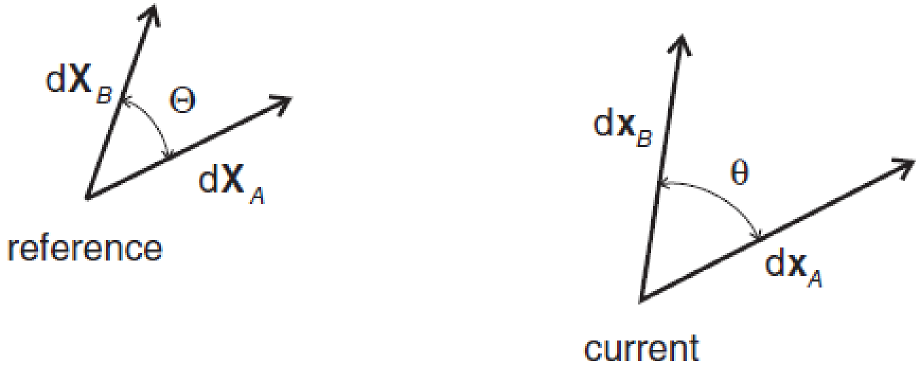
$$d\boldsymbol{X} = \boldsymbol{F}^{-1} \cdot d\boldsymbol{x} = d\boldsymbol{x} \cdot \boldsymbol{F}^{-T}$$

$$\Rightarrow (dS)^2 = d\boldsymbol{X} \cdot d\boldsymbol{X} = d\boldsymbol{x} \cdot (\boldsymbol{F}^{-T} \cdot \boldsymbol{F}^{-1}) \cdot d\boldsymbol{x}$$

$$\Rightarrow \lambda^2 = \left(\frac{dS}{ds}\right)^2 = \Lambda^{-2} = \boldsymbol{n} \cdot \underline{(\boldsymbol{F}^{-T} \cdot \boldsymbol{F}^{-1})} \cdot \boldsymbol{n}$$

Cauchy-Green tensor $\boldsymbol{B} = \boldsymbol{F} \cdot \boldsymbol{F}^T$

Change in Angle between Vectors



- Calculate angle after deformation:

$$\cos\theta = \frac{d\boldsymbol{x}_A \cdot d\boldsymbol{x}_B}{|d\boldsymbol{x}_A||d\boldsymbol{x}_B|}$$

$$\Rightarrow \cos\theta = \frac{dS_A \boldsymbol{N}_A \cdot \boldsymbol{F}^T \cdot \boldsymbol{F} \cdot \boldsymbol{N}_B dS_B}{dS_A\sqrt{\boldsymbol{N}_A \cdot \boldsymbol{C} \cdot \boldsymbol{N}_A}\sqrt{\boldsymbol{N}_B \cdot \boldsymbol{C} \cdot \boldsymbol{N}_B}dS_B} = \frac{\boldsymbol{N}_A \cdot \boldsymbol{C} \cdot \boldsymbol{N}_B}{\Lambda_A \Lambda_B}$$

- Change in angle (shear):

$$\gamma = \Theta - \theta = \cos^{-1}(\boldsymbol{N}_A \cdot \boldsymbol{N}_B) - \cos^{-1}\left(\frac{\boldsymbol{N}_A \cdot \boldsymbol{C} \cdot \boldsymbol{N}_B}{\Lambda_A \Lambda_B}\right)$$

Change in Area Enveloped by Two Vectors

- Utilize vector product, before deformation:

$$\boldsymbol{N}dA = d\boldsymbol{X}_A \times d\boldsymbol{X}_B = \boldsymbol{e}_i \epsilon_{ijk} (d\boldsymbol{X}_A)_j (d\boldsymbol{X}_B)_k$$

- After deformation:

$$\boldsymbol{n}da = d\boldsymbol{x}_A \times d\boldsymbol{x}_B = (\boldsymbol{F} \cdot d\boldsymbol{X}_A) \times (\boldsymbol{F} \cdot d\boldsymbol{X}_B)$$

$$\Rightarrow \boldsymbol{n}da = \boldsymbol{e}_i \epsilon_{ijk} F_{jr} (d\boldsymbol{X}_A)_r F_{ks} (d\boldsymbol{X}_B)_s$$

$$\boldsymbol{F} = F_{ij} \boldsymbol{e}_i \boldsymbol{e}_j$$

$$\Rightarrow \boldsymbol{n} \cdot \boldsymbol{F}da = \boldsymbol{e}_i \epsilon_{ijk} F_{jr} (d\boldsymbol{X}_A)_r F_{ks} (d\boldsymbol{X}_B)_s \cdot F_{tu} \boldsymbol{e}_t \boldsymbol{e}_u$$

$$\Rightarrow \boldsymbol{n} \cdot \boldsymbol{F}da = \epsilon_{ijk} F_{jr} (d\boldsymbol{X}_A)_r F_{ks} (d\boldsymbol{X}_B)_s F_{iu} \boldsymbol{e}_u$$

$$\Rightarrow \boldsymbol{n} \cdot \boldsymbol{F}da = \epsilon_{ijk} F_{iu} F_{jr} F_{ks} (d\boldsymbol{X}_A)_r (d\boldsymbol{X}_B)_s \boldsymbol{e}_u$$

$$\det(\boldsymbol{F}) = \det(\boldsymbol{F}^T) = \epsilon_{ijk} F_{i1} F_{j2} F_{k3}$$

*Take different values to u, r and s:

$$\Rightarrow \boldsymbol{n} \cdot \boldsymbol{F}da = \epsilon_{urs} \det(\boldsymbol{F}) (d\boldsymbol{X}_A)_r (d\boldsymbol{X}_B)_s \boldsymbol{e}_u$$

$$\Rightarrow \boldsymbol{n} \cdot \boldsymbol{F}da = \det(\boldsymbol{F})\, d\boldsymbol{X}_A \times d\boldsymbol{X}_B = \det(\boldsymbol{F})\, \boldsymbol{N}dA$$

$$\Rightarrow \boldsymbol{n}da = \det(\boldsymbol{F})\, \boldsymbol{N} \cdot \boldsymbol{F}^{-1} dA$$

$$\Rightarrow \frac{da}{dA} = \det(\boldsymbol{F})\, \boldsymbol{n} \cdot \boldsymbol{N} \cdot \boldsymbol{F}^{-1}$$

