香港中文大學
The Chinese University of Hong Kong

MAEG5160: Design for Additive Manufacturing

Lecture 10: Extensions and Applications – continue

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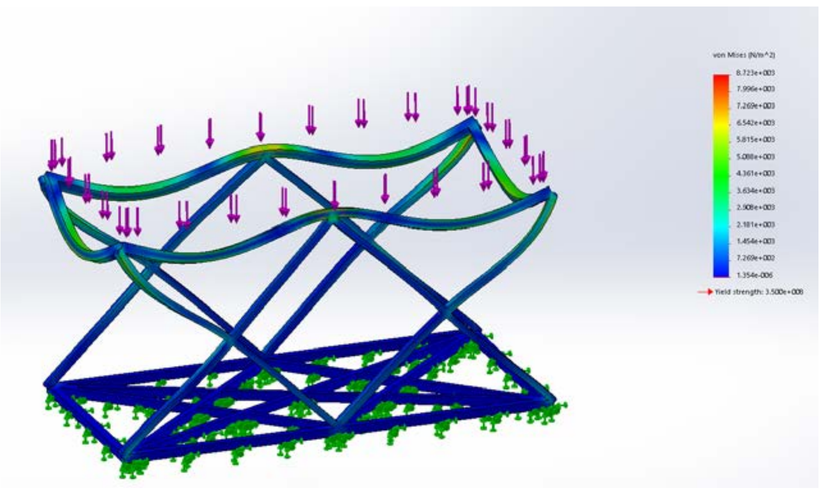
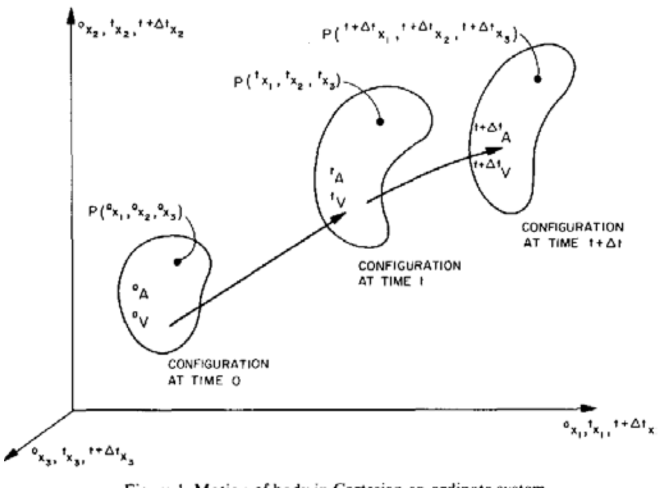
Extensions and Applications – continue

5. Geometrically non-linear problems

For compliance minimization problems displacements are typically small and the problems may be modelled using linear finite element theory. For soft structures, slender structures and mechanisms, however, it is imperative that the problems are modelled using geometrically non-linear finite element analysis. This section discusses objective functions and modelling issues related to stiffness optimization of structures undergoing finite displacements. Later sections will discuss compliant mechanism design, crashworthiness design and other design problems involving geometrical non-linearities. Structures undergoing large displacements mayor may not be subject to large strains. In this section, we assume that strains are small and hence material non-linearity can be ignored.





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5.1 Problem formulation and objective functions

The general topology optimization problem for situations with geometrical non-linearities can in broad terms be written as

$$\min_{\boldsymbol{\rho}} c(\boldsymbol{\rho})$$
$$\text{s.t.} : \mathbf{r} = \mathbf{0} \ ,$$
$$\sum_{e=1}^{N} v_e \rho_e \le V, \qquad 0 < \rho_{min} \le \rho_e \le 1, \quad e = 1, \ldots, N \ ,$$

where $\mathbf{r}$ is the residual in obtaining the structural equilibrium, $c(\boldsymbol{\rho})$ is the objective functions to be defined later and all other symbols have been defined previously.

This topology optimization problem only differs from the standard topology optimization problems in that the equilibrium r = 0 must be found using an iterative procedure. For the linear analysis problems discussed previously, the equilibrium is found from the solution of a linear system of (finite element) equations.

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In the following we use the (non-linear) Green-Lagrange strain measure to model the strain-displacement relations, that is

$$\eta_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i} + u_{k,i}\ u_{k,j})$$

where u is the point-wise displacement and subscript "$_{,j}$" means differentiation with respect to coordinate j. For later use we define the displacement dependent matrix $\mathbf{B}$ as the matrix that transforms a change in displacement $d\mathbf{u}$ into a change in strain, i.e.

$$d\eta = \mathbf{B}(\mathbf{u})\ d\mathbf{u}\ ,$$

where $\mathbf{u}$ is the finite element displacement vector.

The (linear) Hooke's law for Piola-Kirchhoff stresses and Green-Lagrange strains with SIMP interpolations is written as

$$s_{ij} = \rho^p\ E^0_{ijkl}\ \eta_{kl}\ ,$$

where E^0_{ijkl} is the constitutive tensor for a solid, linear, isotropic material.

