Computational Mechanics

Chapter 9 Finite Element Formulation for Vector Field Problems – 2D Linear Elasticity

Introduction to Linear Elasticity

- Assumptions in linear elasticity:

  1. Small deformation – less than 10^{-2} of the body dimensions;
  2. Linear behavior of the material – Hooke's law;
  3. Neglection of dynamic effects – equilibrium state;
  4. Compatibility – no gaps or overlaps within the solid.

Examples of large deformation:
- Car crash
- Skin deformation under massage
- Metal forming process

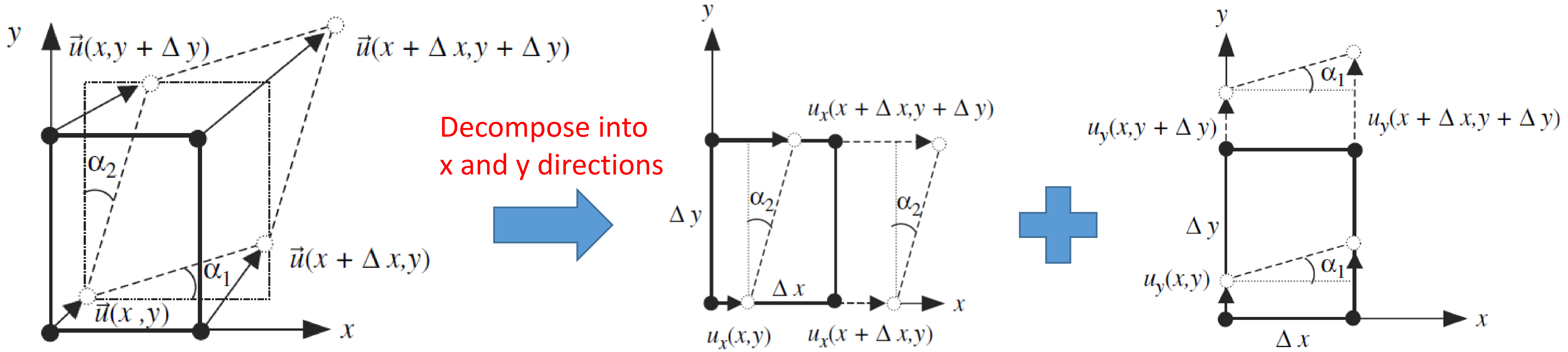
- Widely applied for industrial stress analysis under normal working conditions.

Kinematics

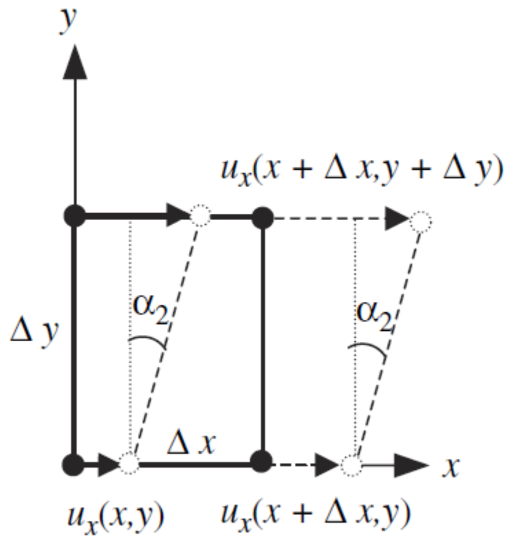
- Expression of 2D displacement vectors:

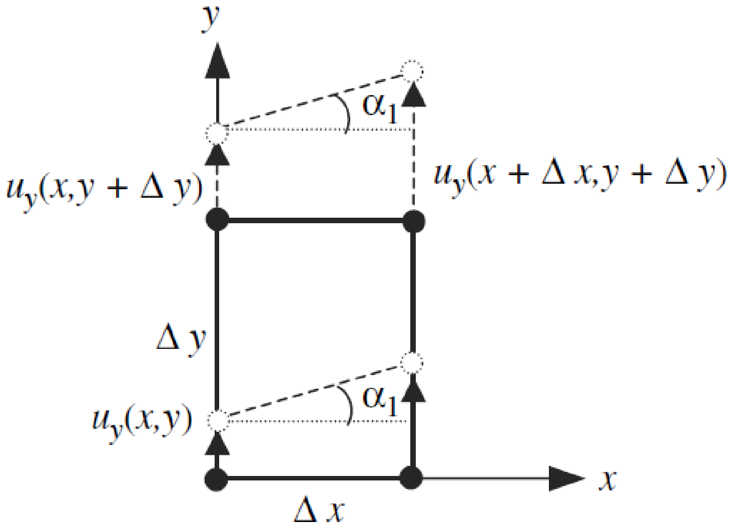
$$\boldsymbol{u} = \begin{bmatrix} u_x \\ u_y \end{bmatrix}, \vec{u} = u_x\vec{\imath} + u_y\vec{\jmath}$$

