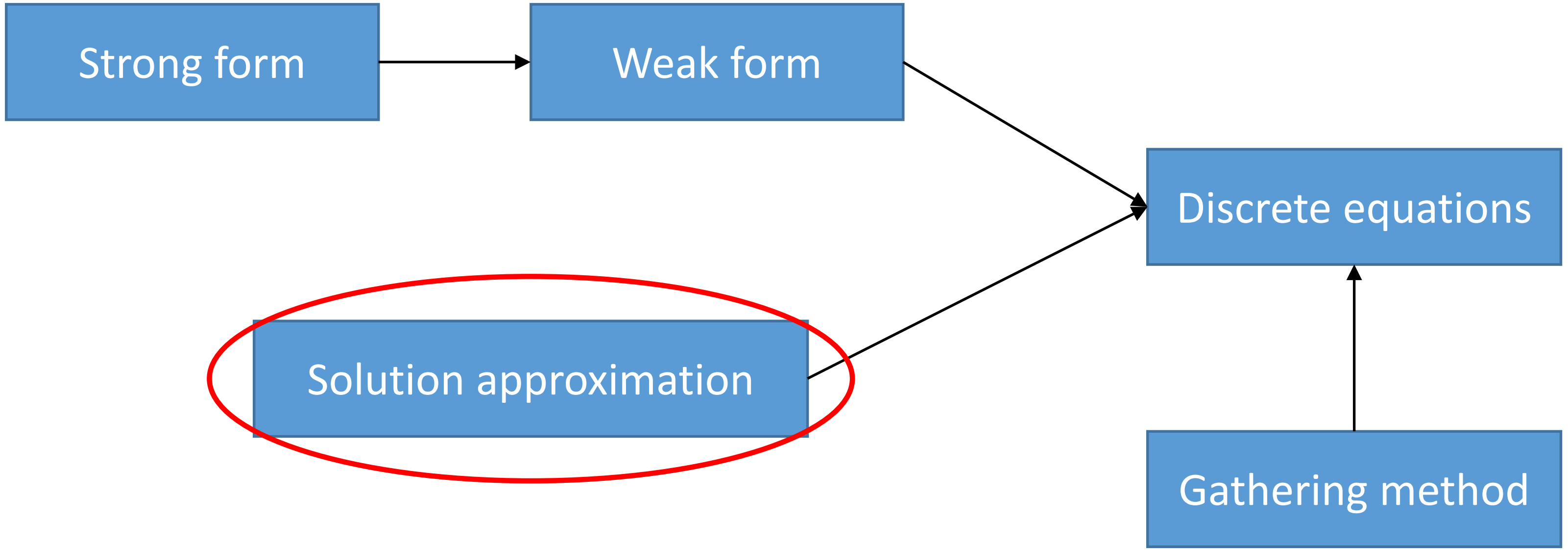
Computational Mechanics

Chapter 7 Trial Solutions, Weight Functions and Gauss Quadrature for Multidimensional Problems

Components for Formulation FEM Equations



Completeness and Continuity Requirement

- Completeness – approximation of solutions and (1st order) derivatives converge to arbitrary constants.
- 2D Completeness examination by the Pascal's triangle:

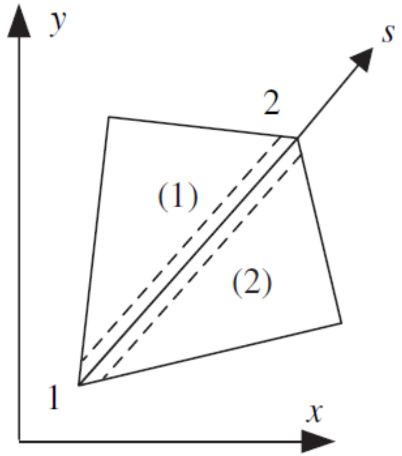
		1		constant
	x		y	linear
x^2		xy	y^2	quadratic
x^3	x^2y	xy^2	y^3	cubic

(a) $\theta^e(x,y) = \alpha_1^e + \alpha_2^e x + \alpha_3^e y$ ✓ Linear

(b) $\theta^e(x,y) = \alpha_1^e + \alpha_2^e x + \alpha_3^e y^2$ ✗

(c) $\theta^e(x,y) = \underbrace{\alpha_1^e + \alpha_2^e x + \alpha_3^e y}_{\text{Linear}} + \alpha_4^e x^2 y^2 + \alpha_5^e xy + \alpha_6^e y^3$ ✗

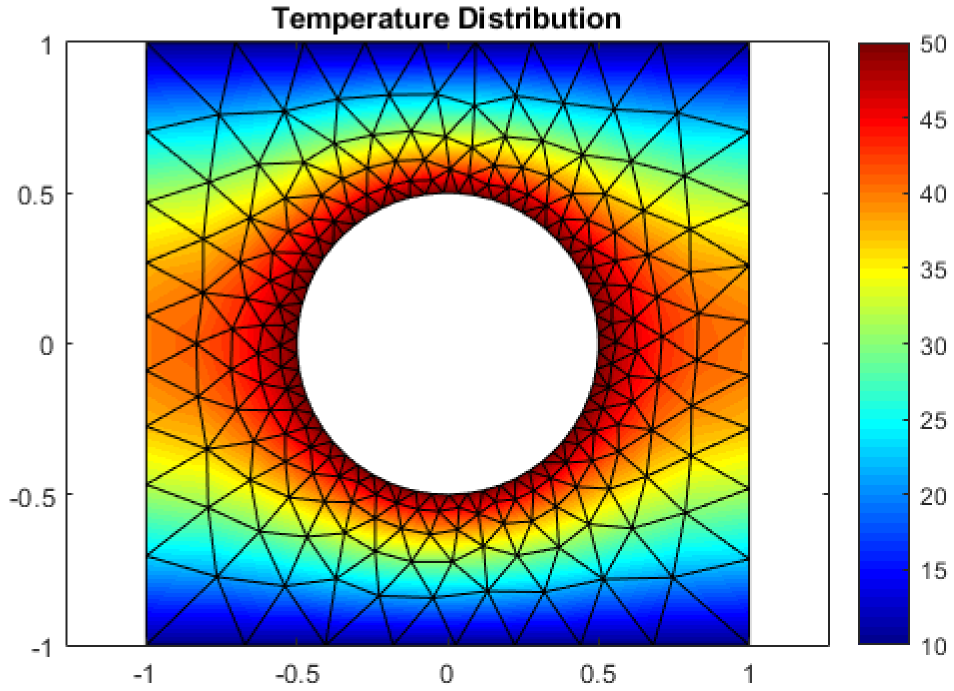
- Continuity - C^0 at the interfaces between elements:



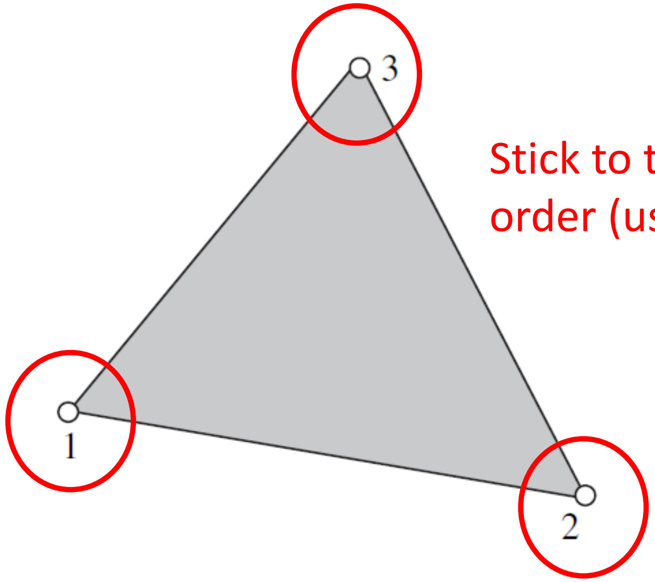
$$\theta^{(1)}(s) = \theta^{(2)}(s)$$

Introduction to 3-Node Triangular Elements

- Triangular elements handle complex geometry:



