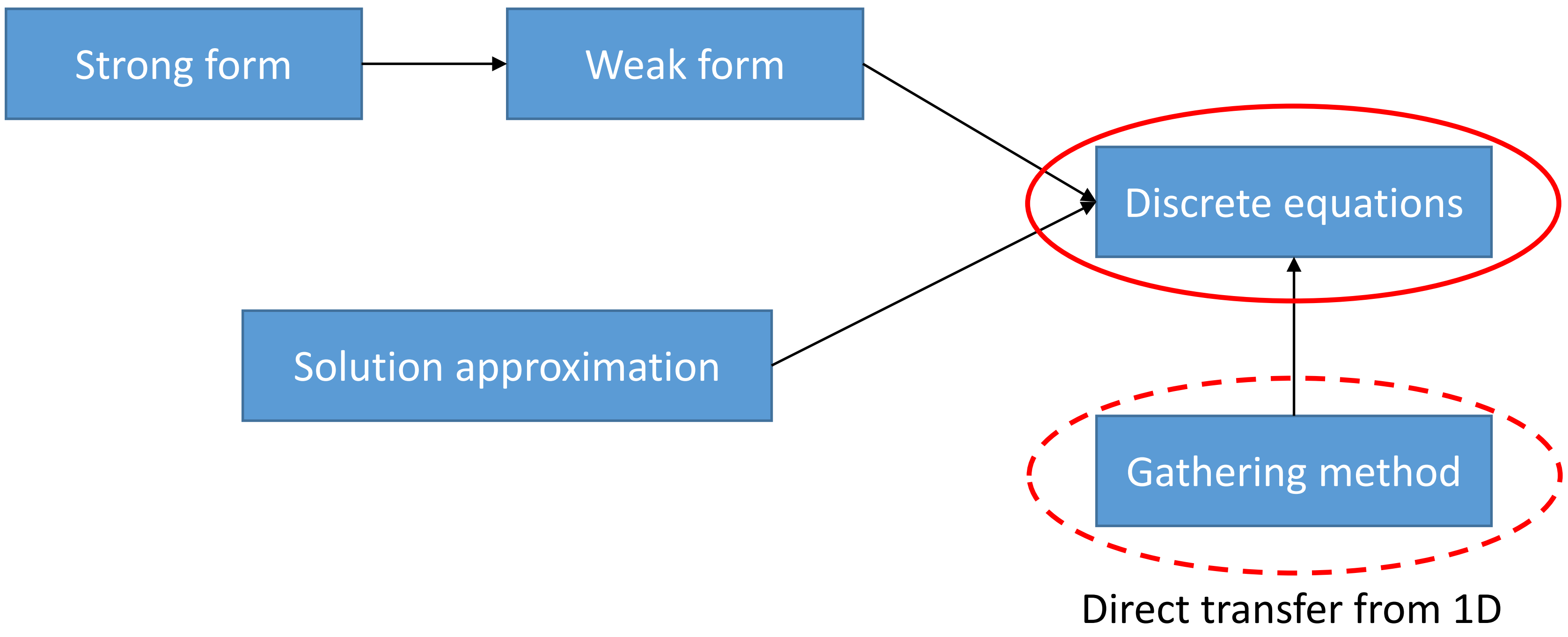
Computational Mechanics

Chapter 8 Finite Element Formulation for Multidimensional Scalar Field Problems

Components for Formulation FEM Equations



2D heat conduction problems

Discretized Weak Form for 2D Heat Conduction

- Weak form of multidimensional heat conduction:

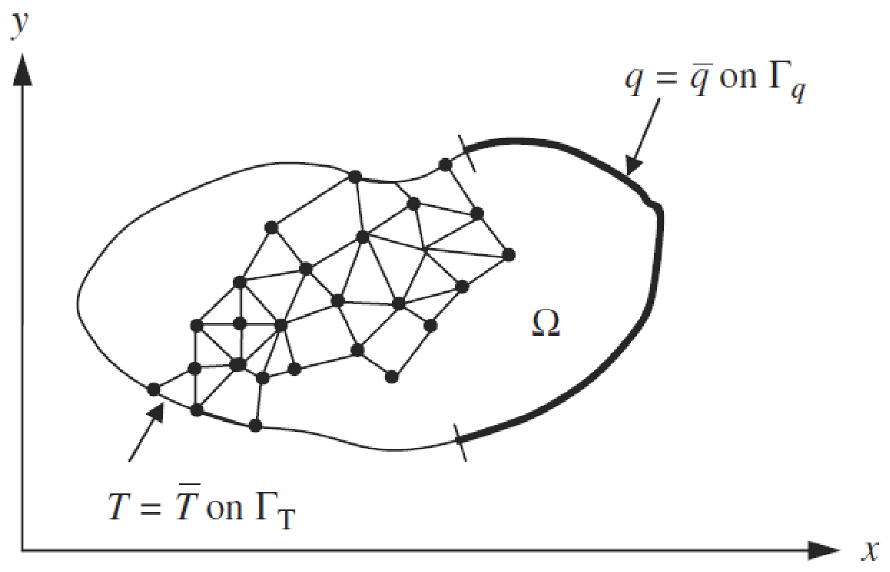
Find $T(x, y) \in U$ so that:

$$\int_{\Omega} (\boldsymbol{\nabla} w)^T \boldsymbol{D} \boldsymbol{\nabla} T d\Omega = -\int_{\Gamma_q} w^T \bar{q} d\Gamma + \int_{\Omega} w^T s d\Omega$$

$$\forall w \in U_0$$

$$\boldsymbol{\nabla} T = \begin{bmatrix} \frac{\partial T}{\partial x} \\ \frac{\partial T}{\partial y} \end{bmatrix}, \qquad \boldsymbol{D} = \begin{bmatrix} k_{xx} & k_{xy} \\ k_{yx} & k_{yy} \end{bmatrix}$$

- Discretization of the complex 2D domain:



- Integration by elements:

$$0 = \sum_{e=1}^{n_{el}} \left(\int_{\Omega^e} (\boldsymbol{\nabla} w^e)^T \boldsymbol{D}^e \boldsymbol{\nabla} T^e d\Omega + \int_{\Gamma_q^e} w^{eT} \bar{q} d\Gamma - \int_{\Omega^e} w^{eT} s d\Omega \right)$$

Solution Approximation of 2D Heat Conduction

- Temperature field estimation in each element:

$$T(x,y) \approx T^e(x,y) = \boldsymbol{N}^e(x,y)\boldsymbol{d}^e = \sum_{I=1}^{n_{en}} N_I^e T_I^e$$

- Weight function estimation in each element:

$$w(x,y) \approx w^e(x,y) = \boldsymbol{N}^e(x,y)\boldsymbol{w}^e = \sum_{I=1}^{n_{en}} N_I^e w_I^e$$

Independent variables are expressed using ξ and η in isoparametric elements.