- 2D deformation of a control volume:



Strain – Displacement Equations





- Small deformation assumption – expression of strains:

$$\varepsilon_{xx} = \lim_{\Delta x \to 0} \frac{u_x(x + \Delta x, y) - u_x(x,y)}{\Delta x} = \frac{\partial u_x}{\partial x}, \qquad \varepsilon_{yy} = \lim_{\Delta y \to 0} \frac{u_y(x, y + \Delta y) - u_y(x,y)}{\Delta y} = \frac{\partial u_y}{\partial y}$$

$$\underline{\gamma_{xy}} = \alpha_1 + \alpha_2 = \lim_{\Delta y \to 0} \frac{u_x(x, y + \Delta y) - u_x(x,y)}{\Delta y} + \lim_{\Delta x \to 0} \frac{u_y(x + \Delta x, y) - u_y(x,y)}{\Delta x} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} = \underline{2\varepsilon_{xy}}$$

Engineering shear strain

Tensor/true shear strain

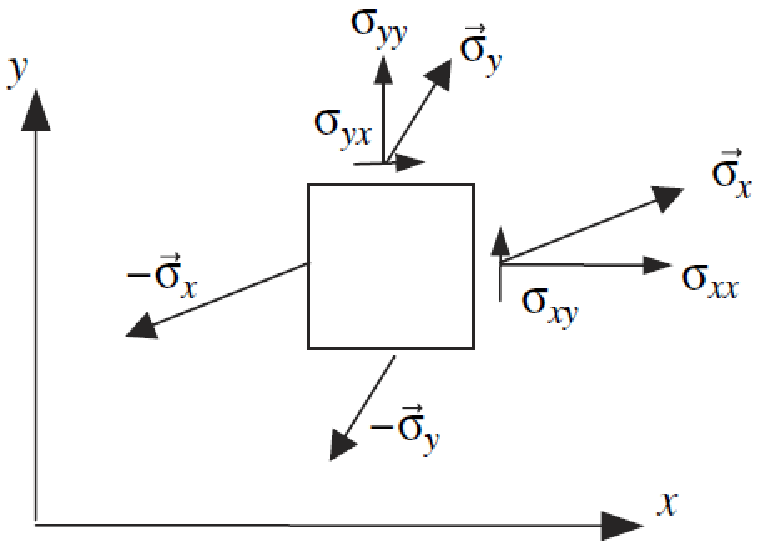
Notation for Strain-Displacement Equations

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x}, \qquad \varepsilon_{yy} = \frac{\partial u_y}{\partial y}, \qquad \gamma_{xy} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}$$

$$\Rightarrow \boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \begin{bmatrix} u_x \\ u_y \end{bmatrix} = \boldsymbol{\nabla}_s \boldsymbol{u}$$

Stress and Traction

- Traction – force/area that acts on the plane:
  - ➢ Traction – measurement of forces.
  - ➢ Stress – measurement of states to represent arbitrary surfaces.
- Connection between 2D traction and stress:



$$\vec{\sigma}_x = \sigma_{xx}\vec{i} + \sigma_{xy}\vec{j}, \qquad \vec{\sigma}_y = \sigma_{yx}\vec{i} + \sigma_{yy}\vec{j}$$