Triangular elements for heat conduction analysis [1]



Stick to the same numbering order (usually counterclockwise)

- Approximation of linear 2D elements:

$$\theta^e(x,y) = \alpha_0^e + \alpha_1^e x + \alpha_2^e y = [1 \quad x \quad y]\begin{bmatrix}\alpha_0^e \\ \alpha_1^e \\ \alpha_2^e\end{bmatrix} = \boldsymbol{p}\boldsymbol{\alpha}^e$$

3 constants – 3 nodes in triangular elements

[1] https://numfactory.upc.edu/web/FiniteElements.html

Shape Functions of 3-Node Triangular Elements (1/2)

$$\theta^e(x,y) = \begin{bmatrix} 1 & x & y \end{bmatrix} \begin{bmatrix} \alpha_0^e \\ \alpha_1^e \\ \alpha_2^e \end{bmatrix} = \boldsymbol{p}(x,y) \cdot \boldsymbol{\alpha}^e$$

- Nodal value conditions:

$$\theta_1^e = \begin{bmatrix} 1 & x_1^e & y_1^e \end{bmatrix} \begin{bmatrix} \alpha_0^e \\ \alpha_1^e \\ \alpha_2^e \end{bmatrix}$$

$$\theta_2^e = \begin{bmatrix} 1 & x_2^e & y_2^e \end{bmatrix} \begin{bmatrix} \alpha_0^e \\ \alpha_1^e \\ \alpha_2^e \end{bmatrix}$$

$$\theta_3^e = \begin{bmatrix} 1 & x_3^e & y_3^e \end{bmatrix} \begin{bmatrix} \alpha_0^e \\ \alpha_1^e \\ \alpha_2^e \end{bmatrix}$$

$$\Rightarrow \underbrace{\begin{bmatrix} \theta_1^e \\ \theta_2^e \\ \theta_3^e \end{bmatrix}}_{\boldsymbol{d}^e} = \underbrace{\begin{bmatrix} 1 & x_1^e & y_1^e \\ 1 & x_2^e & y_2^e \\ 1 & x_3^e & y_3^e \end{bmatrix}}_{\boldsymbol{M}^e} \underbrace{\begin{bmatrix} \alpha_0^e \\ \alpha_1^e \\ \alpha_2^e \end{bmatrix}}_{\boldsymbol{\alpha}^e}$$

$$\Rightarrow \boldsymbol{\alpha}^e = (\boldsymbol{M}^e)^{-1} \cdot \boldsymbol{d}^e$$

$$\Rightarrow \theta^e(x,y) = \boldsymbol{p}(x,y) \cdot \boldsymbol{\alpha}^e = \underbrace{\boldsymbol{p}(x,y) \cdot (\boldsymbol{M}^e)^{-1}}_{\boldsymbol{N}^e(x,y)} \cdot \boldsymbol{d}^e$$

$$\Rightarrow \boldsymbol{p}_{1\times 3} \cdot (\boldsymbol{M}^e)^{-1}_{3\times 3} = \boldsymbol{N}^e_{1\times 3} = \begin{bmatrix} N_1^e & N_2^e & N_3^e \end{bmatrix}$$

Shape Functions of 3-Node Triangular Elements (2/2)

$$[N_1^e \quad N_2^e \quad N_3^e] = \boldsymbol{p} \cdot (\boldsymbol{M}^e)^{-1} = [1 \quad x \quad y]\begin{bmatrix} 1 & x_1^e & y_1^e \\ 1 & x_2^e & y_2^e \\ 1 & x_3^e & y_3^e \end{bmatrix}^{-1}$$

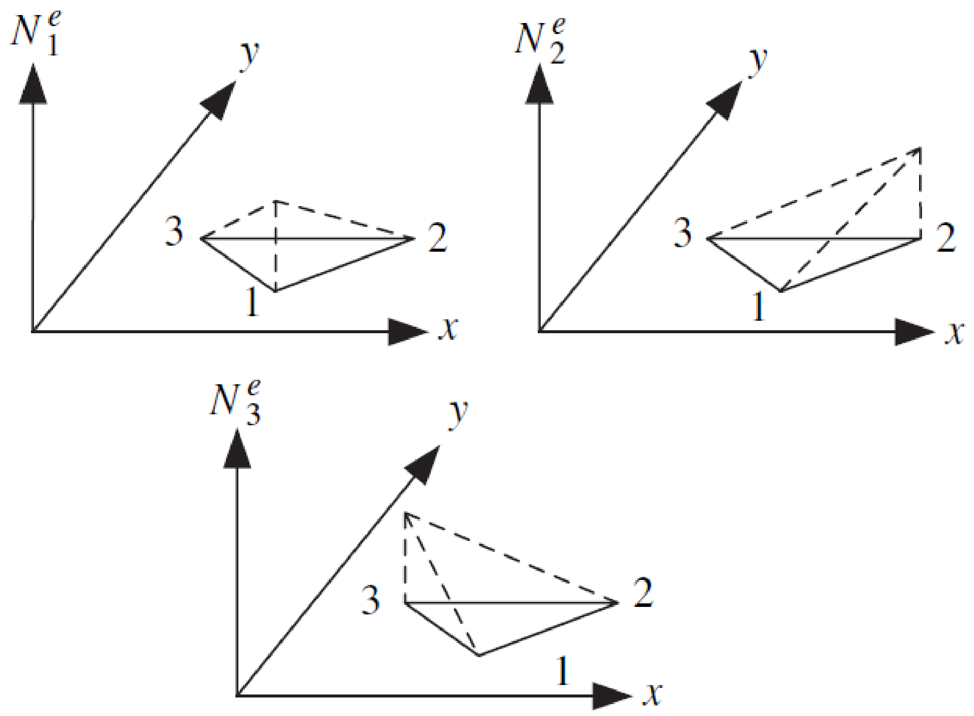
$$\Rightarrow N_1^e = \frac{1}{2A^e}[x_2^e y_3^e - x_3^e y_2^e + (y_2^e - y_3^e)x + (x_3^e - x_2^e)y]$$

$$N_2^e = \frac{1}{2A^e}[x_3^e y_1^e - x_1^e y_3^e + (y_3^e - y_1^e)x + (x_1^e - x_3^e)y]$$

$$N_3^e = \frac{1}{2A^e}[x_1^e y_2^e - x_2^e y_1^e + (y_1^e - y_2^e)x + (x_2^e - x_1^e)y]$$

- Kronecker delta property of shape functions:

$$N_I^e\left(x_J^e, y_J^e\right) = \delta_{IJ}$$



$\boldsymbol{B}^e$ of 3-Node Triangular Elements

$$\boldsymbol{N}^e = [N_1^e \quad N_2^e \quad N_3^e], \qquad N_1^e = \frac{1}{2A^e}[x_2^e y_3^e - x_3^e y_2^e + (y_2^e - y_3^e)x + (x_3^e - x_2^e)y]$$

$$N_2^e = \frac{1}{2A^e}[x_3^e y_1^e - x_1^e y_3^e + (y_3^e - y_1^e)x + (x_1^e - x_3^e)y], N_3^e = \frac{1}{2A^e}[x_1^e y_2^e - x_2^e y_1^e + (y_1^e - y_2^e)x + (x_2^e - x_1^e)y]$$