The residual is defined as the error in obtaining the equilibrium

$$\mathbf{r}(\mathbf{u}) = \mathbf{f} - \int_V \mathbf{B}(\mathbf{u})^T \mathbf{s}(\mathbf{u})\ dV$$

where $\mathbf{f}$ is the external force vector and $\mathbf{s}$ is the Piola-Kirchhoff stress written in vector form. Following a Total Lagrangian approach, the integration is performed over the undeformed volume. The equilibrium has been found when the residual vector is equal to the zero vector. This finite element equilibrium may be found incrementally or in one load step using the iterative Newton-Raphson method. Both kinds of methods require the determination of the tangent stiffness matrix

$$\mathbf{K}_T = -\frac{\partial \mathbf{r}}{\partial \mathbf{u}}.$$

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5.2 Choice of objective function for stiffness optimization

The first goal we consider is to maximize the stiffness of a structure undergoing large deformations. Several different objective functions may be considered in order to solve this task and we will here deal with three possibilities, namely: minimization of end-compliance, minimization of a weighted sum of end-compliances and minimization of the complementary elastic work. These objective functions are discussed in the following.

Defining end-compliance as the compliance of a structure in its equilibrium configuration, the objective function can be written as

$$c(\boldsymbol{\rho}) = \mathbf{f}^T \mathbf{u}$$

where $\mathbf{u}$ is the displacement vector for the structure in its equilibrium position. Assuming design independent loads, the sensitivity of the end-compliance with respect to a change in element density may be found by the adjoint method to be

$$\frac{dc}{d\rho_e} = \boldsymbol{\lambda}^T \frac{\partial \mathbf{r}}{\partial \rho_e} ,$$

where the adjoint field $\boldsymbol{\lambda}$ is the solution to the linear adjoint problem $\mathbf{K}_T \, \boldsymbol{\lambda} = \mathbf{f}$.

Solving the adjoint system is computationally cheap because the factorized tangent stiffness matrix already has been found during the equilibrium iterations and a solution only requires one extra forward/ backward substitution.

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Multiple loading cases For multiple loading cases the objective function is simply a weighted sum of end-compliances

$$c(\boldsymbol{\rho}) = \sum_{i=1}^{M} w_i \; \mathbf{f}_i^T \; \mathbf{u}_i$$

where M is the number of loading cases, $\mathbf{f}_i$ and $\mathbf{u}_i$ are the load and displacement vectors of loading case i, respectively and w_i are weighting factors ($\sum_{i=1}^{M} w_i = 1$). The sensitivity analysis corresponds to a simple weighting of the sensitivities of the individual loading cases.

Minimization of complementary work The last objective we consider is the minimization of the complementary elastic work. Using the trapezoidal method for numerical integration, the complementary work of the external forces can be calculated as

$$c(\boldsymbol{\rho}) = W^C \approx \Delta\mathbf{f}^T \left[\frac{1}{2}\,\mathbf{u}(\mathbf{f}_0) \; + \; \sum_{i=1}^{n-1} \mathbf{u}(\mathbf{f}_i) \; + \; \frac{1}{2}\,\mathbf{u}(\mathbf{f}_n) \right]$$

where n is the number of increments in the load vector. The size of the increments is determined by $\Delta\mathbf{f} = (\mathbf{f}_n - \mathbf{f}_0)/n$, where $\mathbf{f}_0$ and $\mathbf{f}_n$ is the zero and maximum load vectors, respectively. The sensitivity analysis for the complementary work is again found using the adjoint method as described for the end compliance. This results in

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$$\frac{dc}{d\rho_e} = \Delta \mathbf{f}^T \left[\frac{1}{2}\, \boldsymbol{\lambda}_0^T \frac{\partial \mathbf{r}_0}{\partial \rho_e} + \sum_{i=1}^{n-1} \boldsymbol{\lambda}_i^T \frac{\partial \mathbf{r}_i}{\partial \rho_e} + \frac{1}{2}\, \boldsymbol{\lambda}_n^T \frac{\partial \mathbf{r}_n}{\partial \rho_e} \right] \qquad (2.17)$$

where $\boldsymbol{\lambda}_i$ and $\mathbf{r}_i$ are the vectors of adjoint and residuals, respectively, for the load increment i . This means that for the sensitivity analysis we simply have to perform one extra forward/backward substitution for each load step and sum the results in Eq. 2.17.

5.3 Numerical problems and ways to resolve them

In the non-linear finite element analysis, we save computational time by reusing the displacement solution from a previous topology iteration in the new Newton-Raphson equilibrium iteration. This saves a considerable number of finite element iterations, especially when the topology changes get smaller near convergence. The computational time highly depends on the size of the applied force. For relatively small forces, obtaining the optimal solution takes 1.5 to 2 times the time to obtain a solution using linear modelling. For larger loads where local buckling can be observed, the time in which the optimal solution is found can be 5 to 10 times higher than for the linear case. When the finite element analysis is based on the Green-Lagrange or other non-linear strain measures, large displacements may cause the tangent stiffness matrix to become indefinite or even negative definite. This phenomena is observed frequently during the topology optimization process and results in non-convergence of the equilibrium iterations. Numerical experiments show that the problem occurs in low-density elements with minimum or close to minimum stiffness. The problem is "artificial" since the elements with minimum stiffness represent void and therefore their behaviour should not influence the structural response. Since the problem is an artefact of the numerical model, different schemes may be devised to circumvent the problem.