Plane normal

Force direction

Stress vector

- 2D Moment equilibrium:

$$\sigma_{yx}l^2 = \sigma_{xy}l^2 \Rightarrow \sigma_{xy} = \sigma_{yx}$$

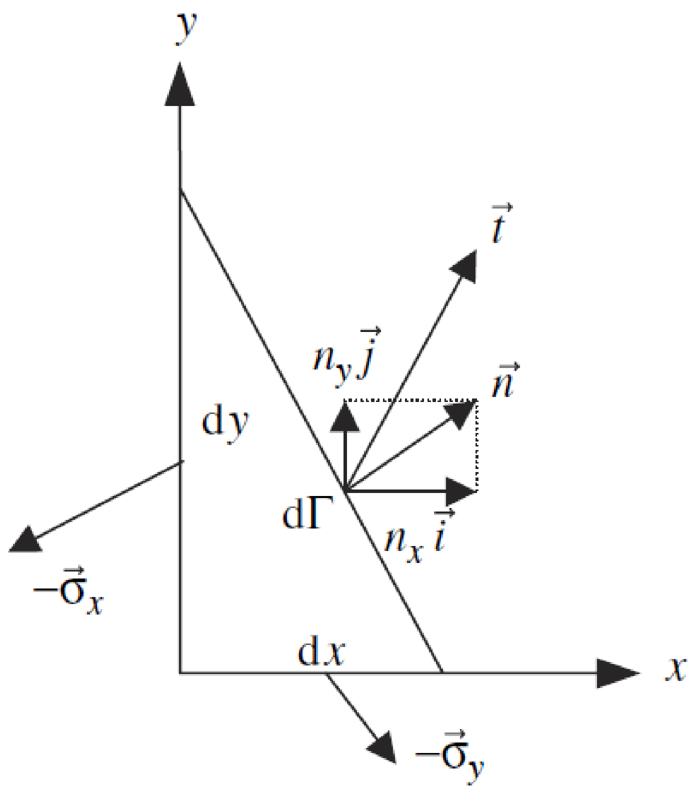
- 1D matrix format expression of stress:

$$\boldsymbol{\sigma}^T = [\sigma_{xx} \quad \sigma_{yy} \quad \sigma_{xy}]$$

- 2D tensor format expression of stress:

$$\boldsymbol{\tau} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} \end{bmatrix}$$

Traction on a specific surface



$$\vec{\sigma}_x = \sigma_{xx}\vec{\imath} + \sigma_{xy}\vec{\jmath}, \qquad \vec{\sigma}_y = \sigma_{yx}\vec{\imath} + \sigma_{yy}\vec{\jmath}, \qquad \sigma_{xy} = \sigma_{yx}$$

- 2D Force equilibrium condition:

$$\vec{t}d\Gamma - \vec{\sigma}_x dy - \vec{\sigma}_y dx = \vec{0}$$

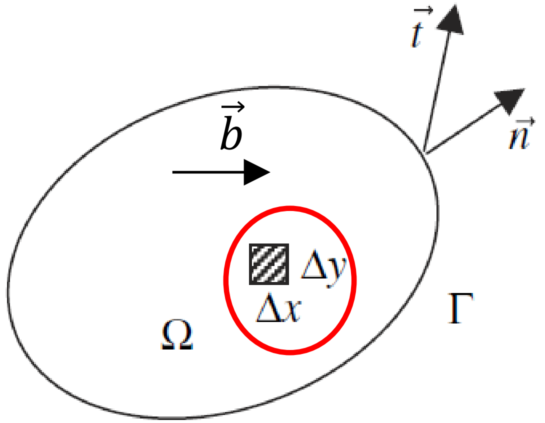
$$\Rightarrow \vec{0} = \vec{t} - \vec{\sigma}_x \frac{dy}{d\Gamma} - \vec{\sigma}_y \frac{dx}{d\Gamma} = \vec{t} - \vec{\sigma}_x n_x - \vec{\sigma}_y n_y$$

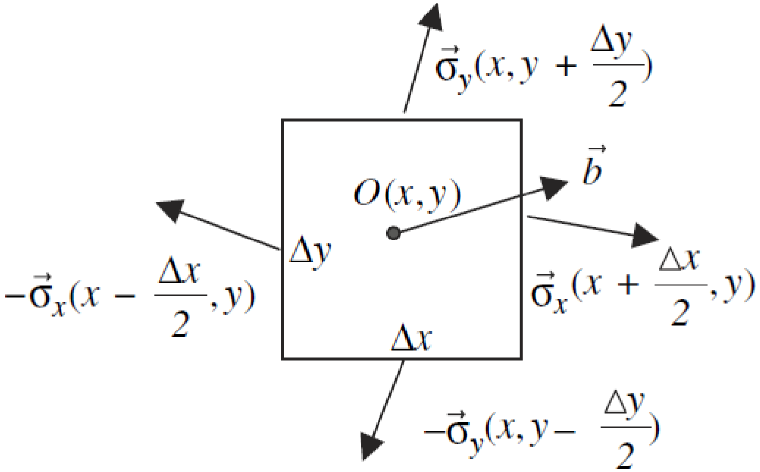
$$\Rightarrow t_x = \vec{\imath}\vec{\sigma}_x n_x + \vec{\imath}\vec{\sigma}_y n_y = \sigma_{xx} n_x + \sigma_{yx} n_y = \sigma_{xx} n_x + \sigma_{xy} n_y = \vec{\sigma}_x \cdot \vec{n}$$