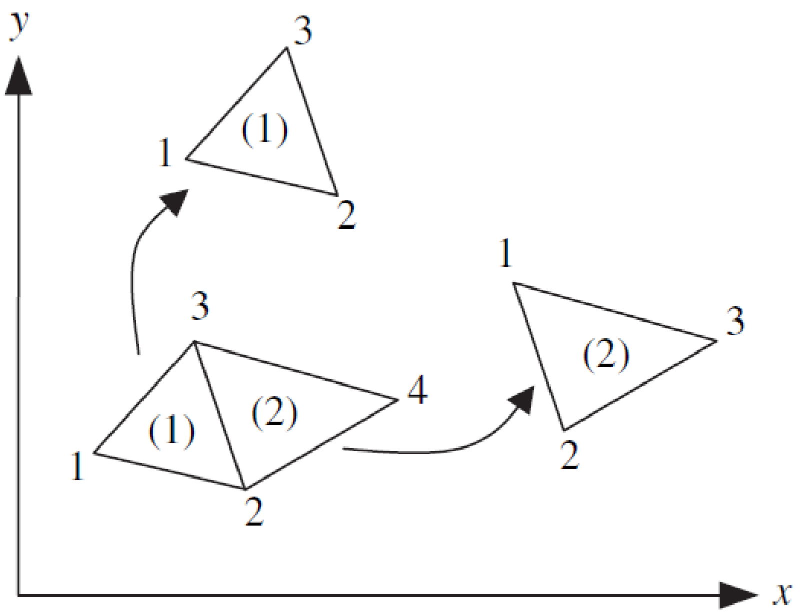
$$\theta^e(x,y) = \boldsymbol{N}^e \cdot \boldsymbol{d}^e = N_1^e\theta_1 + N_2^e\theta_2 + N_3^e\theta_3$$

- Gradient of the trial solution:

Linear elements lead to constant $\boldsymbol{B}^e$ that are only dependent on element coordinates

$$\boldsymbol{\nabla}\theta = \begin{bmatrix} \frac{\partial\theta}{\partial x} \\ \frac{\partial\theta}{\partial y} \end{bmatrix} = \begin{bmatrix} \frac{\partial N_1^e}{\partial x}\theta_1 + \frac{\partial N_2^e}{\partial x}\theta_2 + \frac{\partial N_3^e}{\partial x}\theta_3 \\ \frac{\partial N_1^e}{\partial y}\theta_1 + \frac{\partial N_2^e}{\partial y}\theta_2 + \frac{\partial N_3^e}{\partial y}\theta_3 \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{\partial N_1^e}{\partial x} & \frac{\partial N_2^e}{\partial x} & \frac{\partial N_3^e}{\partial x} \\ \frac{\partial N_1^e}{\partial y} & \frac{\partial N_2^e}{\partial y} & \frac{\partial N_3^e}{\partial y} \end{bmatrix}}_{\frac{1}{2A^e}\begin{bmatrix} (y_2^e - y_3^e) & (y_3^e - y_1^e) & (y_1^e - y_2^e) \\ (x_3^e - x_2^e) & (x_1^e - x_3^e) & (x_2^e - x_1^e) \end{bmatrix}} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix} = \boldsymbol{B}^e\boldsymbol{d}^e$$

Global Continuity of 3-Node Triangular Elements



- Parametric equations for global edge 2-3:

$$x = x_2 + (x_3 - x_2)s, \qquad y = y_2 + (y_3 - y_2)s$$

$$0 \le s \le 1$$

- Linear shape functions on global edge 2-3:

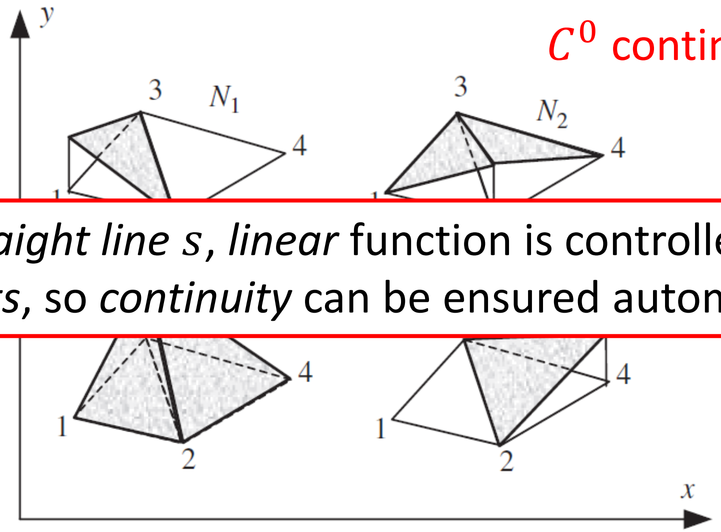
$$\theta^{(1)} = \beta_0^{(1)} + \beta_1^{(1)} s, \qquad \theta^{(2)} = \beta_0^{(2)} + \beta_1^{(2)} s$$

- Nodal value conditions:

$$\theta_2 = \beta_0^{(1)} = \beta_0^{(2)}, \qquad \theta_3 = \beta_0^{(1)} + \beta_1^{(1)} = \beta_0^{(2)} + \beta_1^{(2)}$$

$$\Rightarrow \beta_0^{(1)} = \beta_0^{(2)} = \theta_2, \qquad \beta_1^{(1)} = \beta_1^{(2)} = \theta_3 - \theta_2$$

C^0 continuity

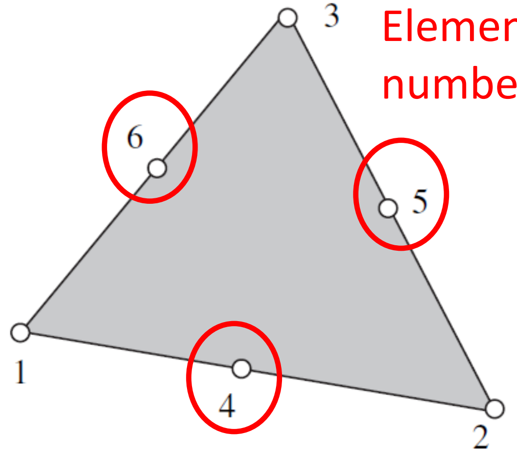


Along *straight line* s, *linear* function is controlled by *two points*, so *continuity* can be ensured automatically

Higher Order Triangular Elements

		1		constant
	x		y	linear
x^2		xy	y^2	quadratic
x^3	x^2y	xy^2	y^3	cubic

Similar format



Elements have the same numbering pattern

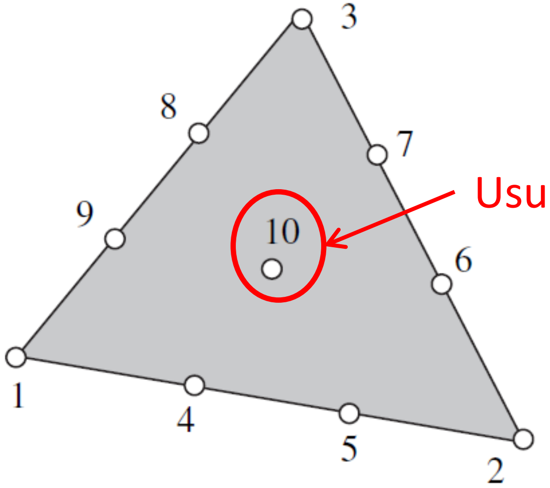
- Quadratic element:
  $$\theta^e = \alpha_1^e + \alpha_2^e x + \alpha_3^e y + \alpha_4^e x^2 + \alpha_5^e xy + \alpha_6^e y^2$$

- Projection to a straight line along the edge:
  $$x = a + bs, \qquad y = c + ds$$

  $$\Rightarrow \theta^e = \underline{\beta_0^e + \beta_1^e s + \beta_2^e s^2}$$

  3 nodes on the edges are essential for continuity.

