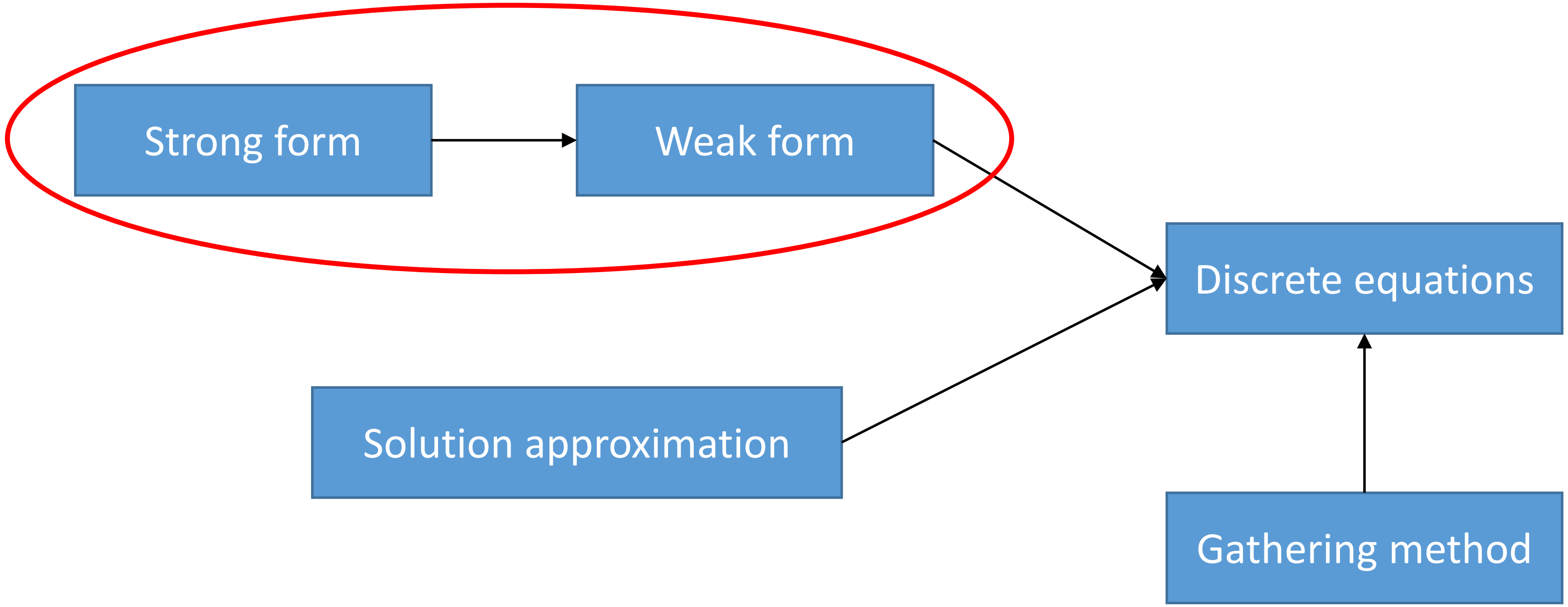
# Computational Mechanics

Chapter 6 Strong and Weak Forms of Multidimensional Scalar Fields

# Components for Formulation FEM Equations




- 1D → 2D → 3D
- Scalar fields → vector fields

- 2D fields, e.g., temperature fields
- post-processing like gradient will lead to vector fields

# Operation of 2D Vectors

- Matrix expression of vectors:

$$\vec{q} = q_x\vec{\imath} + q_y\vec{\jmath} \Rightarrow \boldsymbol{q} = \begin{bmatrix} q_x \\ q_y \end{bmatrix}$$

- Scalar product of vectors as matrices:

$$\vec{q} \cdot \vec{r} = q_x r_x + q_y r_y = \begin{bmatrix} q_x & q_y \end{bmatrix} \begin{bmatrix} r_x \\ r_y \end{bmatrix} = \boldsymbol{q}^T \boldsymbol{r}$$

- Gradient operator:

$$\vec{\nabla} = \frac{\partial}{\partial x}\vec{\imath} + \frac{\partial}{\partial y}\vec{\jmath}, \qquad \boldsymbol{\nabla} = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{bmatrix}$$