Change in Volume Enveloped by Three Vectors

- Before deformation (right-hand rule):

$$dV = d\boldsymbol{X}_A \cdot (d\boldsymbol{X}_B \times d\boldsymbol{X}_C) = d\boldsymbol{X}_A \cdot \epsilon_{ijk}\boldsymbol{e}_i (d\boldsymbol{X}_B)_j (d\boldsymbol{X}_C)_k = \epsilon_{ijk}(d\boldsymbol{X}_A)_i (d\boldsymbol{X}_B)_j (d\boldsymbol{X}_C)_k$$

- After deformation:

$$dv = d\boldsymbol{x}_A \cdot (d\boldsymbol{x}_B \times d\boldsymbol{x}_C) = \epsilon_{rst}(d\boldsymbol{x}_A)_r (d\boldsymbol{x}_B)_s (d\boldsymbol{x}_C)_t$$

- Application of deformation gradient $\boldsymbol{F}$:

$$dv = \epsilon_{rst}F_{ri}F_{sj}F_{tk}(d\boldsymbol{X}_A)_i (d\boldsymbol{X}_B)_j (d\boldsymbol{X}_C)_k$$

$$\Rightarrow dv = \det(\boldsymbol{F})\, \underbrace{\epsilon_{ijk}(d\boldsymbol{X}_A)_i (d\boldsymbol{X}_B)_j (d\boldsymbol{X}_C)_k}_{dV}$$

- Volume change ratio:

$$J = \frac{dv}{dV} = \det(\boldsymbol{F})$$

(Jacobian) determinant

Rigid Body Rotation

- Large rigid rotation causes complexity/nonlinearity even under small actual deformation
- Expression of rigid body rotation $\boldsymbol{R}(t)$ to a material point:

$$\boldsymbol{x}(\boldsymbol{X}, t) = \boldsymbol{R}(t) \cdot \boldsymbol{X} + \underline{\boldsymbol{X}_T(t)}$$

Translation motion

- Rigid body rotation matrix is orthogonal:

$$(dx)^2 = d\boldsymbol{x} \cdot d\boldsymbol{x} = d\boldsymbol{X} \cdot \boldsymbol{R}^T \cdot \boldsymbol{R} \cdot d\boldsymbol{X} = (dX)^2 = d\boldsymbol{X} \cdot d\boldsymbol{X} \Rightarrow \boldsymbol{R}^T \cdot \boldsymbol{R} = \boldsymbol{I}$$

- Rigid rotation of vectors:

$$\boldsymbol{e}_i' = \boldsymbol{R} \cdot \boldsymbol{e}_i = \boldsymbol{e}_i \cdot \boldsymbol{R}^T \Rightarrow \boldsymbol{v}' = \boldsymbol{R} \cdot \boldsymbol{v} = \boldsymbol{v} \cdot \boldsymbol{R}^T$$

- Rigid rotation of 2nd order tensors:

$$\boldsymbol{U} = U_{ij}\boldsymbol{e}_i\boldsymbol{e}_j \Rightarrow \boldsymbol{U}' = U_{ij}\boldsymbol{e}_i'\boldsymbol{e}_j' = U_{ij}\boldsymbol{R} \cdot \boldsymbol{e}_i\boldsymbol{e}_j \cdot \boldsymbol{R}^T = \boldsymbol{R} \cdot U_{ij}\boldsymbol{e}_i\boldsymbol{e}_j \cdot \boldsymbol{R}^T = \underline{\boldsymbol{R} \cdot \boldsymbol{U} \cdot \boldsymbol{R}^T}$$

Becomes a *new tensor* in the *same coordinate*, different from coordinate rotation

Principal Values and Principal Directions

- Stretch along and only along principal directions:

$$\lambda \boldsymbol{\mu} = \boldsymbol{F} \cdot \boldsymbol{\mu}$$

$$\Rightarrow (\boldsymbol{F} - \lambda \boldsymbol{I}) \cdot \boldsymbol{\mu} = \boldsymbol{0}$$

- $\boldsymbol{\mu}$ is nonzero along principal directions, so:

$$|\boldsymbol{F} - \underline{\lambda} \boldsymbol{I}| = 0$$

Principal value(s)

- 2nd order (3x3 matrix) tensors lead to 3 principal values, corresponding to 3 principal directions:

$$\lambda_K \boldsymbol{\mu}_K = \boldsymbol{F} \cdot \underline{\boldsymbol{\mu}_K} \ (K = I, II \ and \ III, no \ sum \ on \ K)$$

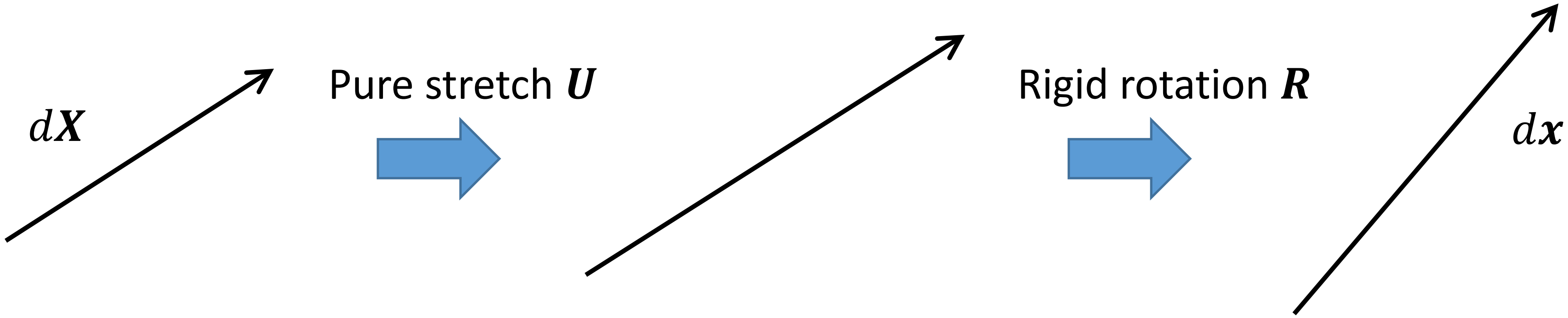
Unit principal vectors $|\boldsymbol{\mu}_K| = 1$