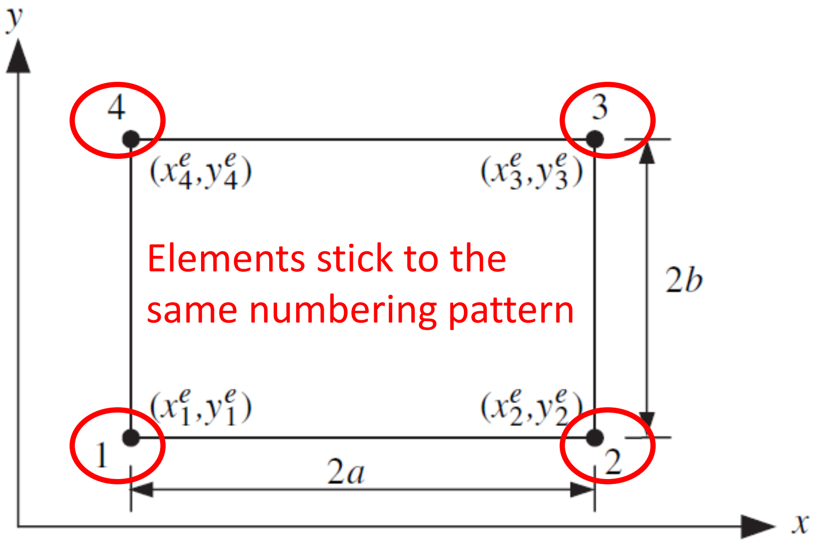
- Inner nodes for higher order elements (seldomly used as not well conditioned):



Usually centroid

4-Node Rectangular Elements

- Avoid shear locking issue
- Counterclockwise nodal numbering is often used:



Bilinear term for edge continuity

			1				constant
		x		y			linear
	x^2		xy		y^2		quadratic
x^3		x^2y		xy^2		y^3	cubic

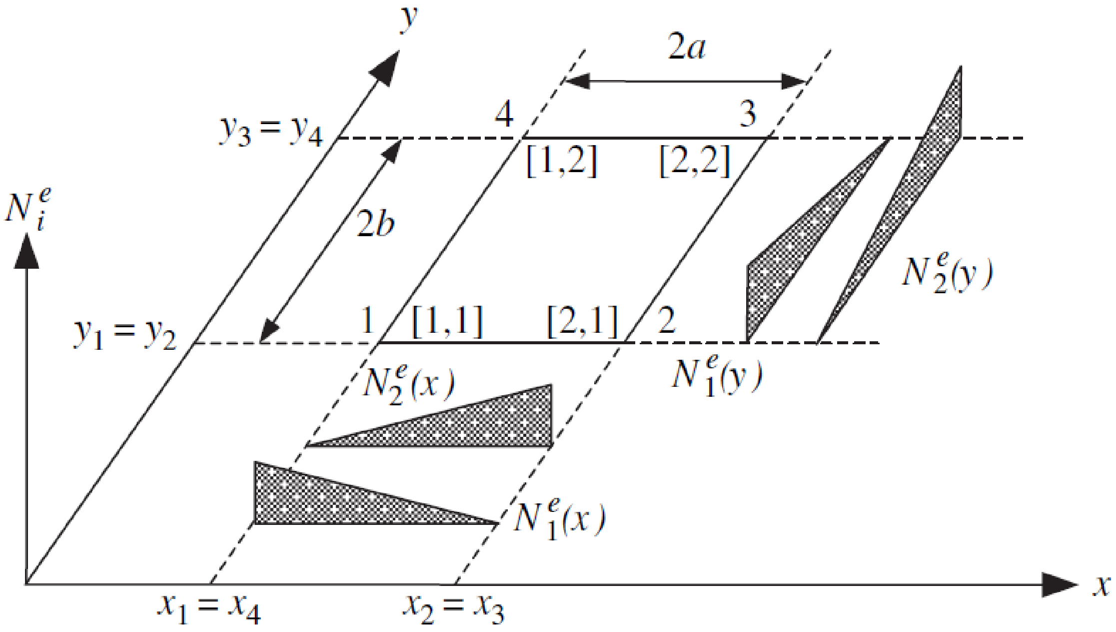
- Element approximation:

$$\theta^e(x, y) = \alpha_1^e + \alpha_2^e x + \alpha_3^e y + \alpha_4^e xy$$

- Direct shape function construction is essential to obtain $(\alpha_1^e, \alpha_2^e, \alpha_3^e, \alpha_4^e) = \boldsymbol{\alpha}(\theta_1^e, \theta_2^e, \theta_3^e, \theta_4^e)$ for manual calculation.

Tensor Product Method

- Increase dimension and keep interpolation by multiplication of shape functions:



- Shape functions in dyadic notation:

$$N_{[I,J]}^e(x,y) = N_I^e(x)N_J^e(y)$$

- Kronecker delta properties:

$$N_{[I,J]}^e(x_M^e, y_L^e) = N_I^e(x_M^e)N_J^e(y_L^e) = \delta_{IM}\delta_{JL}$$

- Shape functions in regular notation:

$$N_1^e(x,y) = \frac{x - x_2^e}{x_1^e - x_2^e}\frac{y - y_4^e}{y_1^e - y_4^e} = \frac{1}{A^e}(x - x_2^e)(y - y_4^e)$$

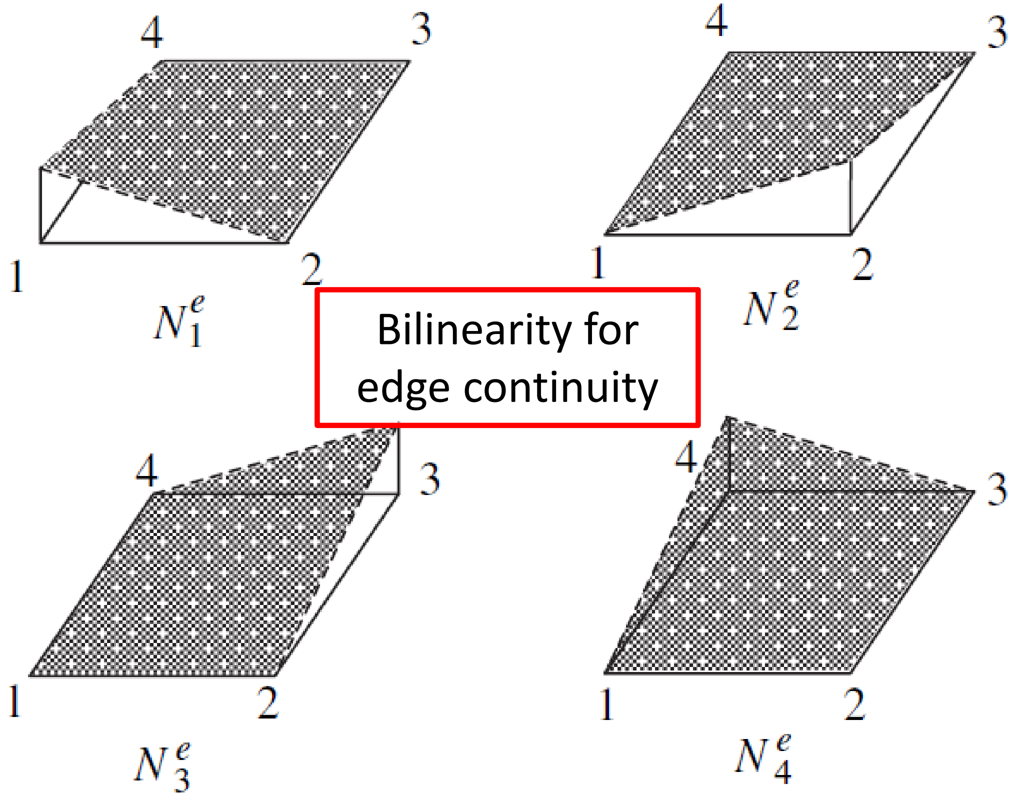
$$N_2^e(x,y) = \frac{x - x_1^e}{x_2^e - x_1^e}\frac{y - y_4^e}{y_1^e - y_4^e} = \frac{-1}{A^e}(x - x_1^e)(y - y_4^e)$$

$$N_3^e(x,y) = \frac{x - x_1^e}{x_2^e - x_1^e}\frac{y - y_1^e}{y_4^e - y_1^e} = \frac{1}{A^e}(x - x_1^e)(y - y_1^e)$$

$$N_4^e(x,y) = \frac{x - x_2^e}{x_1^e - x_2^e}\frac{y - y_1^e}{y_4^e - y_1^e} = \frac{-1}{A^e}(x - x_2^e)(y - y_1^e)$$