$$t_y = \vec{\jmath}\vec{\sigma}_x n_x + \vec{\jmath}\vec{\sigma}_y n_y = \sigma_{xy} n_x + \sigma_{yy} n_y = \sigma_{yx} n_x + \sigma_{yy} n_y = \vec{\sigma}_y \cdot \vec{n}$$

$$\Rightarrow \begin{bmatrix} t_x \\ t_y \end{bmatrix} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} \end{bmatrix} \begin{bmatrix} n_x \\ n_y \end{bmatrix} \Rightarrow \boldsymbol{t} = \boldsymbol{\tau}\boldsymbol{n}$$

Force Equilibrium within a 2D Body





- Equilibrium of the infinitesimal 2D domain:

$$-\vec{\sigma}_x\left(x-\frac{\Delta x}{2},y\right)\Delta y+\vec{\sigma}_x\left(x+\frac{\Delta x}{2},y\right)\Delta y-\vec{\sigma}_y\left(x,y-\frac{\Delta y}{2}\right)\Delta x$$
$$+\vec{\sigma}_y\left(x,y+\frac{\Delta y}{2}\right)\Delta x+\vec{b}(x,y)\Delta x\Delta y=\vec{0}$$

$$\Rightarrow\frac{1}{\Delta x}\left[\vec{\sigma}_x\left(x+\frac{\Delta x}{2},y\right)-\vec{\sigma}_x\left(x-\frac{\Delta x}{2},y\right)\right]$$
$$+\frac{1}{\Delta y}\left[\vec{\sigma}_y\left(x,y+\frac{\Delta y}{2}\right)--\vec{\sigma}_y\left(x,y-\frac{\Delta y}{2}\right)\right]+\vec{b}(x,y)=\vec{0}$$

- Let $\Delta x\to 0$ and $\Delta y\to 0$:

$$\frac{\partial\vec{\sigma}_x}{\partial x}+\frac{\partial\vec{\sigma}_y}{\partial y}+\vec{b}=\vec{0}$$

Governing equation

Expressions of Equilibrium (Governing) Equations

$$\frac{\partial \vec{\sigma}_x}{\partial x} + \frac{\partial \vec{\sigma}_y}{\partial y} + \vec{b} = \vec{0}, \qquad \vec{\sigma}_x = \sigma_{xx}\vec{\imath} + \sigma_{xy}\vec{\jmath}, \qquad \vec{\sigma}_y = \sigma_{yx}\vec{\imath} + \sigma_{yy}\vec{\jmath}$$

- Equilibrium (governing) equations along base vectors:

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + b_x = 0 \Rightarrow \vec{\nabla} \cdot \vec{\sigma}_x + b_x = 0$$

$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + b_y = 0 \Rightarrow \vec{\nabla} \cdot \vec{\sigma}_y + b_y = 0$$

$$\Rightarrow \mathbf{0} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & \frac{\partial}{\partial y} \\ 0 & \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} + \begin{bmatrix} b_x \\ b_y \end{bmatrix} = \boldsymbol{\nabla}_s^T \boldsymbol{\sigma} + \boldsymbol{b}$$

Constitutive Equations

- 2D isotropic materials:
  - Plane stress:

$$\boldsymbol{C} = \frac{E}{1-v^2}\begin{bmatrix} 1 & v & 0 \\ v & 1 & 0 \\ 0 & 0 & \frac{1-v}{2} \end{bmatrix}$$

  - Plane strain:

$$\boldsymbol{C} = \frac{E}{(1+v)(1-2v)}\begin{bmatrix} 1-v & v & 0 \\ v & 1-v & 0 \\ 0 & 0 & \frac{1-2v}{2} \end{bmatrix}$$