- Determine principal values and directions:

$$|\boldsymbol{F} - \lambda \boldsymbol{I}| = 0 \rightarrow (\boldsymbol{F} - \lambda \boldsymbol{I}) \cdot \boldsymbol{\mu} = \boldsymbol{0}$$

Components of Deformation

$$d\boldsymbol{x} = \boldsymbol{F} \cdot d\boldsymbol{X} = \boldsymbol{R} \cdot \boldsymbol{U} \cdot \boldsymbol{dX}$$



- Rigid rotation is orthogonal:

$$\boldsymbol{R}^T \cdot \boldsymbol{R} = \boldsymbol{I}$$

- Calculation of stretch ratio from Green deformation tensor $\boldsymbol{C}$:

$$\Lambda = \frac{ds}{dS} = \sqrt{\boldsymbol{N} \cdot \boldsymbol{C} \cdot \boldsymbol{N}}$$

$$\Rightarrow \boldsymbol{U} = \sqrt{\boldsymbol{C}}$$

Note: square root operation only applicable when in principal direction coordinates and $\boldsymbol{C}$ is expressed as a diagonal matrix!!!

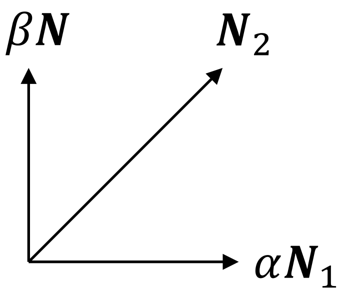
Properties of Principal Directions

- Prove principal directions with different principal values are orthogonal:

  Assume two principal directions are not orthogonal:

$$\boldsymbol{C}\cdot\boldsymbol{N}_K = \mu_K \boldsymbol{N}_K \ (K = 1, 2 \ and \ \mu_1 \neq \mu_2)$$

$$\boldsymbol{N}_2 = \alpha\boldsymbol{N}_1 + \beta\boldsymbol{N} \ (\alpha \neq 0, \boldsymbol{N}_1 \perp \boldsymbol{N})$$



$$\Rightarrow \boldsymbol{N}_1\cdot(\boldsymbol{C}\cdot\boldsymbol{N}_2) = \boldsymbol{N}_1\cdot(\alpha\mu_1\boldsymbol{N}_1 + \beta\boldsymbol{C}\cdot\boldsymbol{N}) = \alpha\mu_1 = \boldsymbol{N}_1\cdot\mu_2\boldsymbol{N}_2 = \boldsymbol{N}_1\cdot(\alpha\mu_2\boldsymbol{N}_1 + \beta\mu_2\boldsymbol{N}) = \alpha\mu_2$$ ✗

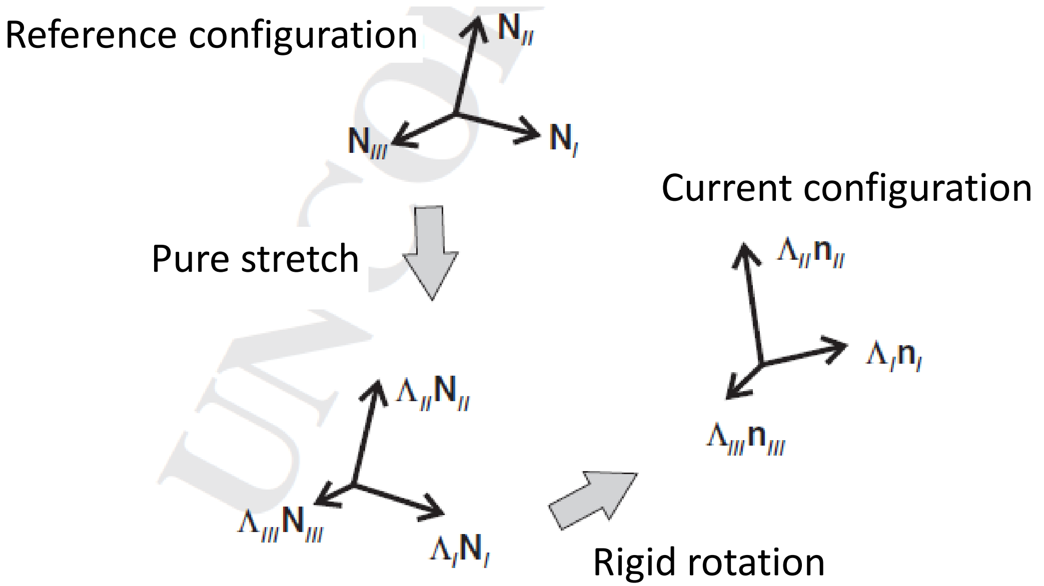
- Principal directions with different principal values can be base axes of a Cartesian system
- Orthogonal principal directions will keep orthogonal after deformation:

$$\cos\theta = \cos < \boldsymbol{n}_1, \boldsymbol{n}_2 > = \frac{\boldsymbol{N}_1\cdot\boldsymbol{C}\cdot\boldsymbol{N}_2}{\Lambda_1\Lambda_2} = \frac{\boldsymbol{N}_1\cdot\mu_2\boldsymbol{N}_2}{\Lambda_1\Lambda_2} = 0$$

$$\Rightarrow \theta = \frac{\pi}{2} rad$$

Polar Decomposition

- Geometrical meaning of polar decomposition with principal direction axes:



- Stretch tensor expressed with principal vectors:

$$\boldsymbol{U} = \Lambda_I \boldsymbol{N}_I \boldsymbol{N}_I + \Lambda_{II} \boldsymbol{N}_{II} \boldsymbol{N}_{II} + \Lambda_{III} \boldsymbol{N}_{III} \boldsymbol{N}_{III}$$

$$\Rightarrow \boldsymbol{U}^T = \boldsymbol{U}$$

- Rigid rotation tensor in principal vectors:

$$\boldsymbol{n}_K = \boldsymbol{R} \cdot \boldsymbol{N}_K \Rightarrow \boldsymbol{R} = \underline{\boldsymbol{n}_K \boldsymbol{N}_K}$$