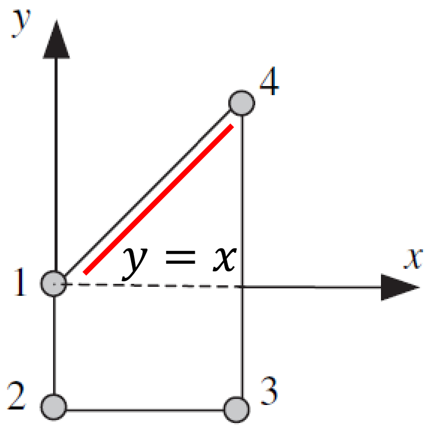
Properties of Bilinear Shape Functions

- Geometry of the shape functions:



Bilinearity for edge continuity

- Drawbacks of the specific shape functions – cannot work for arbitrary bilateral elements:



$$N_1^e(x,y) = \frac{1}{A^e}(x - x_2^e)(y - y_4^e)$$

$$N_2^e(x,y) = \frac{-1}{A^e}(x - x_1^e)(y - y_4^e)$$

$$N_3^e(x,y) = \frac{1}{A^e}(x - x_1^e)(y - y_1^e)$$

$$N_4^e(x,y) = \frac{-1}{A^e}(x - x_2^e)(y - y_1^e)$$

The *quadratic function* N_4^e on the edge needs *3 nodes on the edge* to ensure continuity

Introduction to Isoparametric Elements

- Linear mapping from physical domains to a regular parent domain:

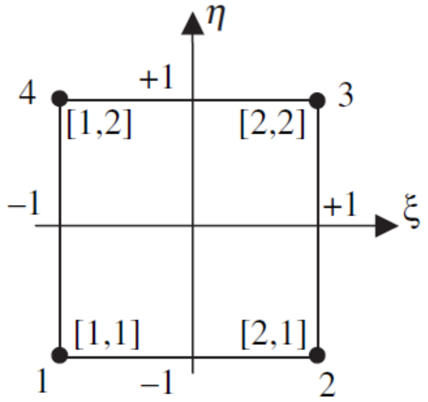
$$x = x_1^e N_1^e(\xi) + x_2^e N_2^e(\xi) = x_1^e \frac{1-\xi}{2} + x_2^e \frac{1+\xi}{2}, \qquad \xi \in [-1, 1]$$

Isoparametric element

- Linear approximation of θ by ξ instead of x – direct construction:

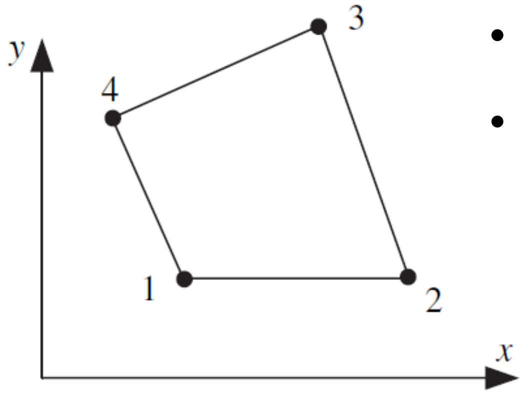
$$\theta^e = \theta_1^e N_1^e(\xi) + \theta_2^e N_2^e(\xi) = \theta_1^e \frac{1-\xi}{2} + \theta_2^e \frac{1+\xi}{2}, \qquad \xi \in [-1, 1]$$

- Quadrilateral elements in the regular parent domain:



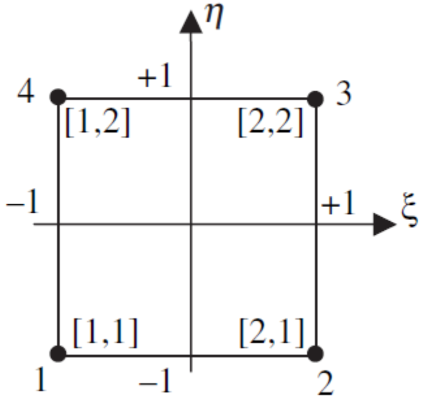
$$x = \boldsymbol{N}^{4Q}(\xi, \eta)\boldsymbol{x}^e$$

$$y = \boldsymbol{N}^{4Q}(\xi, \eta)\boldsymbol{y}^e$$



- $\boldsymbol{N}^{4Q}(\xi, \eta)$: 4-node shape functions.
- $\boldsymbol{x}^e = [x_1^e \quad x_2^e \quad x_3^e \quad x_4^e]$
- $\boldsymbol{y}^e = [y_1^e \quad y_2^e \quad y_3^e \quad y_4^e]$
  - ➢ Physical coordinates are mapped by the same functions as those for trial solutions
  - ➢ Straight *edges* to straight *edges*

Components in $\boldsymbol{N}^{4Q}(\xi, \eta)$



- Tensor product method for shape functions:

$$N_1^e(\xi, \eta) = \frac{1}{4}(\xi - 1)(\eta - 1)$$
$$N_2^e(\xi, \eta) = \frac{-1}{4}(\xi + 1)(\eta - 1)$$
$$N_3^e(\xi, \eta) = \frac{1}{4}(\xi + 1)(\eta + 1)$$
$$N_4^e(\xi, \eta) = \frac{-1}{4}(\xi - 1)(\eta + 1)$$

Independent of physical nodal positions.

- Trial solution approximation:

$$\theta^e = \boldsymbol{N}^{4Q}(\xi, \eta)\boldsymbol{d}^e$$

Same as mapping – isoparametric.

- Bilinearity of shape functions:

$$\theta^e(\xi, \eta) = \alpha_1^e + \alpha_2^e \xi + \alpha_3^e \eta + \alpha_4^e \xi \eta$$

Linear on the edge – C^0 continuity along physical edges

Derivative of Isoparametric Shape Functions (1/2)

$$\theta^e = \boldsymbol{N}^{4Q}(\xi, \eta)\boldsymbol{d}^e = \sum_{i=1}^{4} N_i^{4Q} d_i^e$$

- Gradient of the trial solution:

$$\boldsymbol{\nabla}\theta^e = \begin{bmatrix} \dfrac{\partial \theta^e}{\partial x} \\ \dfrac{\partial \theta^e}{\partial y} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial \sum_{i=1}^{4} N_i^{4Q} d_i^e}{\partial x} \\ \dfrac{\partial \sum_{i=1}^{4} N_i^{4Q} d_i^e}{\partial y} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial N_1^{4Q}}{\partial x} & \dfrac{\partial N_2^{4Q}}{\partial x} & \dfrac{\partial N_3^{4Q}}{\partial x} & \dfrac{\partial N_4^{4Q}}{\partial x} \\ \dfrac{\partial N_1^{4Q}}{\partial y} & \dfrac{\partial N_2^{4Q}}{\partial y} & \dfrac{\partial N_3^{4Q}}{\partial y} & \dfrac{\partial N_4^{4Q}}{\partial y} \end{bmatrix} \begin{bmatrix} d_2^e \\ d_2^e \\ d_3^e \\ d_4^e \end{bmatrix} = \boldsymbol{B}^e \boldsymbol{d}^e$$