Steepest descent direction of the field

- Gradient of a scalar function:

$$\vec{\nabla}\theta = \frac{\partial \theta}{\partial x}\vec{\imath} + \frac{\partial \theta}{\partial y}\vec{\jmath}$$

- Divergence of a vector field:

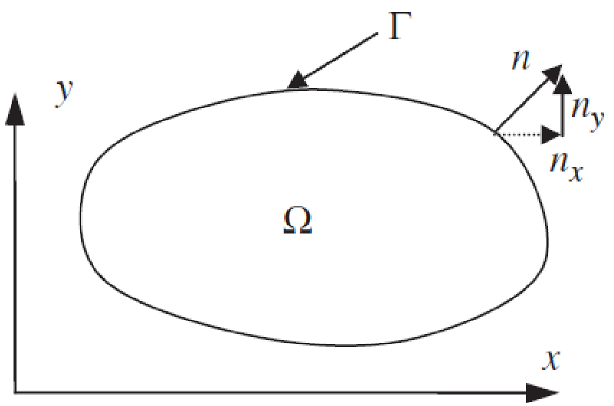
$$div\vec{q} = \vec{\nabla} \cdot \vec{q} = \frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y}$$

Flow leaving a point

- Matrix formats of gradient and divergence:

$$\boldsymbol{\nabla}\theta = \begin{bmatrix} \frac{\partial \theta}{\partial x} \\ \frac{\partial \theta}{\partial y} \end{bmatrix}, \boldsymbol{\nabla}^T \cdot \boldsymbol{q} = \begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \end{bmatrix} \begin{bmatrix} q_x \\ q_y \end{bmatrix} = \frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y}$$

# 2D Divergence Theorem




- Normal vector pointing to the outside:

$$\vec{n} = n_x \vec{i} + n_y \vec{j}$$

- Goal of transformation – reduce 2D area integral to 1D closed curve line integral

- 2D Green's theorem:

  If $\theta(x, y) \in C^0$ and integrable, then

$$\int_\Omega \vec{\nabla}\theta \, d\Omega = \oint_\Gamma \theta \vec{n} d\Gamma \;\; or \;\; \int_\Omega \boldsymbol{\nabla}^T \theta \, d\Omega = \oint_\Gamma \theta \boldsymbol{n} d\Gamma$$

$$\Rightarrow \begin{cases} \int_\Omega \frac{\partial \theta}{\partial x} d\Omega = \oint_\Gamma \theta n_x d\Gamma \\ \int_\Omega \frac{\partial \theta}{\partial y} d\Omega = \oint_\Gamma \theta n_y d\Gamma \end{cases}$$

$$\Rightarrow \int_\Omega \left( \frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} \right) d\Omega = \oint_\Gamma (q_x n_x + q_y n_y) d\Gamma$$

$$\Rightarrow \int_\Omega \vec{\nabla} \cdot \vec{q} \, d\Omega = \oint_\Gamma \vec{q} \cdot \vec{n} d\Gamma$$

Divergence theorem

# 2D Green's Formula

$$\vec{\nabla}\cdot(w\vec{q}) = \frac{\partial w q_x}{\partial x} + \frac{\partial w q_y}{\partial y} = \frac{\partial w}{\partial x} q_x + w\frac{\partial q_x}{\partial x} + \frac{\partial w}{\partial y} q_y + w\frac{\partial q_y}{\partial y} = w\underbrace{\left(\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y}\right)}_{\vec{\nabla}\cdot\vec{q}} + \underbrace{\left(\frac{\partial w}{\partial x} q_x + \frac{\partial w}{\partial y} q_y\right)}_{\vec{\nabla}w\cdot\vec{q}}$$