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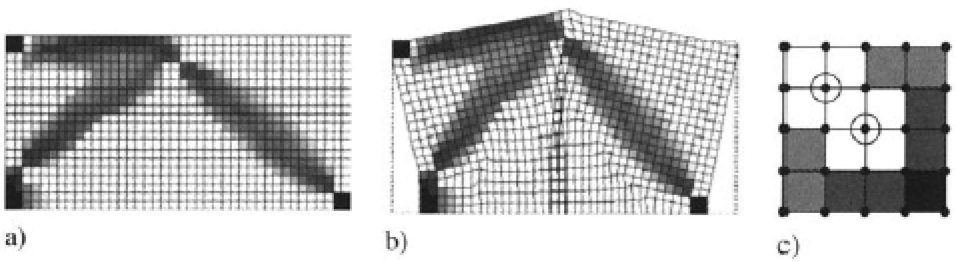
Ignoring convergence in low density elements Usually, the Newton-Raphson iterative scheme is stopped when the changes in nodal displacements get below a certain value. For the topology optimization case, nonconvergence occurs when the displacements oscillate in nodes surrounded by "void" (minimum density) elements. Since these nodes should have no structural importance one can circumvent the problem by relaxing the convergence criterion for these nodes in the equilibrium iterations, that is, those nodes surrounded by void elements are eliminated from the convergence criterion. This solution to the problem is efficient and seldomly causes convergence problems. In the few cases where the procedure does not converge after 20 iterations, the displacement vector is reset to zero and the equilibrium iterations are restarted.

Element removal Another way to circumvent the problem is to remove elements with minimum density from the design domain. Element removal may jeopardize convergence to the right minimum since re-appearance of material in the removed elements is impossible. Examples show that the "re-appearance" of material is crucial for the design process. Therefore one should include a criterion for the "re-appearance" of elements. This can be based on the same type of filtering techniques that are used to ensure mesh independency.



a) Original mesh, b) distortion of finite element mesh causing ill convergence of Newton-Raphson procedure and c) prevention of ill-convergence by ignoring "low-density" nodes (indicated by circles) in the convergence criterion.

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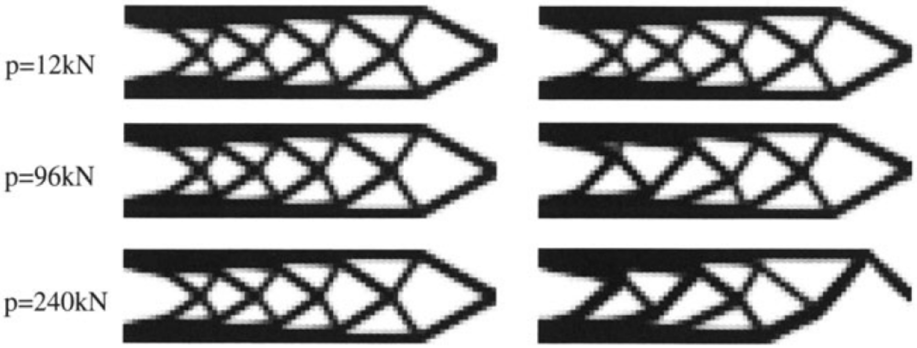
5.4 Examples

It is not typical that structures optimized for stiffness undergo large displacements. Nonetheless it may happen for very slender structures or for structures built from very soft materials such as Nylon.

Optimal topologies for maximum stiffness Results from minimizing the end compliance of a cantilever beam for three different load magnitudes are shown in the right column. The left column shows the topologies obtained using linear modelling which are independent of the load magnitude. We notice that the topology obtained for the large displacement modelling and the smallest load is equal to the topology obtained with a small displacement modelling. We also see that the non-linear topologies become less symmetric for larger loads. Finally, we notice that the optimized topologies become increasingly degenerated for larger loads.



Optimized topologies for end-loaded cantilever example. Left column: Optimized topology for small displacement FE-modelling. Right column: Optimized topologies for large displacement FE-modelling

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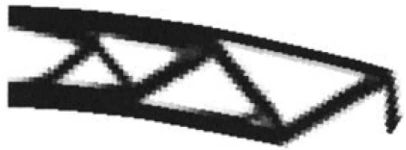

Deformed configuration of the topology optimized for 144 kN in previous case. Note that the right-most bar supporting the load is un-bent in the deformed configuration

The deformed state of the structure optimized for the largest load is shown above. It is seen that the bar in the right side of the structure (which supports the load) is vertical in the deformed configuration. In this configuration the bar is un-bent. For any other load the bar will bend, resulting in a bad compliance for the structure. This example therefore demonstrates, that *minimization of the end-compliance may result in degenerated structures which only can support the load they are designed for.* However, the problem is worse for the non-linear case. Here the structure may not only collapse for a load having another *direction* than the design load, but it may also collapse for a load which just in *magnitude* is different from the design load.

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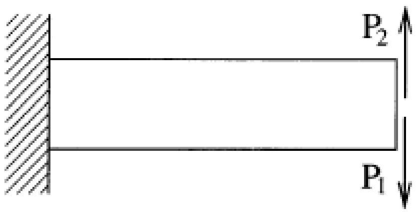
One can partially circumvent the problem of degenerated topologies by applying a minimization of a weighted sum of end-compliances. Figure below shows a design optimized for two loadings, one pointing upwards and one pointing downwards. As expected, the optimal topology is symmetric and in fact cannot be differentiated in topology nor in compliance from the results obtained for small displacement theory. It is interesting to note that the compliance of the symmetric structure is only 2.5% lower than for the non-symmetric one.



two-load case problem with two large loads acting in opposite vertical directions