- Derivatives on different domains – chain rule:

$$\frac{\partial N_I^{4Q}}{\partial \xi} = \frac{\partial N_I^{4Q}}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial N_I^{4Q}}{\partial y}\frac{\partial y}{\partial \xi}, \qquad \frac{\partial N_I^{4Q}}{\partial \eta} = \frac{\partial N_I^{4Q}}{\partial x}\frac{\partial x}{\partial \eta} + \frac{\partial N_I^{4Q}}{\partial y}\frac{\partial y}{\partial \eta}$$

$$\Rightarrow \begin{bmatrix} \dfrac{\partial N_I^{4Q}}{\partial \xi} \\ \dfrac{\partial N_I^{4Q}}{\partial \eta} \end{bmatrix} = \underbrace{\begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{bmatrix}}_{\text{Jacobian } \boldsymbol{J}^e} \begin{bmatrix} \dfrac{\partial N_I^{4Q}}{\partial x} \\ \dfrac{\partial N_I^{4Q}}{\partial y} \end{bmatrix}$$

Derivative of Isoparametric Shape Functions (2/2)

$$\begin{bmatrix} \frac{\partial N_I^{4Q}}{\partial \xi} \\ \frac{\partial N_I^{4Q}}{\partial \eta} \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} \begin{bmatrix} \frac{\partial N_I^{4Q}}{\partial x} \\ \frac{\partial N_I^{4Q}}{\partial y} \end{bmatrix} = \boldsymbol{J}^e \begin{bmatrix} \frac{\partial N_I^{4Q}}{\partial x} \\ \frac{\partial N_I^{4Q}}{\partial y} \end{bmatrix}$$

$$\Rightarrow \underbrace{\begin{bmatrix} \frac{\partial N_I^{4Q}}{\partial x} \\ \frac{\partial N_I^{4Q}}{\partial y} \end{bmatrix}}_{\boldsymbol{\nabla} N_I^{4Q}} = (\boldsymbol{J}^e)^{-1} \underbrace{\begin{bmatrix} \frac{\partial N_I^{4Q}}{\partial \xi} \\ \frac{\partial N_I^{4Q}}{\partial \eta} \end{bmatrix}}_{\boldsymbol{G} N_I^{4Q}}$$

$$\Rightarrow \boldsymbol{\nabla N}^{4Q} = (\boldsymbol{J}^e)^{-1} \boldsymbol{G N}^{4Q}$$

$$x = \boldsymbol{N}^{4Q}(\xi, \eta)\boldsymbol{x}^e, \qquad y = \boldsymbol{N}^{4Q}(\xi, \eta)\boldsymbol{y}^e$$

$$\Rightarrow \boldsymbol{J}^e = \begin{bmatrix} \frac{\partial \boldsymbol{N}^{4Q}\boldsymbol{x}^e}{\partial \xi} & \frac{\partial \boldsymbol{N}^{4Q}\boldsymbol{y}^e}{\partial \xi} \\ \frac{\partial \boldsymbol{N}^{4Q}\boldsymbol{x}^e}{\partial \eta} & \frac{\partial \boldsymbol{N}^{4Q}\boldsymbol{y}^e}{\partial \eta} \end{bmatrix} = \boldsymbol{G N}^{4Q}[\boldsymbol{x}^e \quad \boldsymbol{y}^e]$$

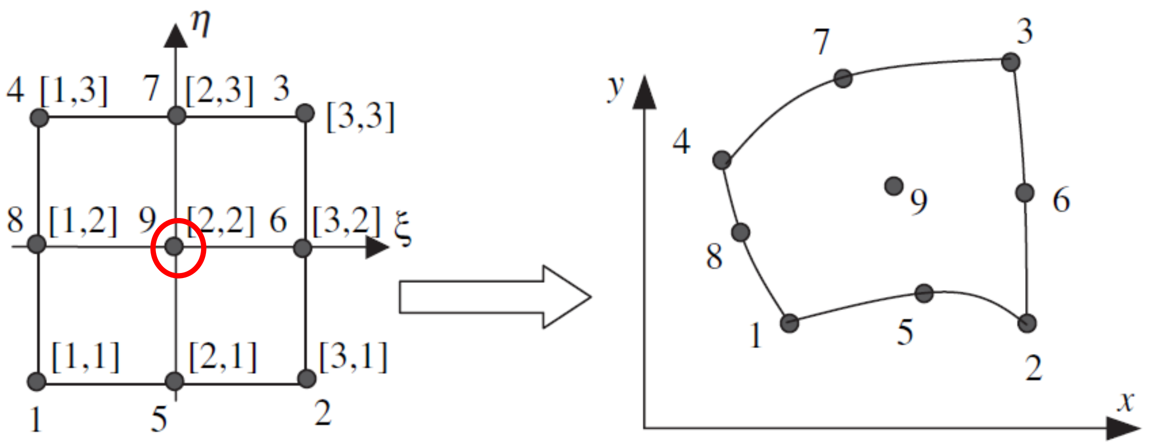
$$\boldsymbol{B}^e = \begin{bmatrix} \frac{\partial N_1^{4Q}}{\partial x} & \frac{\partial N_2^{4Q}}{\partial x} & \frac{\partial N_3^{4Q}}{\partial x} & \frac{\partial N_4^{4Q}}{\partial x} \\ \frac{\partial N_1^{4Q}}{\partial y} & \frac{\partial N_2^{4Q}}{\partial y} & \frac{\partial N_3^{4Q}}{\partial y} & \frac{\partial N_4^{4Q}}{\partial y} \end{bmatrix} = \nabla \boldsymbol{N}^{4Q}$$

$$\Rightarrow \underline{\boldsymbol{B}^e} = \nabla \boldsymbol{N}^{4Q} = \underline{(\boldsymbol{J}^e)}^{-1} \boldsymbol{G N}^{4Q}$$

Depend on physical domains

Higher Order (Quadratic) Quadrilateral Elements

- 3 nodes on *edges* for continuity
- Node 9 for unique solution and *physical coordinates*



$$N_K^{9Q}(\xi,\eta) = N_I^{3L}(\xi)N_J^{3L}(\eta)$$

$$x(\xi,\eta) = \boldsymbol{N}^{9Q}(\xi,\eta)\boldsymbol{x}^e, \qquad y(\xi,\eta) = \boldsymbol{N}^{9Q}(\xi,\eta)\boldsymbol{y}^e$$

- Mapping of curved edges (1-2 edge as an example):

$$x(\xi,\eta = const) = \alpha_0^x + \alpha_1^x\xi + \alpha_2^x\xi^2, \qquad y(\xi,\eta = const) = \alpha_0^y + \alpha_1^y\xi + \alpha_2^y\xi^2$$

Nonlinear (quadratic) relationship between x and y, leading to fewer elements for curved boundary *approximation*

Completeness of Isoparametric Elements

- Isoparamatric elements are at least linear complete – 1D quadratic element example:

$$x(\xi) = \sum_{I=1}^{3} x_I^e N_I^{3L}(\xi) = x_1^e \frac{1}{2}\xi(\xi - 1) + x_2^e(1 - \xi^2) + x_3^e \frac{1}{2}\xi(\xi + 1)$$

$$\theta^e(\xi) = \sum_{I=1}^{3} \theta_I^e N_I^{3L}(\xi) = \theta_1^e \frac{1}{2}\xi(\xi - 1) + \theta_2^e(1 - \xi^2) + \theta_3^e \frac{1}{2}\xi(\xi + 1)$$

Linear terms?