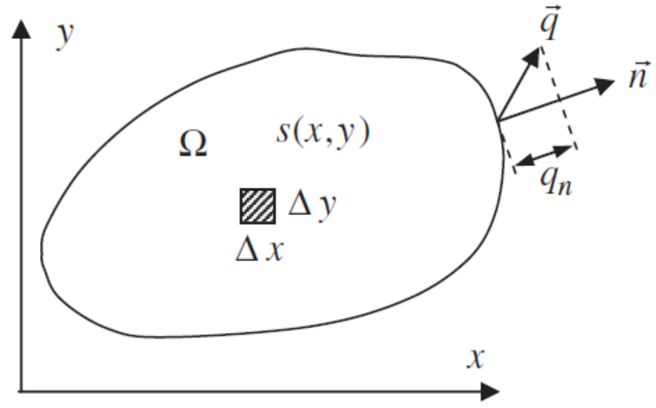
$$\Rightarrow \int_{\Omega} \vec{\nabla}\cdot(w\vec{q})d\Omega = \int_{\Omega} w\vec{\nabla}\cdot\vec{q}d\Omega + \int_{\Omega} \vec{\nabla}w\cdot\vec{q}d\Omega$$

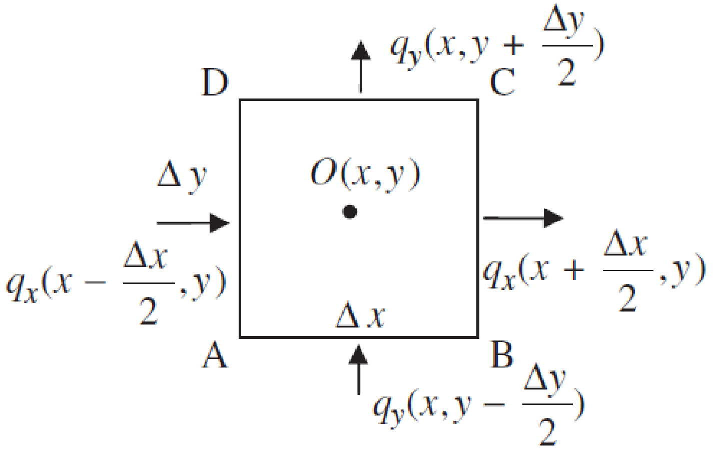
$$\Rightarrow \int_{\Omega} w\vec{\nabla}\cdot\vec{q}d\Omega = \int_{\Omega} \vec{\nabla}\cdot(w\vec{q})d\Omega - \int_{\Omega} \vec{\nabla}w\cdot\vec{q}d\Omega = \oint_{\Gamma} w\vec{q}\cdot\vec{n}d\Gamma - \int_{\Omega} \vec{\nabla}w\cdot\vec{q}d\Omega$$

- Special 1D example with $\vec{q} = q_x\vec{\imath},\ \vec{n} = \pm\vec{\imath},\ n(0) = -\vec{\imath}\ and\ n(l) = \vec{\imath}$, then:

$$\int_{\Omega} w\frac{\partial q_x}{\partial x}d\Omega = \oint_{\Gamma} w q_x n d\Gamma - \int_{\Omega} \frac{\partial w}{\partial x} q_x d\Omega \Rightarrow \underbrace{\int_0^l w\frac{\partial q_x}{\partial x}dx = w q_x|_0^l - \int_0^l \frac{\partial w}{\partial x} q_x dx}_{\text{1D Integration by parts}}$$

# Heat Flux Analysis for 2D Heat Conduction



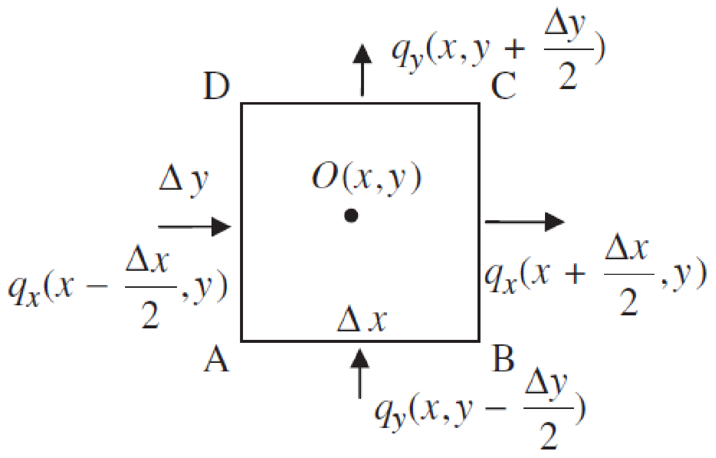

- Effective heat flux is only in normal direction:

$$q_n = \vec{q} \cdot \vec{n} = \boldsymbol{q}^T \cdot \boldsymbol{n}$$

- A small segment with regular rectangular shape:

- Heat flux on the sides:
  - AD: $q_n = -q_x$
  - BC: $q_n = +q_x$
  - AB: $q_n = -q_y$
  - CD: $q_n = +q_y$

# Energy Balance of 2D Heat Conduction




- Energy balance in the control volume:

$$q_x\left(x - \frac{\Delta x}{2}, y\right)\Delta y - q_x\left(x + \frac{\Delta x}{2}, y\right)\Delta y + q_y\left(x, y - \frac{\Delta y}{2}\right)\Delta x - q_x\left(x, y + \frac{\Delta y}{2}\right)\Delta x + s(x, y)\Delta x\Delta y = 0$$

# Governing Equation for 2D Heat Conduction

$$q_x\left(x-\frac{\Delta x}{2},y\right)\Delta y - q_x\left(x+\frac{\Delta x}{2},y\right)\Delta y + q_y\left(x,y-\frac{\Delta y}{2}\right)\Delta x - q_x\left(x,y+\frac{\Delta y}{2}\right)\Delta x + s(x,y)\Delta x\Delta y = 0$$

$$\Rightarrow \frac{q_x\left(x-\frac{\Delta x}{2},y\right) - q_x\left(x+\frac{\Delta x}{2},y\right)}{\Delta x} + \frac{q_y\left(x,y-\frac{\Delta y}{2}\right) - q_x\left(x,y+\frac{\Delta y}{2}\right)}{\Delta y} + s(x,y) = 0$$

$$\lim_{\Delta x\to 0}\frac{q_x\left(x+\frac{\Delta x}{2},y\right) - q_x\left(x-\frac{\Delta x}{2},y\right)}{\Delta x} = \frac{\partial q_x}{\partial x}, \qquad \lim_{\Delta y\to 0}\frac{q_y\left(x,y+\frac{\Delta y}{2}\right) - q_x\left(x,y-\frac{\Delta y}{2}\right)}{\Delta y} = \frac{\partial q_y}{\partial y}$$

$$\Rightarrow \frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} - s = 0, \qquad div\vec{q} - s = \vec{\nabla}\cdot\vec{q} - s = 0, \qquad \boldsymbol{\nabla}^T\cdot\boldsymbol{q} - s = 0$$

# 2D Fourier's Law

$$\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} - s = 0, \qquad div\vec{q} - s = \vec{\nabla} \cdot \vec{q} - s = 0, \qquad \boldsymbol{\nabla}^T \cdot \boldsymbol{q} - s = 0$$

- Isotropic material:

$$\vec{q} = -k\vec{\nabla}T \; or \; \boldsymbol{q} = -k\boldsymbol{\nabla}T$$

$$\Rightarrow k\nabla^2 T + s = 0, \qquad \nabla^2 = \vec{\nabla} \cdot \vec{\nabla} = \boldsymbol{\nabla}^T \cdot \boldsymbol{\nabla} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

- Generalized (can be anisotropic) Fourier's Law:

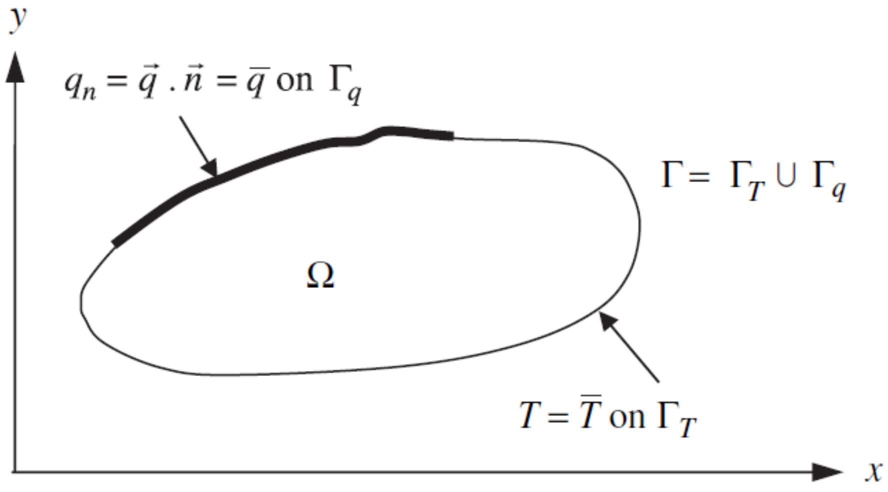
$$\boldsymbol{q} = -\underbrace{\begin{bmatrix} k_{xx} & k_{xy} \\ k_{yx} & k_{yy} \end{bmatrix}}_{\text{Heat conductivity matrix } \boldsymbol{D}} \begin{bmatrix} \frac{\partial T}{\partial x} \\ \frac{\partial T}{\partial y} \end{bmatrix} = -\boldsymbol{D} \cdot \boldsymbol{\nabla}T \Rightarrow \boldsymbol{\nabla}^T \cdot (\boldsymbol{D} \cdot \boldsymbol{\nabla}T) + s = 0$$