- Linear elasticity:

$$\boldsymbol{\sigma} = \underline{\boldsymbol{C}}\boldsymbol{\varepsilon}$$

Constant

Strong Form of 2D Stress Analysis

- Governing equation:

$$\boldsymbol{\nabla}_s^T \boldsymbol{\sigma} + \boldsymbol{b} = \boldsymbol{0} \text{ on } \Omega$$

$$\boldsymbol{\nabla}_s^T = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & \frac{\partial}{\partial y} \\ 0 & \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix}$$

- Strain-displacement equation:

$$\boldsymbol{\varepsilon} = \boldsymbol{\nabla}_s \boldsymbol{u}$$

- Constitutive law:

$$\boldsymbol{\sigma} = \boldsymbol{C}\boldsymbol{\varepsilon}$$

- Natural boundary conditions:

$$\boldsymbol{\tau}\boldsymbol{n} = \boldsymbol{t} = \bar{\boldsymbol{t}} \text{ on } \Gamma_t$$

- Displacement boundary conditions:

$$\boldsymbol{u} = \bar{\boldsymbol{u}} \text{ on } \Gamma_u$$

- Restrictions to boundary regions:

$$\Gamma_t \cap \Gamma_u = \emptyset$$

$$\Gamma_t \cup \Gamma_u = \Gamma$$

Boundary regions can be different on 2 base directions.

Weak Form of 2D Stress Analysis (1/3)

- Derivation from the strong form:

$$\boldsymbol{\nabla}_s^T \boldsymbol{\sigma} + \boldsymbol{b} = \boldsymbol{0} \text{ on } \Omega$$

$$\Rightarrow \vec{\nabla} \cdot \vec{\sigma}_x + b_x = 0, \qquad \vec{\nabla} \cdot \vec{\sigma}_y + b_y = 0$$

$$\Rightarrow \int_\Omega w_x \left(\vec{\nabla} \cdot \vec{\sigma}_x + b_x\right) d\Omega = 0, \qquad \forall w_x \in U_0$$

$$\int_\Omega w_y \left(\vec{\nabla} \cdot \vec{\sigma}_y + b_y\right) d\Omega = 0, \qquad \forall w_y \in U_0$$

$$\vec{w} = w_x \vec{\imath} + w_y \vec{\jmath}$$

- Application of Green's theorem:

$$\int_\Omega w_x \vec{\nabla} \cdot \vec{\sigma}_x \, d\Omega = \oint_\Gamma w_x \vec{\sigma}_x \cdot \vec{n} d\Gamma - \int_\Omega \vec{\nabla} w_x \cdot \vec{\sigma}_x d\Omega$$

$$\int_\Omega w_y \vec{\nabla} \cdot \vec{\sigma}_y \, d\Omega = \oint_\Gamma w_y \vec{\sigma}_y \cdot \vec{n} d\Gamma - \int_\Omega \vec{\nabla} w_y \cdot \vec{\sigma}_y d\Omega$$

$$\Rightarrow \int_\Omega \left(w_x \vec{\nabla} \cdot \vec{\sigma}_x + w_y \vec{\nabla} \cdot \vec{\sigma}_y\right) d\Omega$$

$$= \oint_{\Gamma_t} \left(w_x \vec{\sigma}_x \cdot \vec{n} + w_y \vec{\sigma}_y \cdot \vec{n}\right) d\Gamma - \int_\Omega \left(\vec{\nabla} w_x \cdot \vec{\sigma}_x + \vec{\nabla} w_y \cdot \vec{\sigma}_y\right) d\Omega$$

Weak Form of 2D Stress Analysis (2/3)

$$\int_{\Omega} w_x\left(\vec{\nabla}\cdot\vec{\sigma}_x + b_x\right)d\Omega = 0, \qquad \forall w_x \in U_0$$

$$\int_{\Omega} w_y\left(\vec{\nabla}\cdot\vec{\sigma}_y + b_y\right)d\Omega = 0, \qquad \forall w_y \in U_0$$

$$\int_{\Omega} \left(w_x\vec{\nabla}\cdot\vec{\sigma}_x + w_y\vec{\nabla}\cdot\vec{\sigma}_y\right)d\Omega$$

$$= \oint_{\Gamma_t} \left(w_x\vec{\sigma}_x\cdot\vec{n} + w_y\vec{\sigma}_y\cdot\vec{n}\right)d\Gamma - \int_{\Omega} \left(\vec{\nabla}w_x\cdot\vec{\sigma}_x + \vec{\nabla}w_y\cdot\vec{\sigma}_y\right)d\Omega$$

$$\Rightarrow \int_{\Omega} \vec{w}\cdot\vec{b}\,d\Omega + \oint_{\Gamma_t} \left(w_x\vec{\sigma}_x\cdot\vec{n} + w_y\vec{\sigma}_y\cdot\vec{n}\right)d\Gamma$$

$$= \int_{\Omega} \left(\vec{\nabla}w_x\cdot\vec{\sigma}_x + \vec{\nabla}w_y\cdot\vec{\sigma}_y\right)d\Omega$$