- Global gather matrix:

$$\boldsymbol{d}^e = \boldsymbol{L}^e\boldsymbol{d}, \qquad \boldsymbol{w}^e = \boldsymbol{L}^e\boldsymbol{w}$$

$$\Rightarrow T^e(x,y) = \boldsymbol{N}^e\boldsymbol{d}^e = \boldsymbol{N}^e\boldsymbol{L}^e\boldsymbol{d}$$

$$w^e(x,y) = \boldsymbol{N}^e\boldsymbol{w}^e = \boldsymbol{N}^e\boldsymbol{L}^e\boldsymbol{w}$$

$$\Rightarrow \boldsymbol{\nabla}T^e(x,y) = (\boldsymbol{\nabla}\cdot\boldsymbol{N}^e)\boldsymbol{L}^e\boldsymbol{d} = \boldsymbol{B}^e\boldsymbol{L}^e\boldsymbol{d}$$

$$\boldsymbol{\nabla}w^e(x,y) = (\boldsymbol{\nabla}\cdot\boldsymbol{N}^e)\boldsymbol{L}^e\boldsymbol{w} = \boldsymbol{B}^e\boldsymbol{L}^e\boldsymbol{w}$$

- Global matrices partition for calculation:

$$\boldsymbol{d} = \begin{bmatrix}\overline{\boldsymbol{d}}_E\\ \boldsymbol{d}_F\end{bmatrix}, \qquad \boldsymbol{w} = \begin{bmatrix}\boldsymbol{0}\\ \boldsymbol{w}_F\end{bmatrix}$$

Global node numbering rule.

Weak Form Integration of the Approximation

$$0=\sum_{e=1}^{n_{el}}\left(\int_{\Omega^e}(\boldsymbol{\nabla}w^e)^T\boldsymbol{D}^e\boldsymbol{\nabla}T^e d\Omega+\int_{\Gamma_q^e}w^{eT}\bar{q}d\Gamma-\int_{\Omega^e}w^{eT}s d\Omega\right)$$

$$T^e(x,y)=\boldsymbol{N}^e\boldsymbol{L}^e\boldsymbol{d},\qquad \boldsymbol{\nabla}T^e(x,y)=\boldsymbol{B}^e\boldsymbol{L}^e\boldsymbol{d},\qquad w^e(x,y)=\boldsymbol{N}^e\boldsymbol{L}^e\boldsymbol{w},\qquad \boldsymbol{\nabla}w^e(x,y)=\boldsymbol{B}^e\boldsymbol{L}^e\boldsymbol{w}$$

$$\Rightarrow 0=\sum_{e=1}^{n_{el}}\left(\int_{\Omega^e}(\boldsymbol{B}^e\boldsymbol{L}^e\boldsymbol{w})^T\boldsymbol{D}^e\boldsymbol{B}^e\boldsymbol{L}^e\boldsymbol{d}d\Omega+\int_{\Gamma_q^e}(\boldsymbol{N}^e\boldsymbol{L}^e\boldsymbol{w})^T\bar{q}d\Gamma-\int_{\Omega^e}(\boldsymbol{N}^e\boldsymbol{L}^e\boldsymbol{w})^T s d\Omega\right)$$

$$\Rightarrow 0=\sum_{e=1}^{n_{el}}\left(\int_{\Omega^e}\boldsymbol{w}^T\boldsymbol{L}^{eT}\boldsymbol{B}^{eT}\boldsymbol{D}^e\boldsymbol{B}^e\boldsymbol{L}^e\boldsymbol{d}d\Omega+\int_{\Gamma_q^e}\boldsymbol{w}^T\boldsymbol{L}^{eT}\boldsymbol{N}^{eT}\bar{q}d\Gamma-\int_{\Omega^e}\boldsymbol{w}^T\boldsymbol{L}^{eT}\boldsymbol{N}^{eT}s d\Omega\right)$$

$$\Rightarrow 0=\boldsymbol{w}^T\sum_{e=1}^{n_{el}}\boldsymbol{L}^{eT}\left(\underbrace{\int_{\Omega^e}\boldsymbol{B}^{eT}\boldsymbol{D}^e\boldsymbol{B}^e d\Omega}_{\boldsymbol{K}^e}\boldsymbol{L}^e\boldsymbol{d}+\underbrace{\int_{\Gamma_q^e}\boldsymbol{N}^{eT}\bar{q}d\Gamma}_{-\boldsymbol{f}_\Gamma^e}-\underbrace{\int_{\Omega^e}\boldsymbol{N}^{eT}s d\Omega}_{\boldsymbol{f}_\Omega^e}\right)$$

$\boldsymbol{K}^e$ - element conductance matrix $-\boldsymbol{f}_\Gamma^e$ $\boldsymbol{f}_\Omega^e$

$\boldsymbol{f}^e=\boldsymbol{f}_\Gamma^e+\boldsymbol{f}_\Omega^e$ - element flux matrix

Nodal Value Calculation

$$0 = \boldsymbol{w}^T \sum_{e=1}^{n_{el}} \boldsymbol{L}^{eT}(\boldsymbol{K}^e \boldsymbol{L}^e \boldsymbol{d} - \boldsymbol{f}^e) = \boldsymbol{w}^T \left[\left(\underbrace{\sum_{e=1}^{n_{el}} (\boldsymbol{L}^{eT} \boldsymbol{K}^e \boldsymbol{L}^e)}_{\boldsymbol{K}} \boldsymbol{d} \right) - \underbrace{\sum_{e=1}^{n_{el}} \boldsymbol{L}^{eT} \boldsymbol{f}^e}_{\boldsymbol{f}} \right]$$

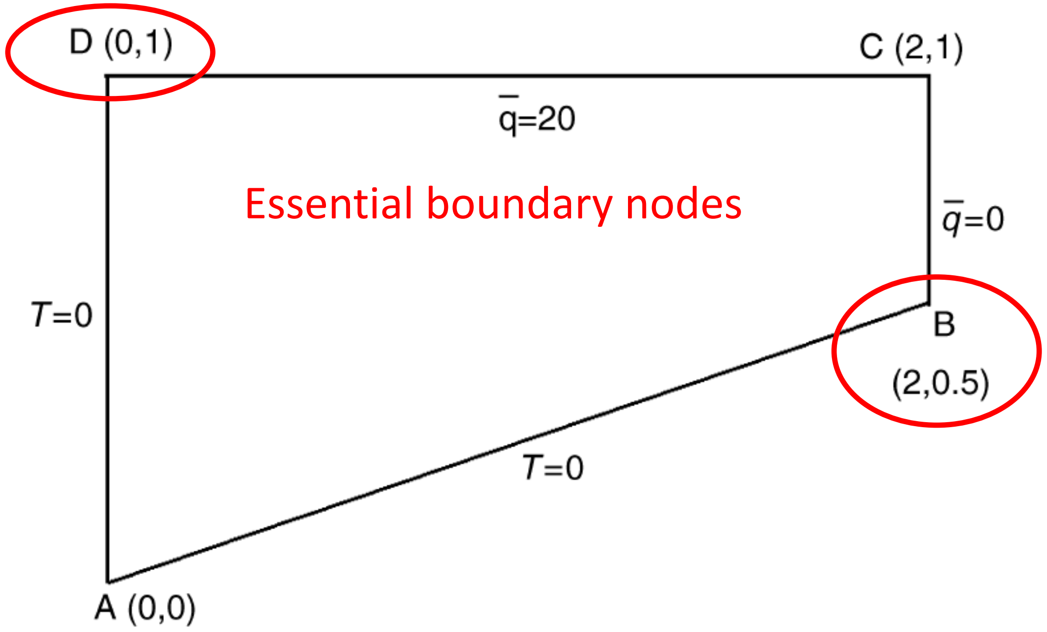
$\boldsymbol{K}$ and $\boldsymbol{f}$ can be also from direct assembly.