Heat conductivity matrix $\boldsymbol{D}$

Isotropic case:

$$\boldsymbol{D} = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix} = k\boldsymbol{I}$$

# Boundary Conditions of 2D Heat Conduction




- Requirements for boundary conditions:

$$\Gamma_q \cup \Gamma_T = \Gamma, \qquad \Gamma_q \cap \Gamma_T = \emptyset$$

- Temperature boundary condition:

$$T(x,y) = \bar{T}(x,y), \qquad (x,y) \in \Gamma_T$$

Essential/Dirichlet boundary condition

- Normal heat flux boundary condition:

$$q_n = \underline{\vec{q}} \cdot \vec{n} = \bar{q}, \qquad (x,y) \in \Gamma_q$$

$-k\vec{\nabla}T$

Natural/Neumann boundary condition

# Strong Form for 2D Heat Conduction

- Vector and matrix notations:

| Equation item | Vector notation | Matrix notation | Affecting region |
|---|---|---|---|
| Energy balance | $\vec{\nabla} \cdot \vec{q} - s = 0$ | $\boldsymbol{\nabla}^T \cdot \boldsymbol{q} - s = 0$ | $\Omega$ |
| Fourier's law | $\vec{q} = -k\vec{\nabla}T$ | $\boldsymbol{q} = -\boldsymbol{D} \cdot \boldsymbol{\nabla}T$ | $\Omega$ |
| Natural boundary | $q_n = \vec{q} \cdot \vec{n} = \bar{q}$ | $q_n = \boldsymbol{q}^T \cdot \boldsymbol{n} = \bar{q}$ | $\Gamma_q$ |
| Essential boundary | $T(x, y) = \bar{T}(x, y)$ | $T(x, y) = \bar{T}(x, y)$ | $\Gamma_T$ |

The governing equations and Fourier's law are not combined for convenience of subsequent derivation

- Both notations can be switched to each other for analysis:
  - Vector notation provides clear physical meaning and is more convenient for derivation
  - Matrix notation enables component analysis in each dimension and is more convenient for computation

# Derivation of Weak Form for 2D Heat Conduction (1/2)

- Governing equation and natural boundary condition (vector notation):

$$\vec{\nabla} \cdot \vec{q} - s = 0, \qquad q_n = \vec{q} \cdot \vec{n} = \bar{q}$$

- Introduction of the weight function and integration:

$$\int_{\Omega} \underline{w(\vec{\nabla} \cdot \vec{q}} - s)d\Omega = 0, \qquad \int_{\Gamma_q} w(\bar{q} - \vec{q} \cdot \vec{n})d\Gamma = 0, \qquad \forall w$$

- Green's formula:

$$\int_{\Omega} w\vec{\nabla} \cdot \vec{q}d\Omega = \oint_{\Gamma} w\vec{q} \cdot \vec{n}d\Gamma - \int_{\Omega} \vec{\nabla}w \cdot \vec{q}d\Omega$$

$$\int_{\Omega} \vec{\nabla}w \cdot \vec{q}d\Omega = \oint_{\Gamma} w\vec{q} \cdot \vec{n}d\Gamma - \int_{\Omega} wsd\Omega = \int_{\Gamma_q} w\vec{q} \cdot \vec{n}d\Gamma + \int_{\Gamma_T} w\vec{q} \cdot \vec{n}d\Gamma - \int_{\Omega} wsd\Omega, \qquad \forall w$$