Optimized topology for minimization of complementary elastic work

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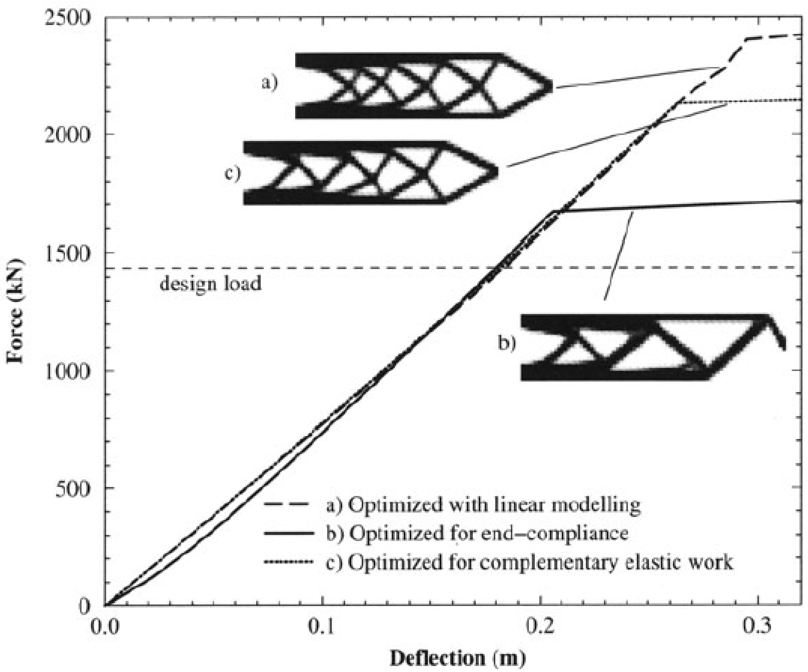
In other situations the results are not so convincing, and one may obtain structures that still become unstable due to buckling at a load which is not a design load; this depends very intricately on the choice of loads. The most effective way to prevent this is to operate with the complementary elastic work. In this way, we can make sure that the structure is stable for any load up to the maximum design load. An example topology obtained for a load of 144 kN and 12 load steps is shown previously and this is a structure seemingly without degeneracies.

A force-displacement diagram for the results obtained for a) small displacement modelling, b) end-compliance and c) complementary work minimization is shown below. Notice that the topology optimized for end-compliance has minimum deflection at the design load as expected, but for smaller loads, it has the maximum deflection. The curves for the designs obtained with linear modelling and with minimization of complementary work are almost coinciding, with the latter designs being slightly stiffer for most of the interval. It is also interesting to note that the topology obtained for linear modelling has a higher maximum load than the two others. This means that obtaining a slightly higher stiffness by using non-linear modelling is achieved at the cost of a more critical response to load perturbations.



Force-displacement diagram for the topologies optimized for a) minimum compliance using small displacement finite element analysis, b) minimum end compliance for large displacement analysis and loading of 144 kN and c) minimum complementary elastic work and end-loading of 144 kN

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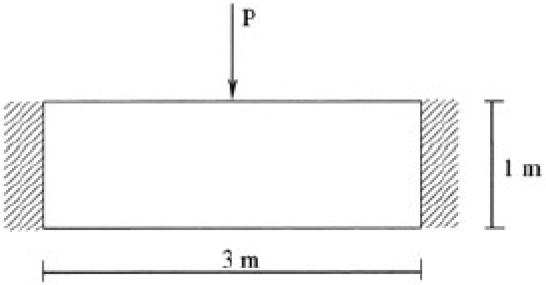
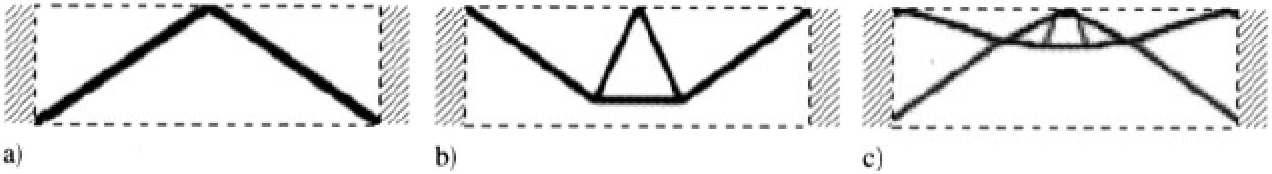
Optimization of a structure with snap-through effect The examples above show that the inclusion of large displacements in the topology optimization process does not significantly affect the resulting topologies. Also, the force-displacement curves obtained for small displacement optimization and complementary work minimization only differ by a few percent. However, in some cases the difference can be extremely large as will be seen in this example.

Thus, for the design problem sketched below we obtain the solutions shown, using linear and non-linear analysis, respectively. It is seen that the two topologies are totally different due to the buckling effects. The topology obtained using linear modelling consists of two long beams under compression and when using non-linear modelling the compressed beams buckle and the whole structure snaps through. Using non-linear modelling in the design process, the resulting topology consists of two longer beams in tension and two short beams in compression. Obviously, the topology in the previous case is optimal also in the non-linear case if the force is applied in the upward direction instead of in the downward direction. To obtain a structure that is stiff for loads in both directions, the topology can be optimized using non-linear modelling and two load-cases, one acting upwards and the other acting downwards. The resulting topology is shown in the final figure and is seen to be a hybrid of the two single-load topologies.