- Application of natural boundary conditions:

$$\boldsymbol{\tau n} = \bar{\boldsymbol{t}} \text{ on } \Gamma_t$$

$$\Rightarrow t_x = \vec{\sigma}_x\cdot\vec{n}, \qquad t_y = \vec{\sigma}_y\cdot\vec{n}$$

$$\Rightarrow \int_{\Omega} \vec{w}\cdot\vec{b}\,d\Omega + \oint_{\Gamma_t} \vec{w}\cdot\vec{t}\,d\Gamma = \int_{\Omega} \left(\vec{\nabla}w_x\cdot\vec{\sigma}_x + \vec{\nabla}w_y\cdot\vec{\sigma}_y\right)d\Omega$$

Weak Form of 2D Stress Analysis (3/3)

$$\int_{\Omega} \vec{w} \cdot \vec{b} d\Omega + \oint_{\Gamma_t} \vec{w} \cdot \vec{t} d\Gamma = \int_{\Omega} \left(\vec{\nabla} w_x \cdot \vec{\sigma}_x + \vec{\nabla} w_y \cdot \vec{\sigma}_y\right) d\Omega$$

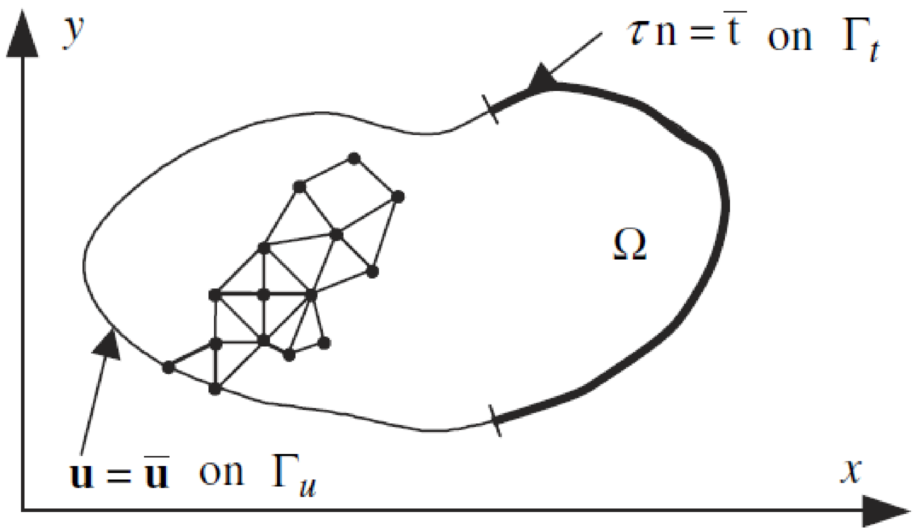
$$\vec{\nabla} w_x \cdot \vec{\sigma}_x + \vec{\nabla} w_y \cdot \vec{\sigma}_y = \frac{\partial w_x}{\partial x}\sigma_{xx} + \frac{\partial w_x}{\partial y}\sigma_{xy} + \frac{\partial w_y}{\partial x}\sigma_{xy} + \frac{\partial w_y}{\partial y}\sigma_{yy} = \underbrace{\begin{bmatrix} \frac{\partial w_x}{\partial x} & \frac{\partial w_y}{\partial y} & \frac{\partial w_x}{\partial y} + \frac{\partial w_y}{\partial x} \end{bmatrix}}_{(\boldsymbol{\nabla}_s \boldsymbol{w})^T} \underbrace{\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix}}_{\boldsymbol{\sigma}}$$

$$\Rightarrow \int_{\Omega} \boldsymbol{w}^T \boldsymbol{b} d\Omega + \oint_{\Gamma_t} \boldsymbol{w}^T \bar{\boldsymbol{t}} d\Gamma = \int_{\Omega} (\boldsymbol{\nabla}_s \boldsymbol{w})^T \boldsymbol{\sigma}\, d\Omega = \int_{\Omega} (\boldsymbol{\nabla}_s \boldsymbol{w})^T \boldsymbol{C} \boldsymbol{\nabla}_s \boldsymbol{u}\, d\Omega$$