$$\boldsymbol{K}\boldsymbol{d} - \boldsymbol{f} = \boldsymbol{r} \Rightarrow 0 = \boldsymbol{w}^T \boldsymbol{r} = \boldsymbol{w}_F^T \boldsymbol{r}_F + \boldsymbol{w}_E^T \boldsymbol{r}_E = \boldsymbol{w}_F^T \boldsymbol{r}_F$$

$$\forall \boldsymbol{w}_F^T \Rightarrow \boldsymbol{r} = \begin{bmatrix} \boldsymbol{r}_E \\ \boldsymbol{0} \end{bmatrix} = \begin{bmatrix} \boldsymbol{K}_E & \boldsymbol{K}_{EF} \\ \boldsymbol{K}_{EF}^T & \boldsymbol{K}_F \end{bmatrix} \begin{bmatrix} \bar{\boldsymbol{d}}_E \\ \boldsymbol{d}_F \end{bmatrix} - \begin{bmatrix} \boldsymbol{f}_E \\ \boldsymbol{f}_F \end{bmatrix}$$

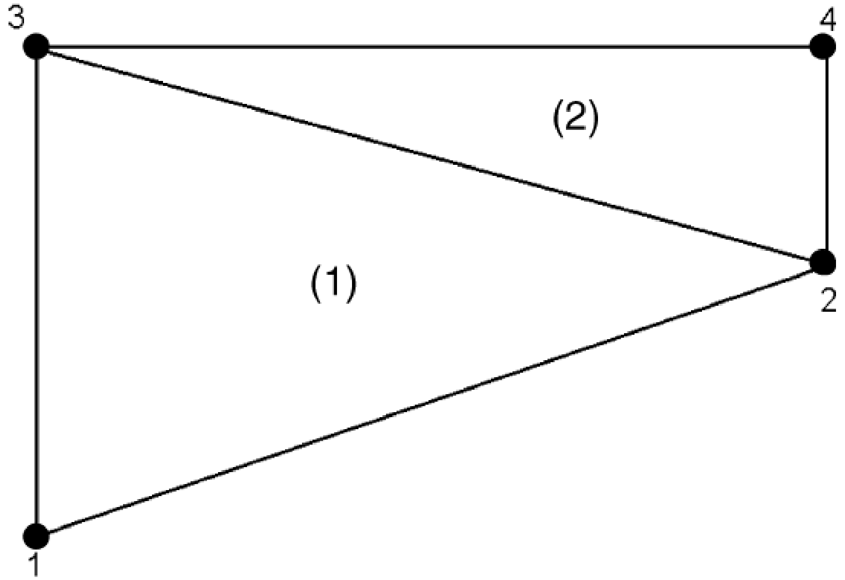
$$\Rightarrow \begin{bmatrix} \boldsymbol{K}_E & \boldsymbol{K}_{EF} \\ \boldsymbol{K}_{EF}^T & \boldsymbol{K}_F \end{bmatrix} \begin{bmatrix} \bar{\boldsymbol{d}}_E \\ \boldsymbol{d}_F \end{bmatrix} = \begin{bmatrix} \boldsymbol{f}_E + \boldsymbol{r}_E \\ \boldsymbol{f}_F \end{bmatrix}$$

Heat Conduction Problem with Triangular Elements



- Unit system: $m - W - C^{\circ}$
- Isotropic heat conductivity: $k = 5W/C^{\circ}$
- Heat source: $s = 6W/m^2$

- Example 1 – two linear triangular elements:

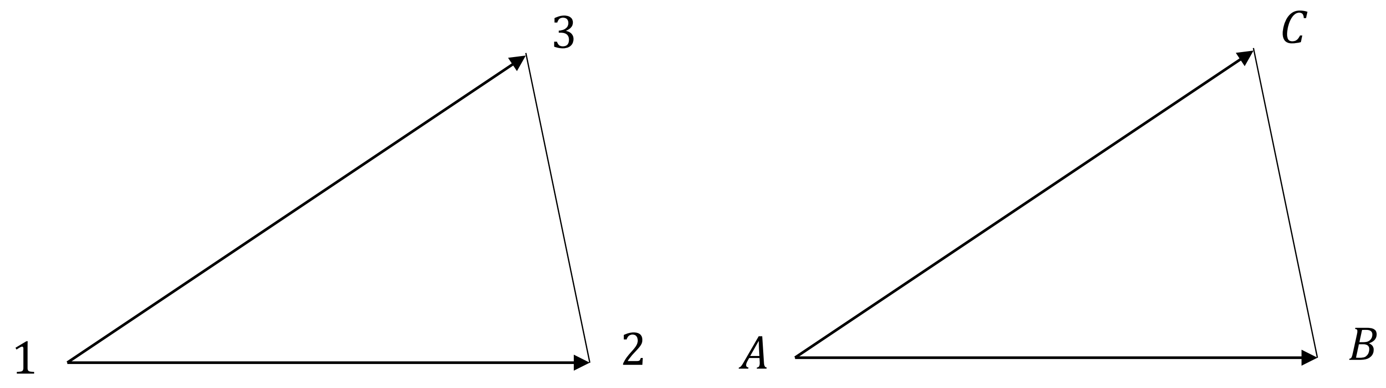


- Calculation of $\boldsymbol{B}^e$:

$$\boldsymbol{B}^e = \frac{1}{2A^e}\begin{bmatrix}(y_2^e - y_3^e) & (y_3^e - y_1^e) & (y_1^e - y_2^e)\\(x_3^e - x_2^e) & (x_1^e - x_3^e) & (x_2^e - x_1^e)\end{bmatrix}$$

Constant

Areas for Triangles



$$2A = |\boldsymbol{AB} \times \boldsymbol{AC}| = |[(x_2 - x_1)\boldsymbol{i} + (y_2 - y_1)\boldsymbol{j}] \times [(x_3 - x_1)\boldsymbol{i} + (y_3 - y_1)\boldsymbol{j}]|$$

$$\Rightarrow A = \frac{1}{2}\left|\begin{vmatrix}(x_2 - x_1) & (y_2 - y_1)\\(x_3 - x_1) & (y_3 - y_1)\end{vmatrix}\right|$$

Element 1 Heat Conductance Matrix

- General element matrices calculation:

$$\boldsymbol{K}^e = \int_{\Omega^e} \boldsymbol{B}^{eT} \boldsymbol{D}^e \boldsymbol{B}^e d\Omega$$