Optimal topologies for the design problem on the left a) optimized topology for small displacement finite element modelling and b) optimized topology for large displacement modelling and c) optimized topology for large displacement modelling and two load-cases

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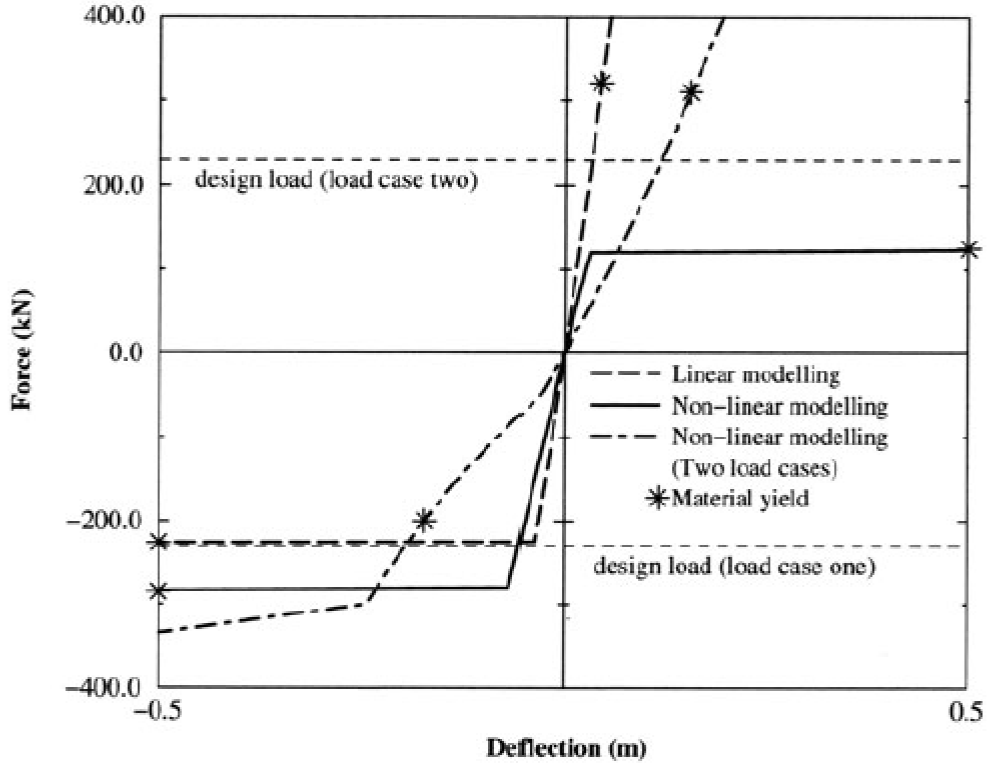
The non-linear responses for the three topologies are shown below. It is seen that the topology which is optimized using linear modelling buckles just below the design load, whereas the buckling load of the design optimized using non-linear modelling is well above the design load. Moreover, the buckling load for the two-load structure is also seen to be higher than the design load.



Force-displacement diagram for the optimized topologies in previous case found using linear, non-linear and two-load non-linear finite element modelling

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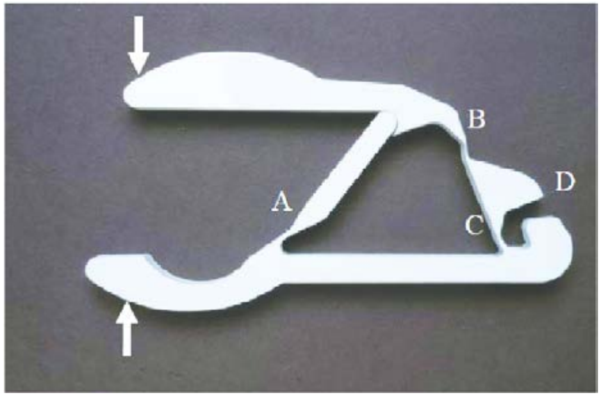
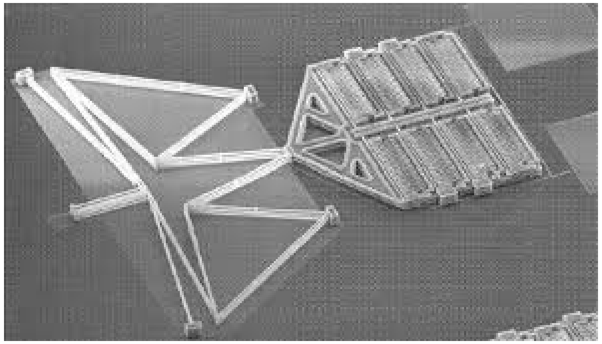
6. Synthesis of compliant mechanisms

Compliant mechanisms attain their mobility from flexibility of their constituents as opposed to their rigid body counterparts that attain their mobility from hinges, bearings and sliders. The main advantages of compliant mechanisms are that they can be built using fewer parts, require fewer assembly processes and need no lubrication. Special care must be taken, however, in designing compliant mechanisms in order to obtain sufficient mobility and safety against failure due to fatigue. An important application of compliant mechanisms lies in Micro Electro-Mechanical Systems (MEMS) which cannot be manufactured using typical assembly processes and may not make use of hinges and bearings since friction dominates at the small (typically submilimeter) scale.





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One of the most important objectives in compliant mechanism synthesis (and rigid-body mechanism synthesis for that sake) is to be able to control the ratios between output and input displacements or output and input forces which are described by the geometrical and mechanical advantages, respectively. It is also important to be able to synthesize mechanisms with prescribed output paths for given inputs.

Topology optimization of compliant mechanisms can be performed based on continuum as well as truss and frame discretizations. Each discretization has advantages and disadvantages. The truss and frame formulations may have crossing members which cannot be manufactured in microscale. On larger scales, however, overlaps are allowed and may result in mechanisms with larger displacement ranges. Here we concentrate on the continuum discretization but the basic procedures apply to truss and frame discretizations as well.