Find $\boldsymbol{u} \in U = \{\boldsymbol{u} | \boldsymbol{u} \in H^1, \boldsymbol{u} = \bar{\boldsymbol{u}} \text{ } on \text{ } \Gamma_u\}$, such that the equation holds for:

$$\forall \boldsymbol{w} \in U_0 = \{\boldsymbol{w} | \boldsymbol{w} \in H^1, \boldsymbol{w} = \boldsymbol{0} \text{ } on \text{ } \Gamma_u\}$$

Discretization for 2D Stress Analysis



- u_x and u_y are generally approximated by the same shape functions.
- 2D nodal value of $\boldsymbol{u} = [u_x \quad u_y]^T$:
$$\boldsymbol{d} = [u_{x1} \quad u_{y1} \quad \cdots \quad u_{xn_{np}} \quad u_{yn_{np}}]^T$$

- Approximation of trial solutions and weight functions at the element level:
$$\boldsymbol{u}(x,y) \approx \boldsymbol{u^e}(x,y) = \boldsymbol{N^e}(x,y)\boldsymbol{d}^e$$

$$\boldsymbol{w}^T(x,y) \approx \boldsymbol{w}^{eT}(x,y) = \boldsymbol{w}^{eT}\boldsymbol{N^e}(x,y)^T$$

$$\boldsymbol{N^e} = \begin{bmatrix} N_1^e & 0 & \dots & N_{n_{en}}^e & 0 \\ 0 & N_1^e & \dots & 0 & N_{n_{en}}^e \end{bmatrix}$$

$$\boldsymbol{d}^e = [u_{x1}^e \quad u_{y1}^e \quad \cdots \quad u_{xn_{en}}^e \quad u_{yn_{en}}^e]^T$$

$$\boldsymbol{w}^e = [w_{x1}^e \quad w_{y1}^e \quad \cdots \quad w_{xn_{en}}^e \quad w_{yn_{en}}^e]^T$$

Nodal Values Calculation (1/2)

- Integration from the weak form:

$$\int_{\Omega} \boldsymbol{w}^T \boldsymbol{b} d\Omega + \oint_{\Gamma_t} \boldsymbol{w}^T \bar{\boldsymbol{t}} d\Gamma = \int_{\Omega} (\boldsymbol{\nabla}_s \boldsymbol{w})^T \boldsymbol{C} \boldsymbol{\nabla}_s \boldsymbol{u} \, d\Omega$$

- Integration by elements:

$$0 = \sum_{e=1}^{nel} \left(\int_{\Omega^e} (\boldsymbol{\nabla}_s \boldsymbol{w}^e)^T \boldsymbol{C}^e \boldsymbol{\nabla}_s \boldsymbol{u}^e \, d\Omega - \int_{\Omega^e} \boldsymbol{w}^{eT} \boldsymbol{b} d\Omega - \oint_{\Gamma_t^e} \boldsymbol{w}^{eT} \bar{\boldsymbol{t}} d\Gamma \right)$$

- Approximation of strain:

$$\boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{bmatrix} = \boldsymbol{\nabla}_s \boldsymbol{u} \approx \boldsymbol{\nabla}_s \boldsymbol{u}^e(x, y) = \underbrace{\boldsymbol{\nabla}_s \boldsymbol{N}^e(x, y)}_{\boldsymbol{B}^e} \boldsymbol{d}^e$$

$$\Rightarrow \boldsymbol{B}^e = \boldsymbol{\nabla}_s \boldsymbol{N}^e = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \begin{bmatrix} N_1^e & 0 & \dots & N_{n_{en}}^e & 0 \\ 0 & N_1^e & \dots & 0 & N_{n_{en}}^e \end{bmatrix}$$

$$\Rightarrow \boldsymbol{B}^e = \begin{bmatrix} \frac{\partial N_1^e}{\partial x} & 0 & \dots & \frac{\partial N_{n_{en}}^e}{\partial x} & 0 \\ 0 & \frac{\partial N_1^e}{\partial y} & \dots & 0 & \frac{\partial N_{n_{en}}^e}{\partial y} \\ \frac{\partial N_1^e}{\partial y} & \frac{\partial N_1^e}{\partial x} & \dots & \frac{\partial N_{n_{en}}^e}{\partial y} & \frac{\partial N_{n_{en}}^e}{\partial x} \end{bmatrix}$$

$$(\boldsymbol{\nabla}_s \boldsymbol{w}^e)^T = (\boldsymbol{\nabla}_s \boldsymbol{N}^e \boldsymbol{w}^e)^T = \boldsymbol{w}^{eT} \boldsymbol{B}^{eT}$$