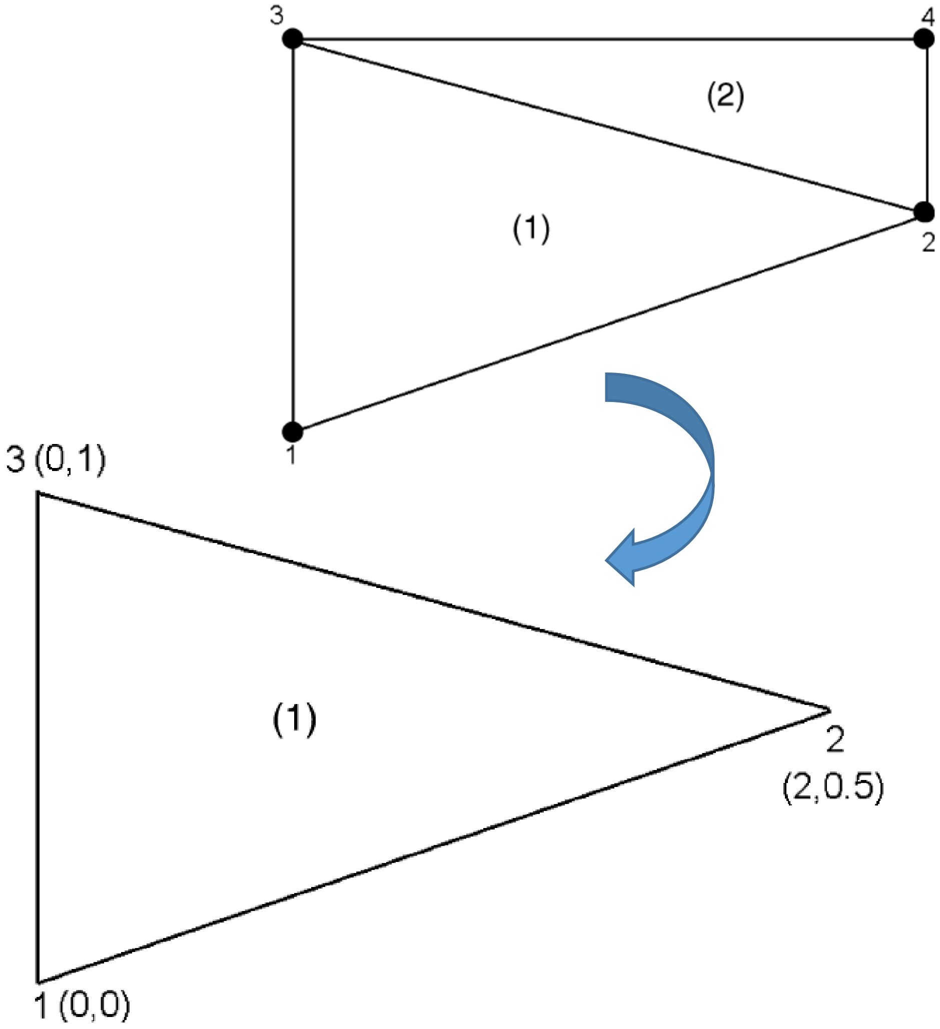
$$\boldsymbol{B}^e = const, \qquad \boldsymbol{D}^e = k\boldsymbol{I} = const$$

$$\Rightarrow \boldsymbol{K}^e = k\boldsymbol{B}^{eT}\boldsymbol{B}^e \int_{\Omega^e} d\Omega = kA^e\boldsymbol{B}^{eT}\boldsymbol{B}^e$$

- Element 1:

$$\boldsymbol{K}^{(1)} = kA^{(1)}\boldsymbol{B}^{(1)T}\boldsymbol{B}^{(1)}$$

$$\Rightarrow \boldsymbol{K}^{(1)} = \begin{array}{c} \begin{matrix} \quad 1 & \qquad 2 & \qquad 3 \end{matrix} \\ \begin{bmatrix} 5.3125 & -0.625 & -4.6875 \\ -0.625 & 1.25 & -0.625 \\ -4.6875 & -0.625 & 5.3125 \end{bmatrix} \end{array} \begin{matrix} 1 \\ 2 \\ 3 \end{matrix}$$



Element 2 and Global Heat Conductance Matrices

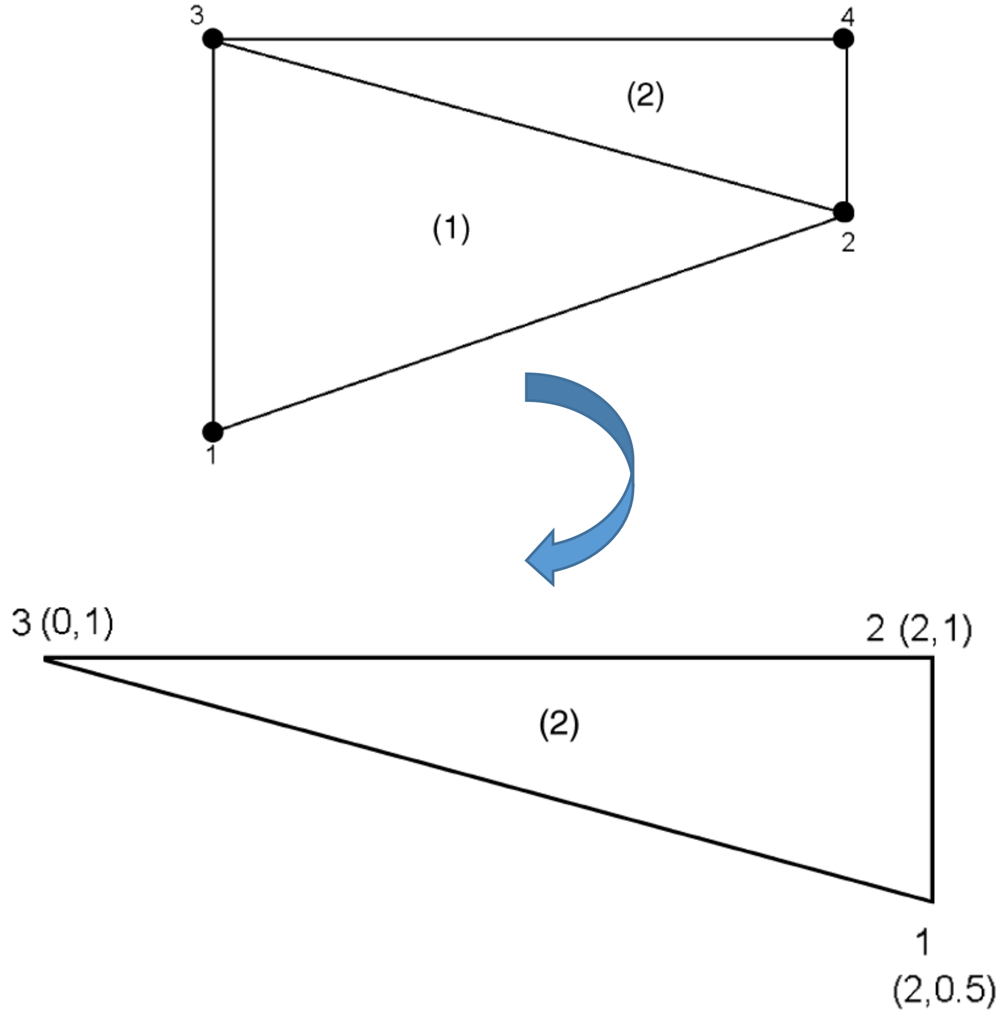
- Element 2:

$$\boldsymbol{K}^{(2)} = kA^{(2)}\boldsymbol{B}^{(2)T}\boldsymbol{B}^{(2)}$$

$$\begin{array}{c} \\ \Rightarrow \boldsymbol{K}^{(2)} = \end{array} \begin{array}{c} \begin{array}{ccc} \color{red}{2} & \color{red}{4} & \color{red}{3} \end{array} \\ \begin{bmatrix} 10 & -10 & 0 \\ -10 & 10.625 & -0.625 \\ 0 & -0.625 & 0.625 \end{bmatrix} \end{array} \begin{array}{c} \\ \color{red}{2} \\ \color{red}{4} \\ \color{red}{3} \end{array}$$

- Global matrix by direct assembly:

$$\boldsymbol{K} = \begin{bmatrix} 5.3125 & -0.625 & -4.6875 & 0 \\ -0.625 & 11.25 & -0.625 & -10 \\ -4.6875 & -0.625 & 5.9375 & -0.625 \\ 0 & -10 & -0.625 & 10.625 \end{bmatrix}$$

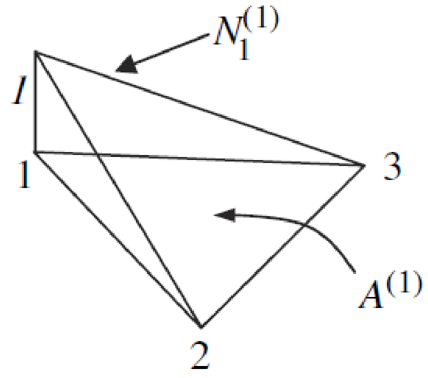


Heat Source Matrices

- Element heat source matrices:

$$\boldsymbol{f}_\Omega^e = \int_{\Omega^e} \boldsymbol{N}^{eT} s\, d\Omega = s \int_{\Omega^e} \boldsymbol{N}^{eT} d\Omega = s \begin{bmatrix} \int_{\Omega^e} N_1^{eT} d\Omega \\ \int_{\Omega^e} N_2^{eT} d\Omega \\ \int_{\Omega^e} N_3^{eT} d\Omega \end{bmatrix}$$

$$\Rightarrow \boldsymbol{f}_\Omega^e = s\frac{A^e}{3}\begin{bmatrix}1\\1\\1\end{bmatrix}$$



$$\Rightarrow \int_{\Omega^e} N_I^{eT} d\Omega = \frac{A^e}{3}$$

$$\Rightarrow \boldsymbol{f}_\Omega^{(1)} = s\frac{A^{(1)}}{3}\begin{bmatrix}1\\1\\1\end{bmatrix} = 6\,\frac{1}{3}\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}2\\2\\2\end{bmatrix} \begin{matrix}1\\2\\3\end{matrix}$$

$$\boldsymbol{f}_\Omega^{(2)} = s\frac{A^{(2)}}{3}\begin{bmatrix}1\\1\\1\end{bmatrix} = 6\,\frac{0.5}{3}\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}1\\1\\1\end{bmatrix} \begin{matrix}2\\4\\3\end{matrix}$$

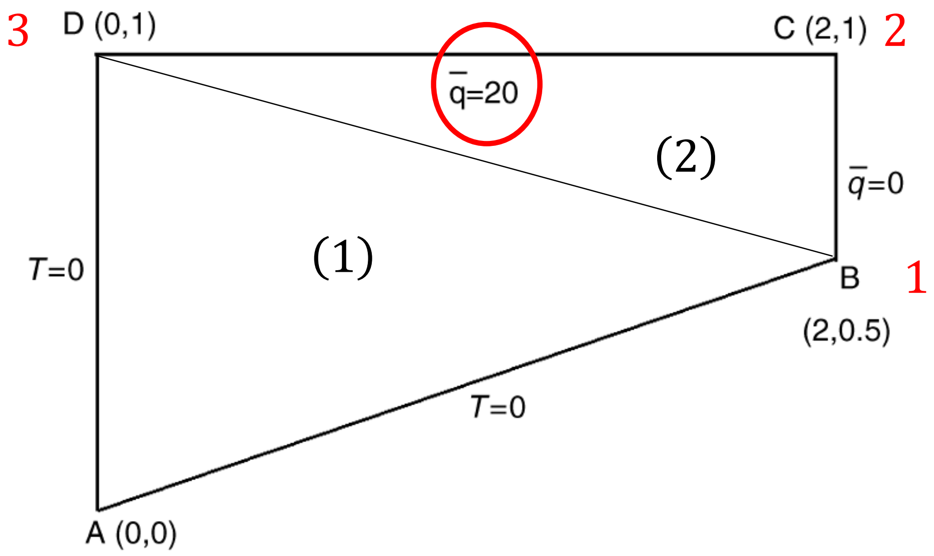
$$\Rightarrow \boldsymbol{f}_\Omega = \begin{bmatrix}2\\3\\3\\1\end{bmatrix}$$

Element Heat Flux Matrix

- Element heat flux matrices:

$$\boldsymbol{f}_\Gamma^e = -\int_{\Gamma_q^e} \boldsymbol{N}^{eT} \bar{q} d\Gamma$$

- Only element 2 contributes to heat flux:



- Calculation of the heat flux matrix in element 2:

$$\boldsymbol{f}_\Gamma^{(2)} = -\int_0^2 \boldsymbol{N}^{(2)T}\Big|_{y=1} \bar{q} dx$$

$$N_1^e = \frac{1}{2A^e}[x_2^e y_3^e - x_3^e y_2^e + (y_2^e - y_3^e)x + (x_3^e - x_2^e)y]$$

$$N_2^e = \frac{1}{2A^e}[x_3^e y_1^e - x_1^e y_3^e + (y_3^e - y_1^e)x + (x_1^e - x_3^e)y]$$

$$N_3^e = \frac{1}{2A^e}[x_1^e y_2^e - x_2^e y_1^e + (y_1^e - y_2^e)x + (x_2^e - x_1^e)y]$$

$$\Rightarrow \boldsymbol{N}^{(2)}\Big|_{y=1} = [0 \quad 0.5x \quad -0.5x + 1]$$

Element and Global Heat Flux Matrices

$$\boldsymbol{f}_{\Gamma}^{(2)} = -\int_0^2 \boldsymbol{N}^{(2)T}\Big|_{y=1} \bar{q}dx\,, \qquad \boldsymbol{N}^{(2)}\Big|_{y=1} = [0 \quad 0.5x \quad -0.5x+1]$$

$$\Rightarrow \boldsymbol{f}_{\Gamma}^{(2)} = -\int_0^2 \begin{bmatrix} 0 \\ 0.5x \\ -0.5x+1 \end{bmatrix} 20dx = \begin{bmatrix} 0 \\ -20 \\ -20 \end{bmatrix} \begin{matrix} 2 \\ 4 \\ 3 \end{matrix}$$

$$\Rightarrow \boldsymbol{f}_{\Gamma} = \begin{bmatrix} 0 \\ 0 \\ -20 \\ -20 \end{bmatrix}$$