Since it is extremely important to use large displacement theory in compliant mechanism design, this section is based on geometrically non-linear modelling. The simplified problem for linear analysis, which may be used as a first step into compliant mechanism design is discussed at the end of this section.

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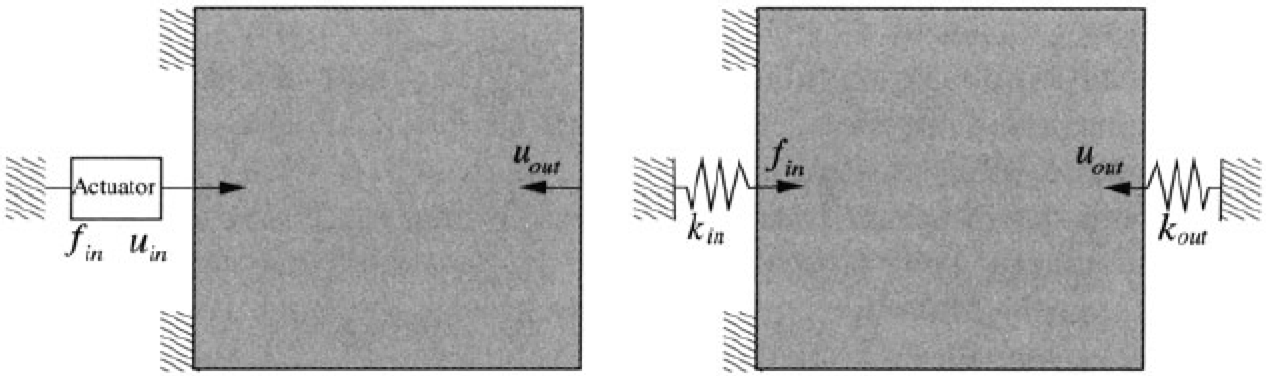
As an example of a compliant mechanism design problem we consider the displacement inverter below. The goal of the topology optimization problem is to design a structure that converts an input displacement on the left edge to a displacement in the opposite direction on the right edge. In order to be able to transfer work from the input port to the output port, the inversion must be performed in a structurally efficient way. Also, it must be possible to control the displacement amplification of the mechanism. Finally, the modelling of the input force and displacements should model physical actuators that may have limited strokes, actuation and blocking forces. In the following, we discuss a formulation that satisfies all of these requirements.



A basic compliant mechanism design problem: the displacement inverter. Left: the basic design problem and Right: spring and load model for the input actuator and workpiece.

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6.1 Problem setting

Assuming that the input actuator is a linear strain based actuator it can be modelled by a spring with stiffness k_{in} and a force f_{in}. Examples of strain based actuation principles are piezoelectric, thermal or electrothermal heating, shape memory alloys, etc., which are characterized by their blocking force (f_{in}) and their free (un-loaded) displacement (f_{in}/k_{in}). An alternative to the linear strain based actuator could be a constant force actuator with a limited stroke. Such an actuator can be modelled by a force f_{in} and a non-linear spring which has a very small stiffness up to the maximum stroke value u_{in} and a very high stiffness after u_{in} so that further displacement is prevented.

The goal of the optimization problem is to maximize the displacement u_{out} (or force or work) performed on a workpiece modelled by a spring with stiffness k_{out}. By specifying different values of k_{out} we can control the displacement amplification. If we specify a low value of k_{out} we get large output displacements and vice versa. In order to maximize the work on the output spring, the available material must be distributed in the structurally most efficient way. An optimization problem incorporating these ideas can be written as

$$
\begin{aligned}
&\max_{\boldsymbol{\rho}} \quad u_{out} \\
&\text{s.t.}: \quad \mathbf{r} = \mathbf{0} \\
&\qquad \sum_{e=1}^{N} v_e \rho_e \le V, \qquad 0 < \rho_{min} \le \rho_e \le 1, \quad e = 1, \dots, N
\end{aligned}
$$

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where r is the finite element residual for the analysis problem with the applied load f_{in}. This optimization problem is very similar to the minimization problem formulated for the minimization of end-compliance.

We now express the displacement at the output point as $u_{out} = \mathbf{l}^T \mathbf{u}$, where $\mathbf{l}$ is a vector with the value 1 at the degree of freedom corresponding to the output point and with zeros at all other places. Then sensitivity of the output displacement can be found to be given as

$$\frac{\partial u_{out}}{\partial \rho_e} = \boldsymbol{\lambda}^T \frac{\partial \mathbf{r}}{\partial \rho_e}$$

where $\boldsymbol{\lambda}^T$ is the solution to the adjoint load problem

$$\mathbf{K_T} \boldsymbol{\lambda} = -\mathbf{l} \ .$$

The simple compliant mechanism optimization problem is of the same form as the compliance minimization problems discussed previously, in the sense that a simple objective function is to be minimized within the limitation of a single linear constraint on volume. Therefore, we may also use an optimality criteria approach to solve it. However, the fixed-point type density update has to be modified since the sensitivity of the objective function may take both positive and negative signs. A (heuristic) modification that results in a fairly stable convergence is

$$\rho_e^{K+1} = \rho_e^K \left[\frac{\max(0, \ -\frac{\partial u_{out}}{\partial \rho_e})}{\lambda \ v_e} \right]^\eta$$

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Whereas the damping coefficient η for linear compliance minimization problems was chosen as 0.5 in order to ensure stable convergence, it sometimes has to be chosen a bit lower to ensure stable convergence in compliant mechanism design problems. The best convergence, however, is obtained using a mathematical programming algorithm like MMA. The problem formulation for compliant mechanism synthesis described so far is very simple and does not allow for multiple inputs or outputs or for a very detailed control of the output ports. The following sections discuss extensions that cater for such aspects.