Dyadic product format

- Polar decomposition of $\boldsymbol{F}$:

$$d\boldsymbol{x} = \boldsymbol{F} \cdot d\boldsymbol{X} = (\boldsymbol{R} \cdot \boldsymbol{U}) \cdot d\boldsymbol{X} \Rightarrow \boldsymbol{F} = \boldsymbol{R} \cdot \boldsymbol{U}$$

$$\boldsymbol{C} = \boldsymbol{F}^T \cdot \boldsymbol{F} = \boldsymbol{U}^T \cdot \boldsymbol{R}^T \cdot \boldsymbol{R} \cdot \boldsymbol{U} = \boldsymbol{U} \cdot \boldsymbol{U} \Rightarrow \boldsymbol{U} = \sqrt{\boldsymbol{C}}$$

Results need to be transferred back to original coordinate

- Alternative route – rotate then deform:

$$d\boldsymbol{x} = (\boldsymbol{V} \cdot \boldsymbol{R}) \cdot d\boldsymbol{X}$$

$$\boldsymbol{V} = \Lambda_I \boldsymbol{n}_I \boldsymbol{n}_I + \Lambda_{II} \boldsymbol{n}_{II} \boldsymbol{n}_{II} + \Lambda_{III} \boldsymbol{n}_{III} \boldsymbol{n}_{III}$$

$$\Rightarrow \boldsymbol{V} = \boldsymbol{R} \cdot \boldsymbol{U} \cdot \boldsymbol{R}^T$$

Requirements for Material Strain Tensors

- Material strain tensors are calculated in respect of reference/initial configuration
- Strain tensors should capture deformation, exclude rigid rotation and cover small-strain scenario:
  - Has the same principal directions as pure stretch tensor $\boldsymbol{U}$:

$$\boldsymbol{E} = f(\Lambda_I)\boldsymbol{N}_I\boldsymbol{N}_I + f(\Lambda_{II})\boldsymbol{N}_{II}\boldsymbol{N}_{II} + \underline{f(\Lambda_{III})}\boldsymbol{N}_{III}\boldsymbol{N}_{III} \Rightarrow \boldsymbol{E}^T = \boldsymbol{E}$$

Smooth and monotonic

  - Exclude pure rotation – f vanishes when $\boldsymbol{U} = \boldsymbol{I}$:

$$f(1) = 0$$

  - Agrees with small strain tensor:

$$f'(1) = 1$$

Taylor expansion to 1st order: $f(\Lambda) = f(1) + f'(1)(\Lambda - 1) + o[(\Lambda - 1)^2]$

$$\Rightarrow f(\Lambda) \approx f(1) + f'(1)(\Lambda - 1) = \underline{\Lambda - 1}$$

Under small deformation, reduce to change in length per unit reference/initial length along principal directions.

Material Green Strain Tensor

$$\boldsymbol{E} = f(\Lambda_I)\boldsymbol{N}_I\boldsymbol{N}_I + f(\Lambda_{II})\boldsymbol{N}_{II}\boldsymbol{N}_{II} + f(\Lambda_{III})\boldsymbol{N}_{III}\boldsymbol{N}_{III}, \qquad f(1) = 0, \qquad f'(1) = 1$$

$$f(\Lambda) = \frac{1}{2}(\Lambda^2 - 1) \Rightarrow \underline{\boldsymbol{E}^G} = \frac{1}{2}(\boldsymbol{U}^T \cdot \boldsymbol{U} - \boldsymbol{I}) = \frac{1}{2}(\boldsymbol{F}^T \cdot \boldsymbol{F} - \boldsymbol{I})$$

Green strain

- Express Green strain in component format:

$$E_{ij}^G = \frac{1}{2}\left(F_{ik}^T \cdot F_{kj} - \delta_{ij}\right) = \frac{1}{2}\left(F_{ki} \cdot F_{kj} - \delta_{ij}\right) = \frac{1}{2}\left(\frac{\partial x_k}{\partial X_i} \cdot \frac{\partial x_k}{\partial X_j} - \delta_{ij}\right)$$

- Before and after deformation, node positions can be linked by displacement vectors:

$$x_k = X_k + u_k \Rightarrow \frac{\partial x_k}{\partial X_i} = \frac{\partial X_k + u_k}{\partial X_i} = \delta_{ki} + \frac{\partial u_k}{\partial X_i}$$

$$\Rightarrow E_{ij}^G = \frac{1}{2}\left[\left(\delta_{ki} + \frac{\partial u_k}{\partial X_i}\right)\left(\delta_{kj} + \frac{\partial u_k}{\partial X_j}\right) - \delta_{ij}\right] = \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i} + \frac{\partial u_k}{\partial X_i}\frac{\partial u_k}{\partial X_j}\right)$$

Other Typical Material Strain Tensors

$$f(\Lambda) = \Lambda - 1 \Rightarrow \boldsymbol{E}^{(1)} = \boldsymbol{U} - \boldsymbol{I}$$

$$f(\Lambda) = 1 - \frac{1}{\Lambda} \Rightarrow \boldsymbol{E}^{(-1)} = \boldsymbol{I} - \boldsymbol{U}^{-1}$$

Logarithm strain: $f(\Lambda) = ln\Lambda \Rightarrow \boldsymbol{E}^{(ln)} = ln\boldsymbol{U}$

Note:
- Format of $f(\Lambda)$ only makes sense along principal directions;
- These tensors cannot be calculated directly from $\boldsymbol{F}$.