Separation of essential and natural boundary conditions

# Derivation of Weak Form for 2D Heat Conduction (2/2)

$$\int_{\Omega} \vec{\nabla} w \cdot \vec{q} d\Omega = \int_{\Gamma_q} w \vec{q} \cdot \vec{n} d\Gamma + \int_{\Gamma_T} w \vec{q} \cdot \vec{n} d\Gamma - \int_{\Omega} w s d\Omega , \qquad q_n = \vec{q} \cdot \vec{n} = \bar{q}$$

$$\Rightarrow \int_{\Omega} \vec{\nabla} w \cdot \vec{q} d\Omega = \int_{\Gamma_q} w \bar{q} d\Gamma + \int_{\Gamma_T} \underbrace{w}_{0} q_n d\Gamma - \int_{\Omega} w s d\Omega$$

$$\Rightarrow \int_{\Omega} \vec{\nabla} w \cdot \vec{q} d\Omega = \int_{\Gamma_q} w \bar{q} d\Gamma - \int_{\Omega} w s d\Omega , \qquad \forall w \in U_0$$

- Matrix format:

$$\Rightarrow \int_{\Omega} (\boldsymbol{\nabla} w)^T \cdot \boldsymbol{q} d\Omega = \int_{\Gamma_q} w \bar{q} d\Gamma - \int_{\Omega} w s d\Omega , \qquad \forall w \in U_0$$

# Weak Form for 2D Heat Conduction

$$\int_{\Omega} (\boldsymbol{\nabla} w)^T \cdot \boldsymbol{q} d\Omega = \int_{\Gamma_q} w\bar{q} d\Gamma - \int_{\Omega} w s d\Omega, \forall w \in U_0, \qquad \boldsymbol{q} = -\boldsymbol{D} \cdot \boldsymbol{\nabla} T$$

- Weak form in matrix form for calculation:

  Find $T \in \underline{U}$ so that:

  Satisfy essential boundary conditions

Indispensable statement in weak form!

$$\int_{\Omega} (\boldsymbol{\nabla} w)^T \cdot \boldsymbol{D} \cdot \boldsymbol{\nabla} T d\Omega = -\int_{\Gamma_q} w\bar{q} d\Gamma + \int_{\Omega} w s d\Omega, \qquad \forall w \in \underline{U_0}$$

Arbitrary smooth functions vanishing on essential boundary

# Equivalence Between Strong and Weak Forms

From weak form to strong form:

$$\int_{\Omega} (\boldsymbol{\nabla} w)^T \cdot \boldsymbol{D} \cdot \boldsymbol{\nabla} T d\Omega = -\int_{\Gamma_q} w\bar{q} d\Gamma + \int_{\Omega} ws d\Omega, \qquad \boldsymbol{q} = -\boldsymbol{D} \cdot \boldsymbol{\nabla} T$$

$$\Rightarrow \int_{\Omega} (\boldsymbol{\nabla} w)^T \cdot \boldsymbol{q} d\Omega = -\int_{\Gamma_q} w\bar{q} d\Gamma + \int_{\Omega} ws d\Omega \;\; or \underline{\int_{\Omega} \vec{\nabla} w \cdot \vec{q} d\Omega} = \int_{\Gamma_q} w\bar{q} d\Gamma - \int_{\Omega} ws d\Omega$$

- Application of Green's formula (to vector format for derivation convenience):

$$\int_{\Omega} \vec{\nabla} w \cdot \vec{q} d\Omega = \oint_{\Gamma} w\vec{q} \cdot \vec{n} d\Gamma - \int_{\Omega} w\vec{\nabla} \cdot \vec{q} d\Omega$$

$$\Rightarrow 0 = \int_{\Omega} w\underbrace{(\vec{\nabla} \cdot \vec{q} - s)}_{0} d\Omega + \int_{\Gamma_q} w\underbrace{(\bar{q} - \vec{q} \cdot \vec{n})}_{0} d\Gamma - \underbrace{\int_{\Gamma_T} w\vec{q} \cdot \vec{n} d\Gamma}_{0}, \qquad \forall w \in U_0$$

# Expansion to 3D Problems

Derivation and expression of strong and weak forms are similar between 2D and 3D problems, the only difference are the specific component dimension of the related vectors/matrices.