Nodal Values Calculation (2/2)

$$0 = \sum_{e=1}^{nel} \left(\int_{\Omega^e} (\boldsymbol{\nabla}_s \boldsymbol{w}^e)^T \boldsymbol{C}^e \boldsymbol{\nabla}_s \boldsymbol{u}^e \, d\Omega - \int_{\Omega^e} \boldsymbol{w}^{eT} \boldsymbol{b} d\Omega - \oint_{\Gamma_t^e} \boldsymbol{w}^{eT} \bar{\boldsymbol{t}} d\Gamma \right)$$

$$\boldsymbol{u}^{\boldsymbol{e}} = \boldsymbol{N}^e \boldsymbol{d}^e, \qquad \boldsymbol{w}^{\boldsymbol{e}T} = \boldsymbol{w}^{eT} \boldsymbol{N}^{eT}, \qquad \boldsymbol{\nabla}_s \boldsymbol{u}^e = \boldsymbol{B}^e \boldsymbol{d}^e, \qquad (\boldsymbol{\nabla}_s \boldsymbol{w}^e)^T = \boldsymbol{w}^{eT} \boldsymbol{B}^{eT}$$

$$\Rightarrow 0 = \sum_{e=1}^{nel} \left(\int_{\Omega^e} \boldsymbol{w}^{eT} \boldsymbol{B}^{eT} \boldsymbol{C}^e \boldsymbol{B}^e \boldsymbol{d}^e d\Omega - \int_{\Omega^e} \boldsymbol{w}^{eT} \boldsymbol{N}^{eT} \boldsymbol{b} d\Omega - \oint_{\Gamma_t^e} \boldsymbol{w}^{eT} \boldsymbol{N}^{eT} \bar{\boldsymbol{t}} d\Gamma \right)$$

$$\Rightarrow 0 = \sum_{e=1}^{nel} \boldsymbol{w}^{eT} \left(\int_{\Omega^e} \boldsymbol{B}^{eT} \boldsymbol{C}^e \boldsymbol{B}^e d\Omega \, \boldsymbol{d}^e - \int_{\Omega^e} \boldsymbol{N}^{eT} \boldsymbol{b} d\Omega - \oint_{\Gamma_t^e} \boldsymbol{N}^{eT} \bar{\boldsymbol{t}} d\Gamma \right), \qquad \forall \boldsymbol{w}_F$$

Direct assembly in 2D:

$$[u_{x1}, u_{y1}, u_{x2}, u_{y2}, \ldots]^T$$

Expansion to 3D

$$\boldsymbol{\nabla}_s^T = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} & 0 \\ 0 & \frac{\partial}{\partial y} & 0 & \frac{\partial}{\partial x} & 0 & \frac{\partial}{\partial z} \\ 0 & 0 & \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \end{bmatrix}$$

$$\boldsymbol{\sigma}^T = [\sigma_{xx} \quad \sigma_{yy} \quad \sigma_{zz} \quad \sigma_{xy} \quad \sigma_{xz} \quad \sigma_{yz}]^T$$

$$\boldsymbol{\varepsilon}^T = [\varepsilon_{xx} \quad \varepsilon_{yy} \quad \varepsilon_{zz} \quad \varepsilon_{xy} \quad \varepsilon_{xz} \quad \varepsilon_{yz}]^T$$

$$\boldsymbol{C} = \frac{E}{(1+v)(1-2v)} \begin{bmatrix} 1-v & v & v & & & \\ v & 1-v & v & & \mathbf{0} & \\ v & v & 1-v & & & \\ & & & \frac{1-2v}{2} & & \\ & \mathbf{0} & & & \frac{1-2v}{2} & \\ & & & & & \frac{1-2v}{2} \end{bmatrix}$$

$v = 0.5$ – incompressible and needs special formulation

All the other matrices follow the general x-y-z order.

The End