- Assumption of a linear field:

$$\theta^e(x) = \alpha_0 + \alpha_1 x \Rightarrow \theta_I^e = \alpha_0 + \alpha_1 x_I^e$$

$$\Rightarrow \theta^e(x) = \sum_{I=1}^{3}(\alpha_0 + \alpha_1 x_I^e) N_I^{3L}(\xi) = \alpha_0 \underbrace{\sum_{I=1}^{3} N_I^{3L}(\xi)}_{1} + \alpha_1 \underbrace{\sum_{I=1}^{3} x_I^e N_I^{3L}(\xi)}_{x}$$

1D linear completeness satisfied, higher order may be possible

Completeness of *2D* Isoparametric Elements

- Coordinate mapping and solution approximation:

$$x = \sum_{I=1}^{n_{en}} x_I^e N_I^e \,, \qquad y = \sum_{I=1}^{n_{en}} y_I^e N_I^e \,, \qquad \theta^e = \sum_{I=1}^{n_{en}} \theta_I^e N_I^e \,, \qquad N_I^e = N_I^e(\xi, \eta)$$

- Assumption of a linear field:

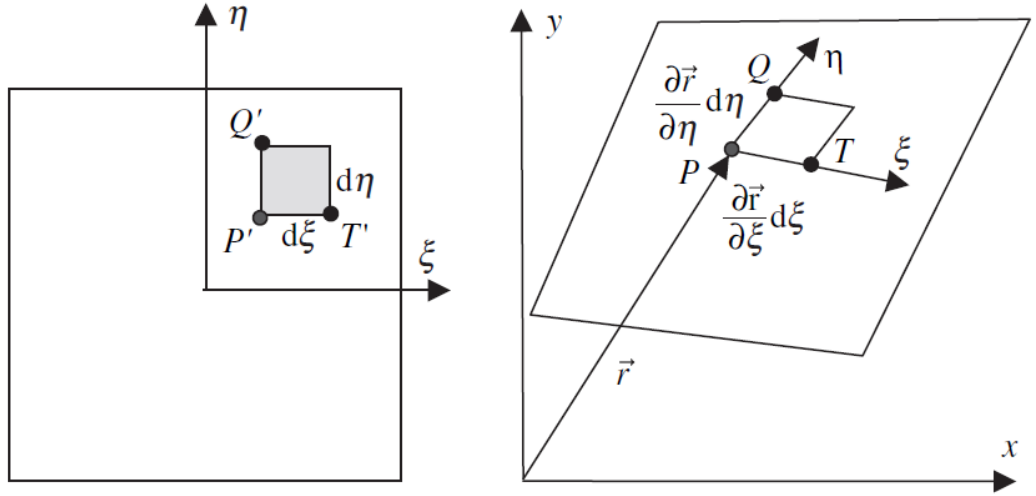
$$\theta^e(x) = \alpha_0 + \alpha_1 x + \alpha_2 y \Rightarrow \theta_I^e = \alpha_0 + \alpha_1 x_I^e + \alpha_2 y_I^e$$

$$\theta^e = \sum_{I=1}^{n_{en}} \theta_I^e N_I^e = \sum_{I=1}^{n_{en}} (\alpha_0 + \alpha_1 x_I^e + \alpha_2 y_I^e) N_I^e = \alpha_0 \underbrace{\sum_{I=1}^{n_{en}} N_I^e}_{1} + \alpha_1 \underbrace{\sum_{I=1}^{n_{en}} x_I^e N_I^e}_{x} + \alpha_2 \underbrace{\sum_{I=1}^{n_{en}} y_I^e N_I^e}_{y}$$

2D linear completeness satisfied, higher order may be possible

Gauss Quadrature for Quadrilateral Elements

- Area mapping:



$$\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j}$$

- Infinitesimal area $d\Omega$:

$$\boldsymbol{PT} = \frac{\partial \boldsymbol{r}}{\partial \xi} d\xi = \frac{\partial x}{\partial \xi} d\xi \boldsymbol{i} + \frac{\partial y}{\partial \xi} d\xi \boldsymbol{j}$$

$$\boldsymbol{PQ} = \frac{\partial \boldsymbol{r}}{\partial \eta} d\eta = \frac{\partial x}{\partial \eta} d\eta \boldsymbol{i} + \frac{\partial y}{\partial \eta} d\eta \boldsymbol{j}$$

$$\Rightarrow d\Omega = |\boldsymbol{PT} \times \boldsymbol{PQ}| = \left| \frac{\partial x}{\partial \xi}\frac{\partial y}{\partial \eta} - \frac{\partial y}{\partial \xi}\frac{\partial x}{\partial \eta} \right| d\xi d\eta = \underbrace{\begin{vmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{vmatrix}}_{\boldsymbol{J}^e} d\xi d\eta$$

$$\Rightarrow I = \iint f(x, y) d\Omega = \iint_{(-1,1)}^{(-1,1)} f(\xi, \eta) |\boldsymbol{J}^e| d\xi d\eta$$

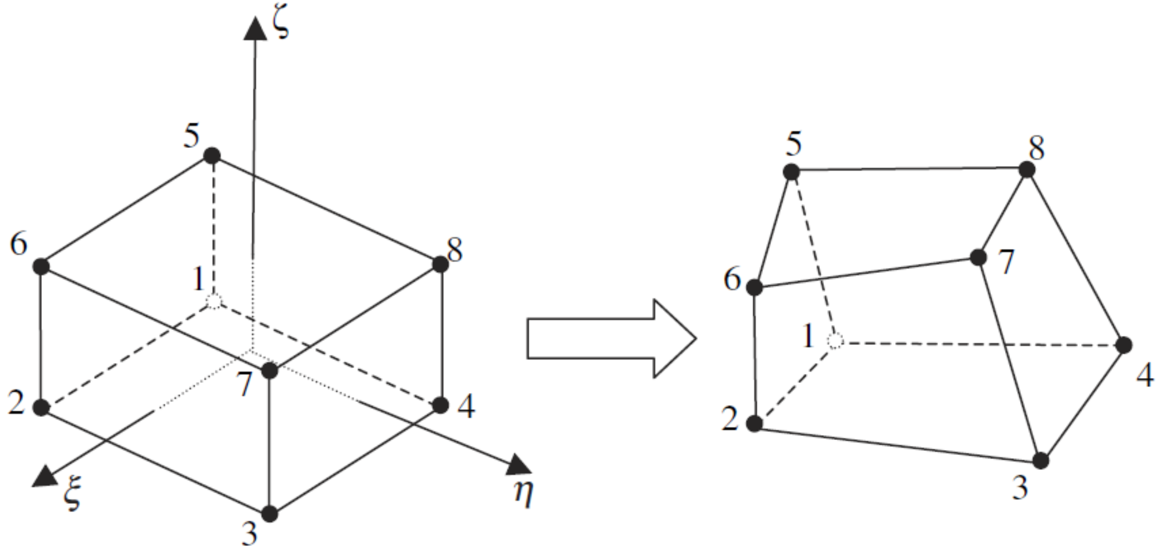
$$\Rightarrow I = \int_{-1}^{1} \underbrace{\left(\sum_{i=1}^{n_{gp}} W_i f(\xi_i, \eta) |\boldsymbol{J}^e(\xi_i, \eta)| \right)}_{\text{Same as 1D Gauss Quadrature}} d\eta = \cdots$$

Hexahedra Elements

- Hexahedra elements are 3D expansion of 2D quadrilateral elements.