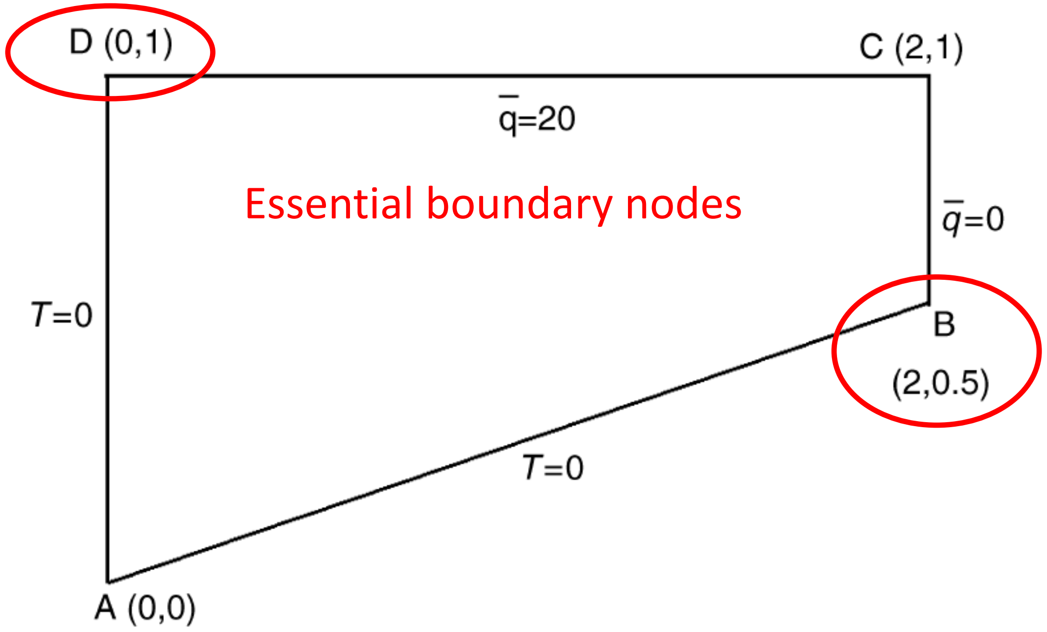
Nodal Value Calculation and Postprocessing

$$\boldsymbol{K} = \begin{bmatrix} 5.3125 & -0.625 & -4.6875 & 0 \\ -0.625 & 11.25 & -0.625 & -10 \\ -4.6875 & -0.625 & 5.9375 & -0.625 \\ 0 & -10 & -0.625 & 10.625 \end{bmatrix}, \quad \boldsymbol{f}_\Omega = \begin{bmatrix} 2 \\ 3 \\ 3 \\ 1 \end{bmatrix}, \quad \boldsymbol{f}_\Gamma = \begin{bmatrix} 0 \\ 0 \\ -20 \\ -20 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 5.3125 & -0.625 & -4.6875 & 0 \\ -0.625 & 11.25 & -0.625 & -10 \\ -4.6875 & -0.625 & 5.9375 & -0.625 \\ 0 & -10 & -0.625 & 10.625 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ T_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 3 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -20 \\ -20 \end{bmatrix} + \begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ 0 \end{bmatrix} = \begin{bmatrix} r_1 + 2 \\ r_2 + 3 \\ r_3 - 17 \\ -19 \end{bmatrix}$$

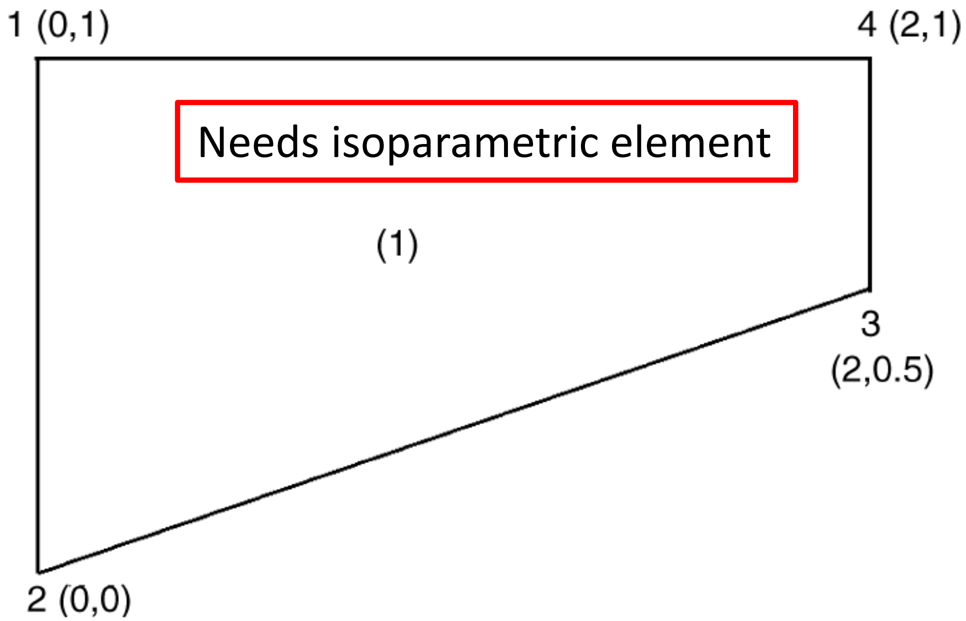
$$\Rightarrow T_4 = -1.788 \Rightarrow \boldsymbol{d}^{(1)} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \boldsymbol{d}^{(2)} = \begin{bmatrix} 0 \\ -1.788 \\ 0 \end{bmatrix} \Rightarrow \boldsymbol{q}^{(1)} = k\boldsymbol{B}^{(1)}\boldsymbol{d}^{(1)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \boldsymbol{q}^{(2)} = k\boldsymbol{B}^{(2)}\boldsymbol{d}^{(2)} = \begin{bmatrix} 4.47 \\ 17.88 \end{bmatrix}$$

Heat Conduction Problem with Quadrilateral Element



- Unit system: $m - W - C^{\circ}$
- Isotropic heat conductivity: $k = 5W/C^{\circ}$
- Heat source: $s = 6W/m^2$

- Example 2 – one linear quadrilateral element:



- Element coordinate matrix:

$$[\boldsymbol{x}^{(1)} \quad \boldsymbol{y}^{(1)}] = \begin{bmatrix} 0 & 0 & 2 & 2 \\ 1 & 0 & 0.5 & 1 \end{bmatrix}^T$$