6.2 Output control

Control of output direction. It is here assumed that the structure is symmetric and therefore the output displacement is a horizontal movement. In other cases where the output displacement does not coincide with a line of symmetry or if an inclined output displacement is specified, the problem formulation does not ensure an output displacement along the desired direction. It only ensures that the component of the output displacement along the desired direction is maximized. This effect is clearly seen in the example below where the output displacement is maximized in the negative horizontal direction. However, the vertical displacement of the resulting topology is actually bigger than the horizontal displacement.

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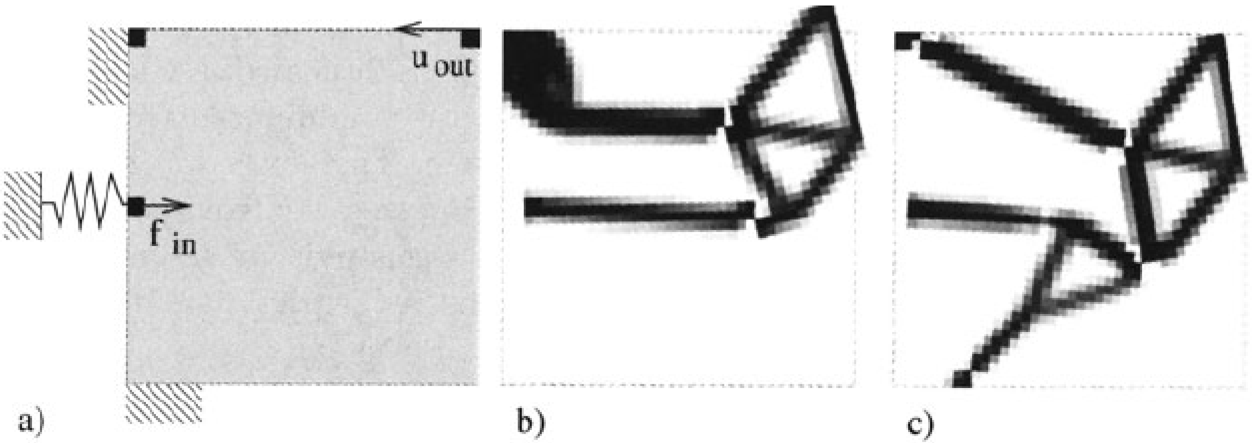

Example with (b) and without (c) cross-sensitivity constraint

This problem can be handled by adding an extra constraint to the optimization problem (2.18). The constraint ensures that the relative displacement $\hat{u}_{out}$ perpendicular to the output displacement u_{out} is below a small number ϵ, i.e.

$$\frac{\hat{u}_{out}^2}{u_{out}^2} \leq \epsilon^2$$

where ϵ is decreased during the design process. Adding an extra constraint to the topology optimization problem is not problematic if one uses mathematical programming (like the MMA) for solving the design problem. Nevertheless, one finds that much work in the area operates with all requirements formulated in one objective function (with weighted multiple objectives) in order to use simple algorithms. This has the disadvantage that it is difficult to have precise control of the behaviour of the resulting mechanisms.

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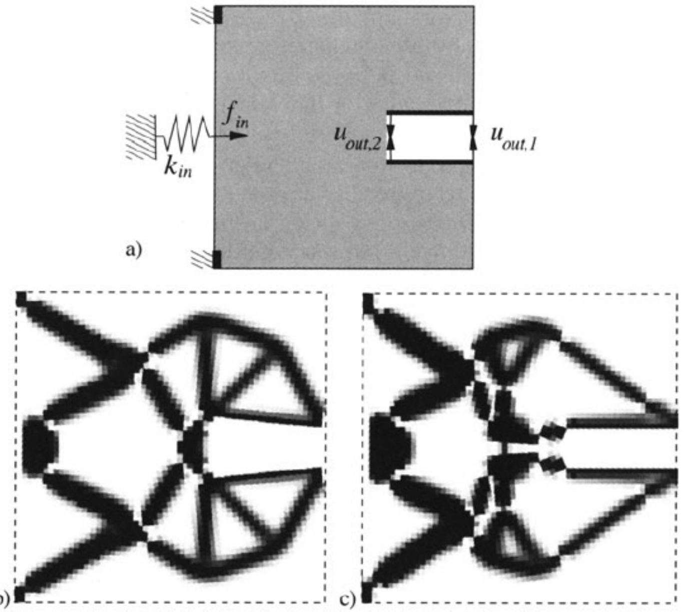
That the constraint manages to control the output displacement as desired is shown in (c). This topology is obtained with ε=0.01, that is, a maximum cross-sensitivity between the two perpendicular output directions of 1 %. The added constraint results in a mechanism with an entirely different topology that ensures that the output point moves horizontally. It is interesting to note that the extra constraint only penalizes the horizontal output by 2% compared to the mechanism in (b) that has an extremely high cross-sensitivity.

Multiple outputs below shows an example design of a gripping mechanism. Here the problem is formulated so that the vertical displacements of the outer "jaws" is maximized, resulting in jaws that open like a pair of scissors. In some cases one may require a parallel movement of the jaws. This can be obtained by reformulating the objective function to a min-max problem

$$\max_{\boldsymbol{\rho}} \min \{u_{out,1},\ u_{out,2}\} \ ,$$

where $u_{out,i}$ is the displacement of the i'th output point. This problem may be solved using a bound formulation for the two objectives $u_{out,1}$, $u_{out,2}$, and gives results as shown



Example with parallel and non-parallel output displacements. a) Design domain, b) the use of one output point results in a non-parallel opening of the 'jaws', while c) the max-min formulation results in a parallel movement of the jaws.