Linear Analysis for Small Strain

- Utilized to demonstrate that material strain tensors can cover small strain scenarios.
- Node positions can be linked by displacement vectors before and after deformation:

$$\boldsymbol{u} = \boldsymbol{x} - \boldsymbol{X}, \qquad u_k = x_k - X_k$$

$$\Rightarrow F_{ij} = \frac{\partial x_i}{\partial X_j} = \delta_{ij} + \frac{\partial u_i}{\partial X_j} \Rightarrow \boldsymbol{F} = \boldsymbol{I} + \underline{(\boldsymbol{\nabla}_X \boldsymbol{u})^T}$$

Displacement gradient tensor

$$\boldsymbol{C} = \boldsymbol{F}^T \cdot \boldsymbol{F} = \left(\boldsymbol{I} + (\boldsymbol{\nabla}_X \boldsymbol{u})\right) \cdot \left(\boldsymbol{I} + (\boldsymbol{\nabla}_X \boldsymbol{u})^T\right) = \boldsymbol{I} + (\boldsymbol{\nabla}_X \boldsymbol{u})^T + (\boldsymbol{\nabla}_X \boldsymbol{u}) + (\boldsymbol{\nabla}_X \boldsymbol{u}) \cdot (\boldsymbol{\nabla}_X \boldsymbol{u})^T$$

$$\Rightarrow C_{ij} = \delta_{ij} + \frac{\partial u_j}{\partial X_i} + \frac{\partial u_i}{\partial X_j} + \underline{\frac{\partial u_j}{\partial X_i}\frac{\partial u_i}{\partial X_j}} \approx \delta_{ij} + \left(\frac{\partial u_j}{\partial X_i} + \frac{\partial u_i}{\partial X_j}\right) = \delta_{ij} + 2\underline{\varepsilon_{ij}}$$

Negligible under small deformation

Small strain

$$\Rightarrow \boldsymbol{C} \approx \boldsymbol{I} + 2\underline{\boldsymbol{\varepsilon}}$$

$$\frac{1}{2}\left((\boldsymbol{\nabla}_X \boldsymbol{u})^T + (\boldsymbol{\nabla}_X \boldsymbol{u})\right)$$

Linear Geometrical Measurement (1/2)

- Stretch ratio along base vector $\boldsymbol{e}_1$:

$$\Lambda = (\boldsymbol{e}_1 \cdot \boldsymbol{C} \cdot \boldsymbol{e}_1)^{\frac{1}{2}} \approx (\boldsymbol{e}_1 \cdot (\boldsymbol{I} + 2\boldsymbol{\varepsilon}) \cdot \boldsymbol{e}_1)^{\frac{1}{2}} = \underline{(1 + 2\boldsymbol{e}_1 \cdot \boldsymbol{\varepsilon} \cdot \boldsymbol{e}_1)^{\frac{1}{2}}} \approx 1 + \boldsymbol{e}_1 \cdot \boldsymbol{\varepsilon} \cdot \boldsymbol{e}_1$$

$$(1 + x)^c \approx 1 + cx + o(x^2)$$

$$\Rightarrow \boldsymbol{e}_1 \cdot \boldsymbol{\varepsilon} \cdot \boldsymbol{e}_1 = \Lambda - 1 = \varepsilon_{11}$$

Same as conventional small strain definition!

- Angle change of orthogonal base vectors $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$:

$$\gamma \approx \cos(\underset{\frac{\pi}{2}}{\underline{\Theta}} - \gamma) = \frac{\boldsymbol{e}_1 \cdot \boldsymbol{C} \cdot \boldsymbol{e}_2}{\Lambda_1 \Lambda_2} \approx \frac{\boldsymbol{e}_1 \cdot (\boldsymbol{I} + 2\boldsymbol{\varepsilon}) \cdot \boldsymbol{e}_2}{\underline{1 \cdot 1}} = 2\boldsymbol{e}_1 \cdot \boldsymbol{\varepsilon} \cdot \boldsymbol{e}_2$$

Small deformation

$$\Rightarrow \boldsymbol{e}_1 \cdot \boldsymbol{\varepsilon} \cdot \boldsymbol{e}_2 = \varepsilon_{12} \approx \frac{\gamma}{2}$$

Linear Geometrical Measurement (2/2)

- Volume change:

$$\frac{dv}{dV} = \det(\boldsymbol{F}) = \epsilon_{ijk} F_{i1} F_{j2} F_{k3} = \epsilon_{ijk} \left(\delta_{i1} + \frac{\partial u_i}{\partial X_1}\right)\left(\delta_{j2} + \frac{\partial u_j}{\partial X_2}\right)\left(\delta_{k3} + \frac{\partial u_k}{\partial X_3}\right)$$

Neglect high order terms due to small deformation

$$\Rightarrow \frac{dv}{dV} \approx \epsilon_{12k} \left(1 + \frac{\partial u_2}{\partial X_2} + \frac{\partial u_1}{\partial X_1}\right)\left(\delta_{k3} + \frac{\partial u_k}{\partial X_3}\right) \approx 1 + \frac{\partial u_1}{\partial X_1} + \frac{\partial u_2}{\partial X_2} + \frac{\partial u_3}{\partial X_3} = 1 + \varepsilon_{ii}$$

Linear Polar Decomposition

- Linear approximation of $\boldsymbol{C}$ along principal directions:

$$\boldsymbol{C} \approx (1+2\varepsilon_I)\boldsymbol{N}_I\boldsymbol{N}_I + (1+2\varepsilon_{II})\boldsymbol{N}_{II}\boldsymbol{N}_{II} + (1+2\varepsilon_{III})\boldsymbol{N}_{III}\boldsymbol{N}_{III}$$

$$\Rightarrow \boldsymbol{U} = \sqrt{\boldsymbol{C}} \approx \sqrt{1+2\varepsilon_I}\boldsymbol{N}_I\boldsymbol{N}_I + \sqrt{1+2\varepsilon_{II}}\boldsymbol{N}_{II}\boldsymbol{N}_{II} + \sqrt{1+2\varepsilon_{III}}\boldsymbol{N}_{III}\boldsymbol{N}_{III}$$

$$\Rightarrow \boldsymbol{U} \approx (1+\varepsilon_I)\boldsymbol{N}_I\boldsymbol{N}_I + (1+\varepsilon_{II})\boldsymbol{N}_{II}\boldsymbol{N}_{II} + (1+\varepsilon_{III})\boldsymbol{N}_{III}\boldsymbol{N}_{III} \Rightarrow \boldsymbol{U} = \boldsymbol{I} + \boldsymbol{\varepsilon}$$

Approximation from small ε_I

$$\Rightarrow \boldsymbol{U}^{-1} \approx \frac{1}{1+\varepsilon_I}\boldsymbol{N}_I\boldsymbol{N}_I + \cdots \approx (1-\varepsilon_I)\boldsymbol{N}_I\boldsymbol{N}_I + \cdots \Rightarrow \boldsymbol{U}^{-1} = \boldsymbol{I} - \boldsymbol{\varepsilon}$$