Shape Functions in the Parent Domain

- Shape functions in the parent domain are generalized:

$$N_1^{4Q}(\xi,\eta) = \frac{1}{4}(\xi-1)(\eta-1)$$

$$N_2^{4Q}(\xi,\eta) = \frac{-1}{4}(\xi+1)(\eta-1)$$

$$N_3^{4Q}(\xi,\eta) = \frac{1}{4}(\xi+1)(\eta+1)$$

$$N_4^{4Q}(\xi,\eta) = \frac{-1}{4}(\xi-1)(\eta+1)$$

- Gradient in the parent domain:

$$\boldsymbol{GN}^{4Q} = \begin{bmatrix} \frac{\partial}{\partial\xi} \\ \frac{\partial}{\partial\eta} \end{bmatrix} \boldsymbol{N}^{4Q} = \frac{1}{4}\begin{bmatrix} \eta-1 & 1-\eta & \eta+1 & -\eta-1 \\ \xi-1 & -\xi-1 & \xi+1 & 1-\xi \end{bmatrix}$$

- Jacobian matrix – 2D mapping:

$$\boldsymbol{J}^{(1)} = \boldsymbol{GN}^{4Q}[\boldsymbol{x}^{(1)} \quad \boldsymbol{y}^{(1)}]$$

$$\Rightarrow \boldsymbol{J}^{(1)} = \frac{1}{4}\begin{bmatrix} \eta-1 & 1-\eta & \eta+1 & -\eta-1 \\ \xi-1 & -\xi-1 & \xi+1 & 1-\xi \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 2 & 0.5 \\ 2 & 1 \end{bmatrix}$$

$$\Rightarrow \boldsymbol{J}^{(1)} = \begin{bmatrix} 0 & 0.125\eta-0.375 \\ 1 & 0.125\xi+0.125 \end{bmatrix}$$

$$\Rightarrow \left|\boldsymbol{J}^{(1)}\right| = 0.375-0.125\eta$$

$$\Rightarrow \left(\boldsymbol{J}^{(1)}\right)^{-1} = \begin{bmatrix} \frac{1+\xi}{3-\eta} & 1 \\ \frac{8}{\eta-3} & 0 \end{bmatrix}$$

$$\boldsymbol{M}^{-1} = \frac{1}{|\boldsymbol{M}|}\begin{bmatrix} M_{22} & -M_{21} \\ M_{12} & M_{11} \end{bmatrix}$$

Conductance Matrix Calculation

- Global conductance matrix:

$$\boldsymbol{K} = \boldsymbol{K}^{(1)} = \int_{\Omega^e} \boldsymbol{B}^{(1)T}\boldsymbol{D}^{(1)}\boldsymbol{B}^{(1)} d\Omega = \iint_{(-1,-1)}^{(1,1)} k\boldsymbol{B}^{(1)T}\boldsymbol{B}^{(1)}\left|\boldsymbol{J}^{(1)}\right| d\xi d\eta$$

$$\boldsymbol{B}^{(1)} = \left(\boldsymbol{J}^{(1)}\right)^{-1}\boldsymbol{G}\boldsymbol{N}^{4Q} = \begin{bmatrix} \frac{1+\xi}{3-\eta} & 1 \\ \frac{8}{\eta-3} & 0 \end{bmatrix} \frac{1}{4}\begin{bmatrix} \eta-1 & 1-\eta & \eta+1 & -\eta-1 \\ \xi-1 & -\xi-1 & \xi+1 & 1-\xi \end{bmatrix}$$

Not polynomial functions, utilize 2 points for integration for demonstration.

$$\Rightarrow \boldsymbol{K} = k\sum_{i=1}^{2}\sum_{j=1}^{2} W_i W_j \boldsymbol{B}^{(1)T}(\xi_i,\eta_j)\boldsymbol{B}^{(1)}(\xi_i,\eta_j)\left|\boldsymbol{J}^{(1)}(\xi_i,\eta_j)\right|, \xi_1 = \eta_1 = -\xi_2 = -\eta_2 = \frac{1}{\sqrt{3}}, W_1 = W_2 = 1$$

$$\Rightarrow \boldsymbol{K} = \begin{bmatrix} 4.76 & -3.51 & -2.98 & 1.73 \\ -3.51 & 4.13 & 1.73 & -2.36 \\ -2.98 & 1.73 & 6.54 & -5.29 \\ 1.73 & -2.36 & -5.29 & 5.91 \end{bmatrix}$$

Heat Source Matrix Calculation

- Global heat source matrix:

$$\boldsymbol{f}_\Omega = \boldsymbol{f}_\Omega^{(1)} = \int_{\Omega^e} \boldsymbol{N}^{(1)T} s d\Omega = s \iint_{(-1,-1)}^{1,1} \boldsymbol{N}^{4QT} \left|\boldsymbol{J}^{(1)}\right| d\xi d\eta = 6 \iint_{(-1,-1)}^{(1,1)} \begin{bmatrix} N_1^{4Q} \\ N_2^{4Q} \\ N_3^{4Q} \\ N_4^{4Q} \end{bmatrix} (0.375 - 0.125\eta)\, d\xi d\eta$$

$$N_1^{4Q}(\xi,\eta) = \frac{1}{4}(\xi - 1)(\eta - 1) \qquad N_3^{4Q}(\xi,\eta) = \frac{1}{4}(\xi + 1)(\eta + 1)$$

$$N_2^{4Q}(\xi,\eta) = \frac{-1}{4}(\xi + 1)(\eta - 1) \qquad N_4^{4Q}(\xi,\eta) = \frac{-1}{4}(\xi - 1)(\eta + 1)$$

- Gauss quadrature:

$$\Rightarrow \boldsymbol{f}_\Omega = \begin{bmatrix} 2.5 \\ 2.5 \\ 2 \\ 2 \end{bmatrix}$$

Review of 2D Gauss Quadrature

- Utilize the 1st component of $\boldsymbol{f}_{\Omega}$ as an example:

$$f_{\Omega 1} = 6 \iint_{(-1,-1)}^{(1,1)} \frac{1}{4}(\xi - 1)(\eta - 1)(0.375 - 0.125\eta\,)d\xi d\eta$$