- Mapping from regular parent to physical domains:

$$x = \boldsymbol{N}^{8H}\boldsymbol{x}^e, \qquad y = \boldsymbol{N}^{8H}\boldsymbol{y}^e, \qquad z = \boldsymbol{N}^{8H}\boldsymbol{z}^e$$



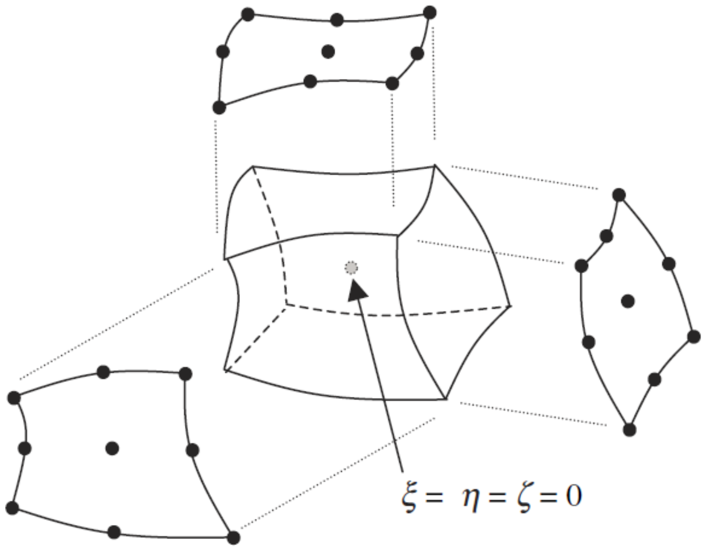
- Construction of shape functions – linear example:

$$N_L^{8H}(\xi,\eta,\varsigma) = N_I^{2L}(\xi)N_J^{2L}(\eta)N_K^{2L}(\varsigma)$$

- Solution approximation:

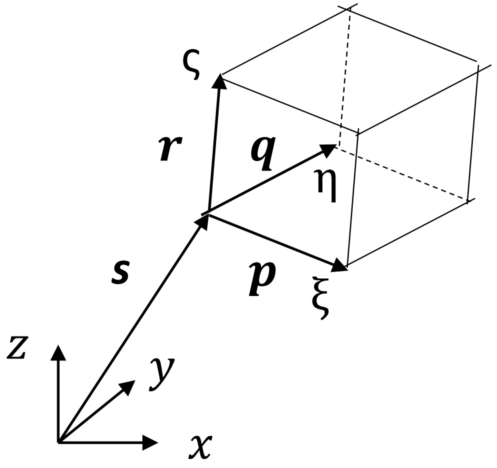
$$\theta^e = \boldsymbol{N}^{8H}\boldsymbol{d}^e$$

- Higher order elements approximate curved surfaces:



Gauss integration for Hexahedral elements

- Infinitesimal volume $d\Omega$ mapping:



$$\boldsymbol{s} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$$

$$\Rightarrow \boldsymbol{p} = \frac{\partial \boldsymbol{s}}{\partial \xi} d\xi = \left(\frac{\partial x}{\partial \xi}\boldsymbol{i} + \frac{\partial y}{\partial \xi}\boldsymbol{j} + \frac{\partial z}{\partial \xi}\boldsymbol{k}\right) d\xi$$

$$\boldsymbol{q} = \frac{\partial \boldsymbol{s}}{\partial \eta} d\eta = \left(\frac{\partial x}{\partial \eta}\boldsymbol{i} + \frac{\partial y}{\partial \eta}\boldsymbol{j} + \frac{\partial z}{\partial \eta}\boldsymbol{k}\right) d\eta$$

$$\boldsymbol{r} = \frac{\partial \boldsymbol{s}}{\partial \varsigma} d\varsigma = \left(\frac{\partial x}{\partial \varsigma}\boldsymbol{i} + \frac{\partial y}{\partial \varsigma}\boldsymbol{j} + \frac{\partial z}{\partial \varsigma}\boldsymbol{k}\right) d\varsigma$$

$$\Rightarrow d\Omega = |\boldsymbol{r} \cdot (\boldsymbol{p} \times \boldsymbol{q})| = |\boldsymbol{J}^e| d\xi d\eta d\varsigma$$

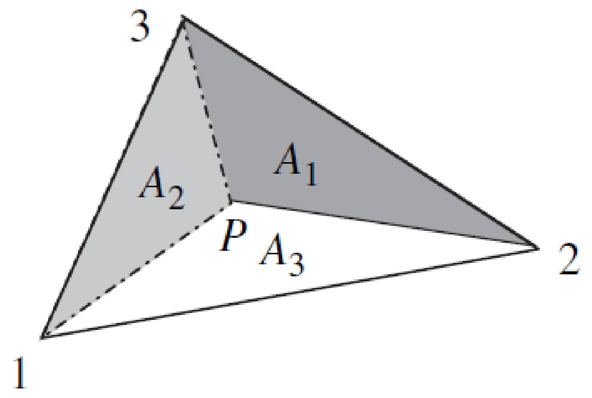
- Gauss integration:

$$I = \iiint_{(-1,-1,-1)}^{(1,1,1)} f(\xi, \eta, \varsigma) |\boldsymbol{J}^e| d\xi d\eta d\varsigma = \cdots$$

Utilize 1D Gauss Quadrature for each dimension

Triangular Coordinates for Linear Elements

- Definition of triangular coordinates:



$$\xi_I = \frac{A_I}{A} \Rightarrow \underline{\xi_1 + \xi_2 + \xi_3 = 1}$$

- Kronecker delta property:

$$\xi_I\left(x_J^e, y_J^e\right) = \delta_{IJ}$$

- Linear mapping:

$$\underline{x = x_1^e \xi_1 + x_2^e \xi_2 + x_3^e \xi_3, y = y_1^e \xi_1 + y_2^e \xi_2 + y_3^e \xi_3}$$

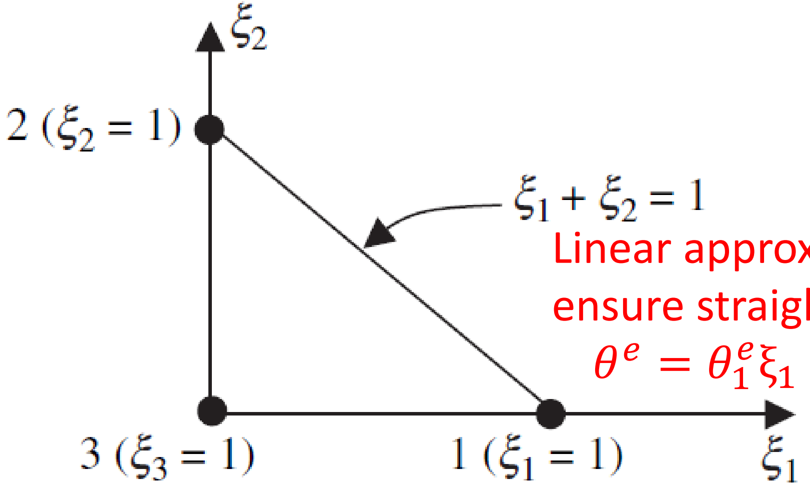
Linear combination of two variables in the parent domain

- Mapping between physical and parent domains:

$$\begin{bmatrix} 1 \\ x \\ y \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 1 & 1 \\ x_1^e & x_2^e & x_3^e \\ y_1^e & y_2^e & y_3^e \end{bmatrix}}_{(\boldsymbol{M}^e)^T} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} = (\boldsymbol{M}^e)^{-T} \begin{bmatrix} 1 \\ x \\ y \end{bmatrix}$$

- Geometry of the parent domain on $\xi_1 - \xi_2$ plane:



Linear approximation to ensure straight edge:

$$\theta^e = \theta_1^e \xi_1 + \theta_2^e \xi_2 + \theta_3^e \xi_3$$

The End