- Vectors ($\vec{q}$) in 3D space:

$$\vec{q} = q_x\vec{\imath} + q_y\vec{\jmath} + q_z\vec{k}, \qquad \boldsymbol{q} = \begin{bmatrix} q_x \\ q_y \\ q_z \end{bmatrix}$$

- Gradient operator in 3D space:

$$\vec{\nabla} = \frac{\partial}{\partial x}\vec{\imath} + \frac{\partial}{\partial y}\vec{\jmath} + \frac{\partial}{\partial z}\vec{k}, \qquad \boldsymbol{\nabla} = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix}$$

- Divergence operator in 3D space:

$$div\vec{q} = \vec{\nabla} \cdot \vec{q} = \frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z}$$

- Laplacian operator in 3D space:

$$\nabla^2 = \vec{\nabla} \cdot \vec{\nabla} = \boldsymbol{\nabla}^T \cdot \boldsymbol{\nabla} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

- Heat conductivity matrix in 3D space:

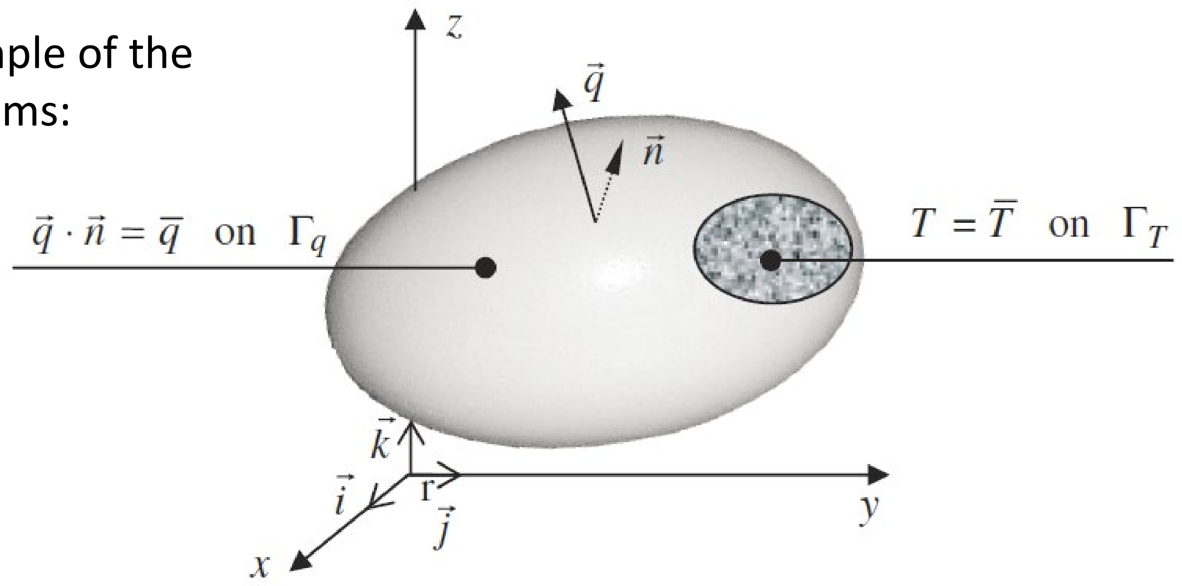
$$\boldsymbol{D} = \begin{bmatrix} k_{xx} & k_{xy} & k_{xz} \\ k_{yx} & k_{yy} & k_{yz} \\ k_{zx} & k_{zy} & k_{zz} \end{bmatrix} = \boldsymbol{D}^T$$

# Dimensionality of Domains and Boundaries

- Strong and weak forms of 1D to 3D problems can be transferred via changing the dimensionality of domains and boundaries:

| Entity | Domain $\Omega$ | Boundary $\Gamma$ |
|---|---|---|
| One dimension (1D) | Line segment | Two end points |
| Two dimensions (2D) | Two-dimensional area | Curve |
| Three dimensions (3D) | Volume | Surface |

One example of the 3D problems:



# The End