$$\xi:\ p = 1 \Rightarrow n_{gp} = 1 \Rightarrow W_1 = 2, \xi_1 = 0$$

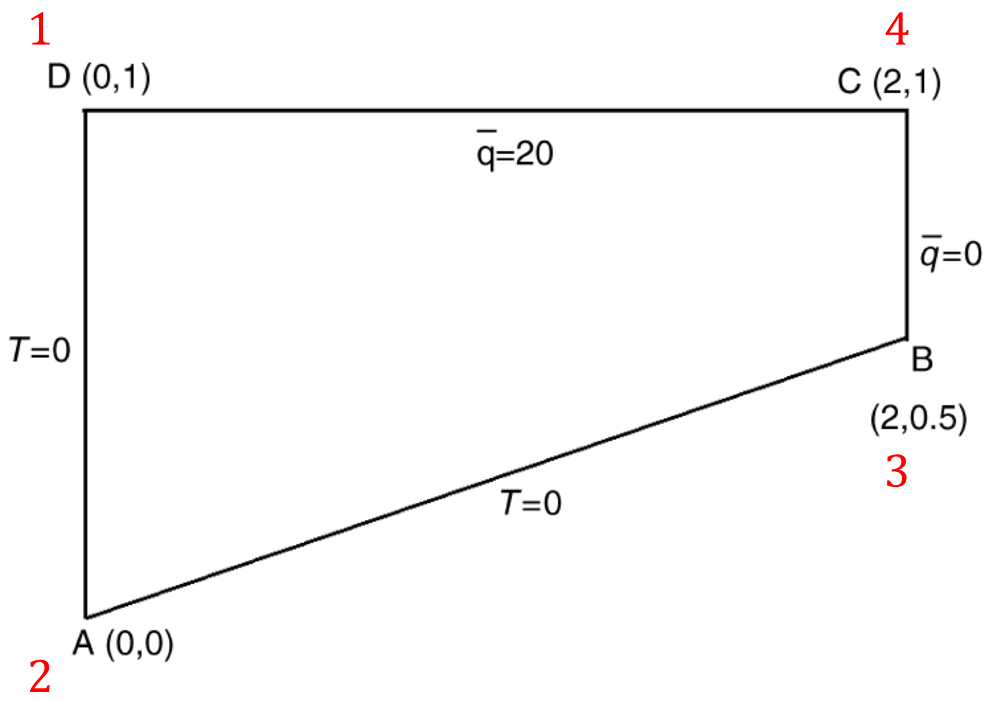
$$\eta:\ p = 2 \Rightarrow n_{gp} = 2 \Rightarrow W_1 = W_2 = 1, \eta_1 = -\eta_2 = \frac{1}{\sqrt{3}}$$

$$\Rightarrow f_{\Omega 1} = \frac{3}{2}\int_{-1}^{1} 2(0-1)(\eta - 1)(0.375 - 0.125\eta\,)d\eta = 3\left[\left(1 - \frac{1}{\sqrt{3}}\right)\left(0.375 - \frac{0.125}{\sqrt{3}}\right) + \left(1 + \frac{1}{\sqrt{3}}\right)\left(0.375 + \frac{0.125}{\sqrt{3}}\right)\right]$$

$$f_{\Omega 1} = 2.5$$

Other components are similar.

Heat Flux Matrix Calculation



- Global boundary flux matrix:

$$\boldsymbol{f}_\Gamma = -\int_{\Gamma_q^e} \boldsymbol{N}^{eT}\bar{q}\,d\Gamma = -20\int_{\boldsymbol{DC}} \boldsymbol{N}^{eT}dx$$

$$\boldsymbol{f}_\Gamma = -20\,\frac{2-0}{2}\int_{-1}^{1}\boldsymbol{N}^{4QT}(\xi=1,\eta)d\eta = -20\int_{-1}^{1}\begin{bmatrix}\frac{1}{2}(1-\eta)\\ 0\\ 0\\ \frac{1}{2}(1+\eta)\end{bmatrix}d\eta$$

$$\eta: p = 1 \Rightarrow n_{gp} = 1 \Rightarrow W_1 = 2, \eta_1 = 0$$

$$\Rightarrow \boldsymbol{f}_\Gamma = [-20 \quad 0 \quad 0 \quad -20]^T$$

Calculation of Nodal Values and Postprocessing

$$\boldsymbol{K} = \begin{bmatrix} 4.76 & -3.51 & -2.98 & 1.73 \\ -3.51 & 4.13 & 1.73 & -2.36 \\ -2.98 & 1.73 & 6.54 & -5.29 \\ 1.73 & -2.36 & -5.29 & 5.91 \end{bmatrix}, \quad \boldsymbol{f}_\Omega = \begin{bmatrix} 2.5 \\ 2.5 \\ 2 \\ 2 \end{bmatrix}, \quad \boldsymbol{f}_\Gamma = \begin{bmatrix} -20 \\ 0 \\ 0 \\ -20 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 4.76 & -3.51 & -2.98 & 1.73 \\ -3.51 & 4.13 & 1.73 & -2.36 \\ -2.98 & 1.73 & 6.54 & -5.29 \\ 1.73 & -2.36 & -5.29 & 5.91 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ T_4 \end{bmatrix} = \begin{bmatrix} r_1 - 17.5 \\ r_2 + 2.5 \\ r_3 + 2 \\ -18 \end{bmatrix} \Rightarrow T_4 = -3.04$$

$$\boldsymbol{q} = -k\boldsymbol{\nabla}T = -k\boldsymbol{B}^{(1)}\boldsymbol{d}^{(1)} = -\frac{5}{4}\begin{bmatrix} \frac{1+\xi}{3-\eta} & 1 \\ \frac{8}{\eta-3} & 0 \end{bmatrix} \begin{bmatrix} \eta-1 & 1-\eta & \eta+1 & -\eta-1 \\ \xi-1 & -\xi-1 & \xi+1 & 1-\xi \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ -3.04 \end{bmatrix}$$

Nodal and also integration point values can be obtained.

Verification and Validation for FEM Models

- Verification – the equations have been solved correctly:
  - ➢ Batch tests for linear code:
  1. Define arbitrary linear temperature field for all boundaries (essential):

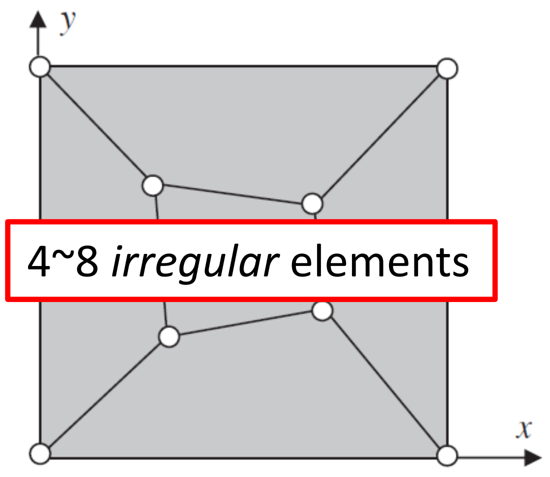
$$T(x,y) = \alpha_0 + \alpha_1 x + \alpha_2 y, \qquad \alpha_i \neq 0$$

  2. Solve the temperature field using finite element analysis:

$$T^h(x,y) = T(x,y)?$$

No heat source + linear field => approximation is the unique exact solution!

1. Nodal values?
2. Heat flux?



4~8 *irregular* elements

  - ➢ Manufactured solution method – construct a solution than determine the source and boundary
  1. Refine mesh to check convergence to the *manufactured* exact solution
  2. Requires enough essential boundary conditions to avoid singularity

- Validation – the models have been setup correctly:
  - ➢ Requires benchmark physical tests

The End