Neglect high order terms due to small deformation

$$\boldsymbol{R} = \boldsymbol{F}\cdot\boldsymbol{U}^{-1} \approx (\boldsymbol{I} + (\nabla_X \boldsymbol{u})^T)\cdot(\boldsymbol{I} - \boldsymbol{\varepsilon}) \approx \boldsymbol{I} - \underline{\boldsymbol{\varepsilon}} + (\nabla_X \boldsymbol{u})^T = \boldsymbol{I} + \underline{\frac{1}{2}\left((\nabla_X \boldsymbol{u})^T - (\nabla_X \boldsymbol{u})\right)}$$

$$\frac{1}{2}\left((\nabla_X \boldsymbol{u})^T + (\nabla_X \boldsymbol{u})\right)$$

Infinitesimal rotation tensor:

$$\Omega_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} - \frac{\partial u_j}{\partial X_i}\right)$$

- Displacement combines deformation and rotation: $(\nabla_X \boldsymbol{u})^T = \boldsymbol{\varepsilon} + \boldsymbol{\Omega}$

Small Strain Compatibility Condition

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_j}{\partial X_i} + \frac{\partial u_i}{\partial X_j}\right)$$

6 strain components from 3 displacement components – has further constraint/relationship.

- Strain compatibility (no crack) – strain leads to single-value displacement field
- Derivatives of single-value and differentiable function:

$$\frac{\partial^n \boldsymbol{u}}{\partial X_{i1} \dots \partial X_{in}} = \frac{\partial^n \boldsymbol{u}}{\partial X_{j1} \dots \partial X_{jn}}, (i1 \dots in \text{ and } j1 \dots jn \text{ are components with different order})$$

Interchangeable differential order

$$\Rightarrow 2\varepsilon_{12,12} = u_{1,212} + u_{2,112} = u_{1,122} + u_{2,211} = \varepsilon_{11,22} + \varepsilon_{22,11}$$

Expand to other components $\Rightarrow \boldsymbol{\nabla_X} \times (\boldsymbol{\nabla_X} \times \boldsymbol{\varepsilon})^T = \boldsymbol{0}$

$$\Rightarrow \boldsymbol{0} = \frac{\partial}{\partial X_m}\boldsymbol{e}_m \times \left(\epsilon_{ijk}\frac{\partial \varepsilon_{jl}}{\partial X_i}\boldsymbol{e}_k\boldsymbol{e}_l\right)^T = \frac{\partial}{\partial X_m}\boldsymbol{e}_m \times \left(\epsilon_{ijl}\frac{\partial \varepsilon_{jk}}{\partial X_i}\boldsymbol{e}_k\boldsymbol{e}_l\right) = \epsilon_{mkn}\epsilon_{ijl}\frac{\partial^2 \varepsilon_{jk}}{\partial X_i \partial X_m}\boldsymbol{e}_n\boldsymbol{e}_l$$

$$0 = \epsilon_{mkn}\epsilon_{ijl}\frac{\partial^2 \varepsilon_{jk}}{\partial X_i \partial X_m}$$

6 equations but only contains 3 independent conditions

Rate of Deformation

- Spatial velocity gradient tensor – variation of velocity field in the current configuration:

$$\boldsymbol{L} = \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{x}} \Rightarrow d\boldsymbol{v} = \boldsymbol{L} \cdot d\boldsymbol{x}$$

$$d\boldsymbol{x} = \boldsymbol{F} \cdot d\boldsymbol{X} \Rightarrow d\boldsymbol{v} = d\dot{\boldsymbol{x}} = \dot{\boldsymbol{F}} \cdot d\boldsymbol{X} = \boldsymbol{L} \cdot d\boldsymbol{x} = \boldsymbol{L} \cdot \boldsymbol{F} \cdot d\boldsymbol{X} \Rightarrow \boldsymbol{L} = \dot{\boldsymbol{F}} \cdot \boldsymbol{F}^{-1}$$

- Symmetric part in $\boldsymbol{L}$:

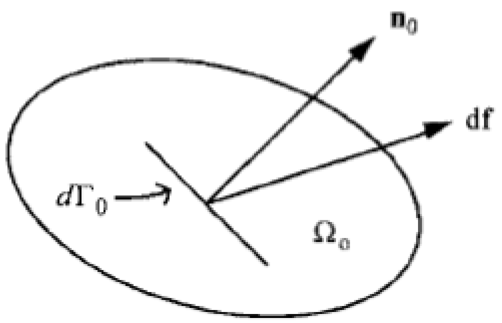
$$\boldsymbol{D} = \frac{1}{2}(\boldsymbol{L} + \boldsymbol{L}^T) = \frac{1}{2}\left(\dot{\boldsymbol{F}} \cdot \boldsymbol{F}^{-1} + \boldsymbol{F}^{-T} \cdot \dot{\boldsymbol{F}}^T\right)$$

$$\dot{\boldsymbol{E}}^G = \frac{1}{2}\left(\dot{\boldsymbol{F}}^T \cdot \boldsymbol{F} + \boldsymbol{F}^T \cdot \dot{\boldsymbol{F}}\right) \Rightarrow \boldsymbol{F}^T \cdot \boldsymbol{D} \cdot \boldsymbol{F} = \frac{1}{2}\left(\dot{\boldsymbol{F}}^T \cdot \boldsymbol{F} + \boldsymbol{F}^T \cdot \dot{\boldsymbol{F}}\right) = \dot{\boldsymbol{E}}^G \Rightarrow \boldsymbol{D} = \boldsymbol{F}^{-T} \cdot \dot{\boldsymbol{E}}^G \cdot \boldsymbol{F}^{-1}$$

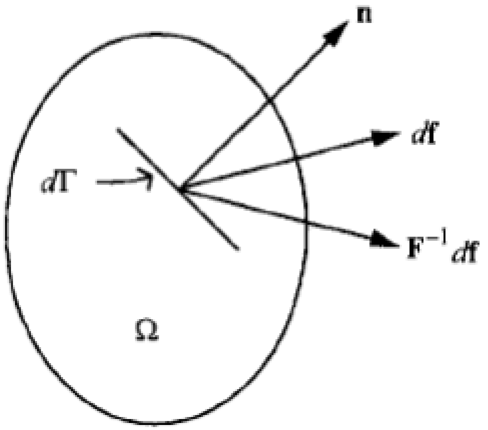
Pull-back (to initial configuration) operation

Push-forward (to current configuration) operation

Common Stress Measures



Reference configuration



Current configuration

- Cauchy stress (true stress in current configuration):

$$\boldsymbol{n} \cdot \boldsymbol{\sigma} d\Gamma = d\boldsymbol{f} = \boldsymbol{t} d\Gamma, \qquad \boldsymbol{\sigma} = \boldsymbol{\sigma}^T$$

- Nominal stress (engineering stress in initial configuration):

$$\boldsymbol{n}_0 \cdot \boldsymbol{P} d\Gamma_0 = d\boldsymbol{f} = \boldsymbol{t}_0 d\Gamma_0, \qquad \boldsymbol{P} \neq \boldsymbol{P}^T$$

- PK2 stress:

$$\boldsymbol{n}_0 \cdot \boldsymbol{S} d\Gamma_0 = \underline{\boldsymbol{F}^{-1}} \cdot \boldsymbol{t}_0 d\Gamma_0$$

  - Make $\boldsymbol{S}$ symmetric
  - Make $\boldsymbol{S}$ conjugate to $\boldsymbol{E}^G$ in power

Corotational Stress and Deformation

- Describe stress and deformation in the coordinate corotates with the body to analyze structure elements and anisotropic materials

- Corotation is rigid and can be measured using $\boldsymbol{R}$ from polar decomposition

- Utilization of coordinate transformation for 2nd order tensors:

$$\hat{\boldsymbol{\sigma}} = \boldsymbol{R}^T \cdot \boldsymbol{\sigma} \cdot \boldsymbol{R}$$ Corotational Cauchy stress

$$\widehat{\boldsymbol{D}} = \boldsymbol{R}^T \cdot \boldsymbol{D} \cdot \boldsymbol{R}$$ Corotational rate-of-deformation

Same tensors expressed in different coordinate systems!

Transformation of Stress Measures

- Stress measures can be transformed using deformation functions, refer to Box 3.2 of the textbook

Box 3.2 Transformations of stresses

Cauchy stress $\boldsymbol{\sigma}$	Nominal stress $\mathbf{P}$	2nd Piola–Kirchhoff stress $\mathbf{S}$	Corotational Cauchy stress $\hat{\boldsymbol{\sigma}}$
$\boldsymbol{\sigma} =$	$J^{-1}\mathbf{F}\cdot\mathbf{P}$	$J^{-1}\mathbf{F}\cdot\mathbf{S}\cdot\mathbf{F}^T$	$\mathbf{R}\cdot\hat{\boldsymbol{\sigma}}\cdot\mathbf{R}^T$
$\mathbf{P} = J\mathbf{F}^{-1}\cdot\boldsymbol{\sigma}$		$\mathbf{S}\cdot\mathbf{F}^T$	$J\mathbf{U}^{-1}\cdot\hat{\boldsymbol{\sigma}}\cdot\mathbf{R}^T$
$\mathbf{S} = J\mathbf{F}^{-1}\cdot\boldsymbol{\sigma}\cdot\mathbf{F}^{-T}$	$\mathbf{P}\cdot\mathbf{F}^{-T}$		$J\mathbf{U}^{-1}\cdot\hat{\boldsymbol{\sigma}}\cdot\mathbf{U}^{-1}$
$\hat{\boldsymbol{\sigma}} = \mathbf{R}^T\cdot\boldsymbol{\sigma}\cdot\mathbf{R}$	$J^{-1}\mathbf{U}\cdot\mathbf{P}\cdot\mathbf{R}$	$J^{-1}\,\mathbf{U}\cdot\mathbf{S}\cdot\mathbf{U}$	
$\boldsymbol{\tau} = J\boldsymbol{\sigma}$	$\mathbf{F}\cdot\mathbf{P}$	$\mathbf{F}\cdot\mathbf{S}\cdot\mathbf{F}^T$	$J\mathbf{R}\cdot\hat{\boldsymbol{\sigma}}\cdot\mathbf{R}^T$

Notes: $d\mathbf{x} = \mathbf{F}\cdot d\mathbf{X} = \mathbf{R}\cdot\mathbf{U}\cdot d\mathbf{X}$

$\mathbf{U}$ is the stretch tensor; see Section 3.7.1

$d\mathbf{x} = \mathbf{R}\cdot d\mathbf{X} = \mathbf{R}\cdot d\hat{\mathbf{x}}$ in rotation

$\boldsymbol{\tau}$ = Kirchhoff stress

Pairing of Stress and Strain Tensors

- Stress and Strain/Deformation tensors can be defined in various ways (coordinates, time derivative, etc.)
- Pairing of stress and strain tensors should follow work conjugate requirement (Box 3.2 in textbook):

Box 3.4 Stress-deformation (strain) rate pairs conjugate in power

Cauchy stress/rate of deformation: $\rho \dot{w}^{\text{int}} = \mathbf{D} : \boldsymbol{\sigma} = \boldsymbol{\sigma} : \mathbf{D} = D_{ij}\sigma_{ij}$

Nominal stress/rate of deformation gradient: $\rho_0 \dot{w}^{\text{int}} = \dot{\mathbf{F}}^T : \mathbf{P} = \mathbf{P}^T : \dot{\mathbf{F}} = \dot{F}_{ij} P_{ji}$

PK2 stress/rate of Green strain: $\rho_0 \dot{w}^{\text{int}} = \dot{\mathbf{E}} : \mathbf{S} = \mathbf{S} : \dot{\mathbf{E}} = \dot{E}_{ij} S_{ij}$

Corotational Cauchy stress/rate-of-deformation: $\rho \dot{w}^{\text{int}} = \hat{\mathbf{D}} : \hat{\boldsymbol{\sigma}} = \hat{\boldsymbol{\sigma}} : \hat{\mathbf{D}} = \hat{D}_{ij}\hat{\sigma}_{ij}$

- Derivation of work conjugate starts from conservation of energy
- Constitutive law/equations of a material should link paired stress and strain tensors

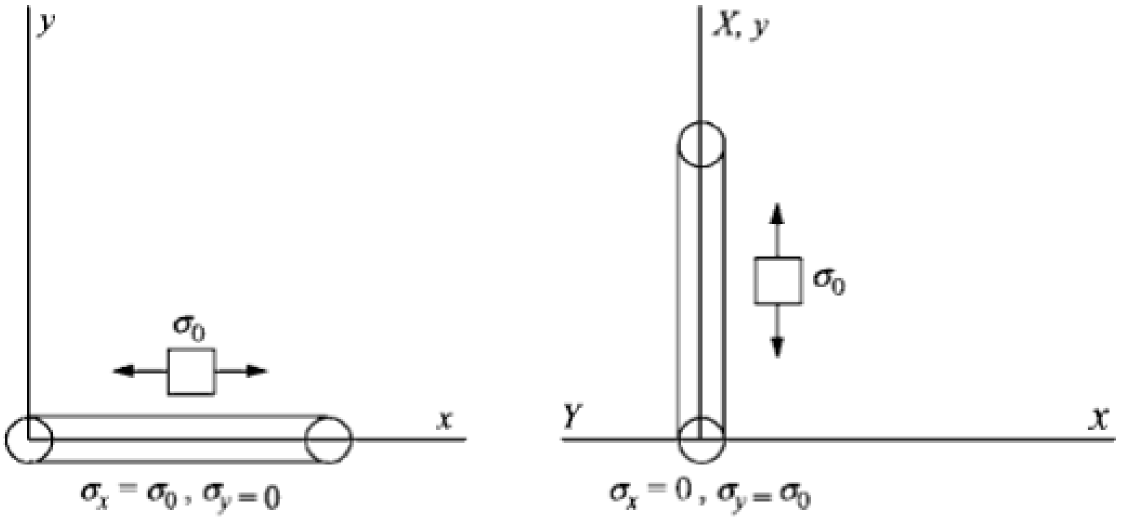
Objective Rates in Constitutive Equations

- Importance of objective rates – test rate form linear elastic law with Cauchy stress tensor:

$$\frac{D\sigma_{ij}}{Dt} = C^{\sigma D}_{ijkl} D_{kl} \quad or \quad \frac{D\boldsymbol{\sigma}}{Dt} = \boldsymbol{C}^{\sigma D} : \boldsymbol{D}$$

Material time derivative in respect to reference configuration

- Change of $\boldsymbol{C}^{\sigma D}$ under rigid rotation with prestress:



- Rigid rotation:

$$\boldsymbol{F} = \boldsymbol{R} \Rightarrow \boldsymbol{D} = \frac{1}{2}\left(\dot{\boldsymbol{R}} \cdot \boldsymbol{R}^T + \boldsymbol{R} \cdot \dot{\boldsymbol{R}}^T\right) = \frac{1}{2}\dot{\boldsymbol{I}} = \boldsymbol{0}$$

- Stress changes in respect to reference configuration:

$$\frac{D\boldsymbol{\sigma}}{Dt} \neq \boldsymbol{0} = \boldsymbol{C}^{\sigma D} : \boldsymbol{D} \quad ✗$$

Jaumann Rate

- Antisymmetric spin tensor:

$$\boldsymbol{W} = \frac{1}{2}(\boldsymbol{L} - \boldsymbol{L}^T) = \boldsymbol{L} - \boldsymbol{D}$$

- Jaumann rate of Cauchy stress:

$$\boldsymbol{\sigma}^{\nabla J} = \frac{D\boldsymbol{\sigma}}{Dt} - \boldsymbol{W} \cdot \boldsymbol{\sigma} - \boldsymbol{\sigma} \cdot \boldsymbol{W}^T$$

- Correct constitutive law:

$$\boldsymbol{\sigma}^{\nabla J} = \boldsymbol{C}^{\sigma J} : \boldsymbol{D}$$

$$\Rightarrow \frac{D\boldsymbol{\sigma}}{Dt} = \underbrace{\boldsymbol{C}^{\sigma J} : \boldsymbol{D}}_{\text{Material}} + \underbrace{\boldsymbol{W} \cdot \boldsymbol{\sigma} - \boldsymbol{\sigma} \cdot \boldsymbol{W}^T}_{\text{Rotation}}$$

Validity of Jaumann rate is demonstrated in Example 3.12 in the textbook.

Common Objective Rates

- Truesdell rate and Green-Naghdi rate are also frequently used due to easy implementation
- Comparison of the objective rates (Box 3.5 in the textbook)

Box 3.5 Objective rates

Jaumann rate

$$\sigma^{\nabla J} = \frac{D\sigma}{Dt} - \mathbf{W}\cdot\sigma - \sigma\cdot\mathbf{W}^T, \quad \sigma_{ij}^{\nabla J} = \frac{D\sigma_{ij}}{Dt} - W_{ik}\sigma_{kj} - \sigma_{ik}W_{kj}^T \qquad \text{(B3.5.1)}$$

Truesdell rate

$$\sigma^{\nabla J} = \frac{D\sigma}{Dt} + \mathrm{div}\ (\mathbf{v})\sigma - \mathbf{L}\cdot\sigma - \sigma\cdot\mathbf{L}^T \qquad \text{(B3.5.2)}$$

$$\sigma_{ij}^{\nabla T} = \frac{D\sigma_{ij}}{Dt} + \frac{\partial v_k}{\partial x_k}\sigma_{ij} - \frac{\partial v_i}{\partial x_k}\sigma_{kj} - \sigma_{ik}\frac{\partial v_j}{\partial x_k} \qquad \text{(B3.5.3)}$$

Green–Naghdi rate

$$\sigma^{\nabla G} = \frac{D\sigma}{Dt} - \Omega\cdot\sigma - \sigma\cdot\Omega^T, \quad \sigma_{ij}^{\nabla G} = \frac{D\sigma_{ij}}{Dt} - \Omega_{ik}\sigma_{kj} - \sigma_{ik}\Omega_{kj}^T \qquad \text{(B3.5.4)}$$

$$\Omega = \dot{\mathbf{R}}\cdot\mathbf{R}^T, \quad \mathbf{L} = \frac{\partial\mathbf{v}}{\partial\mathbf{x}} = \mathbf{D} + \mathbf{W}, \quad L_{ij} = \frac{\partial v_i}{\partial x_j} = D_{ij} + W_{ij} \qquad \text{(B3.5.5)}$$

- Different objective rates utilize different measures of rotation

The End