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6.3 Path generating mechanisms

An important problem in compliant mechanism design is the synthesis of path-generating mechanisms. Here, the output point of the mechanism is required to pass through a number M of precision points defined by given displacement vectors $\mathbf{u}^*_{out,m}$. An objective function formulated as the sum of errors may then be written as

$$c(\boldsymbol{\rho}) = \sum_{k=1}^{K} (\mathbf{u}_{out,k} - \mathbf{u}^*_{out,k})^2 \tag{2.20}$$

where $\mathbf{u}_{out,k}$ is the k'th output displacement corresponding to the input load step k.

For complicated output paths it does not make sense to attach a spring at the output point in order to ensure an efficient force transfer, as done in the previous subsections. Instead, apart from (2.20), we also require the output point to pass through the precision points when loaded with counter loads $p_{k,1}$ and $p_{k,2}$, corresponding to counter-loads against the path and counter loads perpendicular to the path at the points k, respectively. The objective function may then be reformulated to

$$c(\boldsymbol{\rho}) = \sum_{i=0}^{K} \alpha_i \sum_{k=1}^{K} (\mathbf{u}_{out,k,i} - \mathbf{u}^*_{out,k})^2,$$

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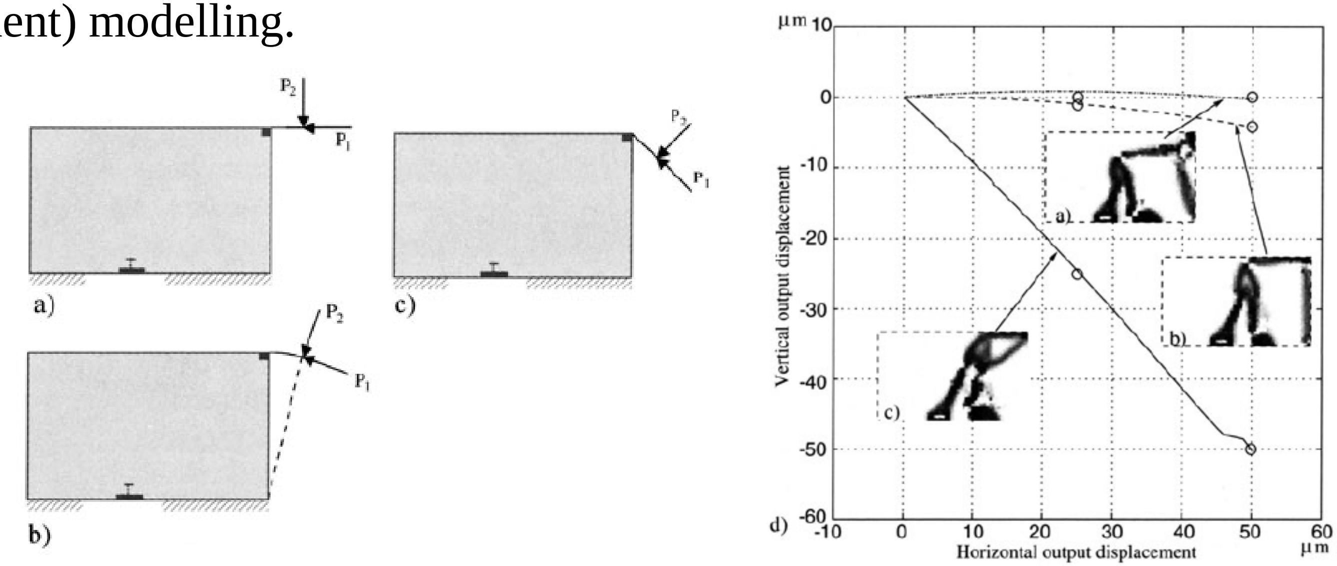
where α_i are weighting factors and $\mathbf{u}_{out,k,i}$, corresponds to the output displacement vectors for no counter load (i=0), for the counter load against the output path (i=1) and for the counter load perpendicular to the output path (i=2).

With the extended optimization formulation which requires the output point to pass through a number of precession points, it is possible to synthesize mechanisms like the ones shown below. Here, the same input displacement can be converted to a straight horizontal output, a straight slanted output and an arch following output, respectively. It is not possible to synthesize such path-generating mechanisms using linear (small displacement) modelling.



Path generating mechanisms with linear inputs. a) Design problem where the output is required to follow a straight horizontal path, b) a straight slanted path and c) an arch. d) Plots of the output paths of the synthesized mechanisms. Path-generating mechanisms cannot be synthesized using linear modelling

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6.4 Linear modelling

A linear version of the compliant mechanism design problem discussed above may be used as an exercise and introduction to compliant mechanism design. However, one must be aware of the severe limitations that such modelling imposes. The linear optimization problem may be written as

$$\max_{\boldsymbol{\rho}} u_{out}$$

$$\text{s.t.} : \mathbf{Ku} = \mathbf{f} \ ,$$

$$\sum_{e=1}^{N} v_e \rho_e \leq V, \qquad 0 < \rho_{min} \leq \rho_e \leq 1, \quad e = 1, \ldots, N$$

If the load vector f is design independent the sensitivities can be found as

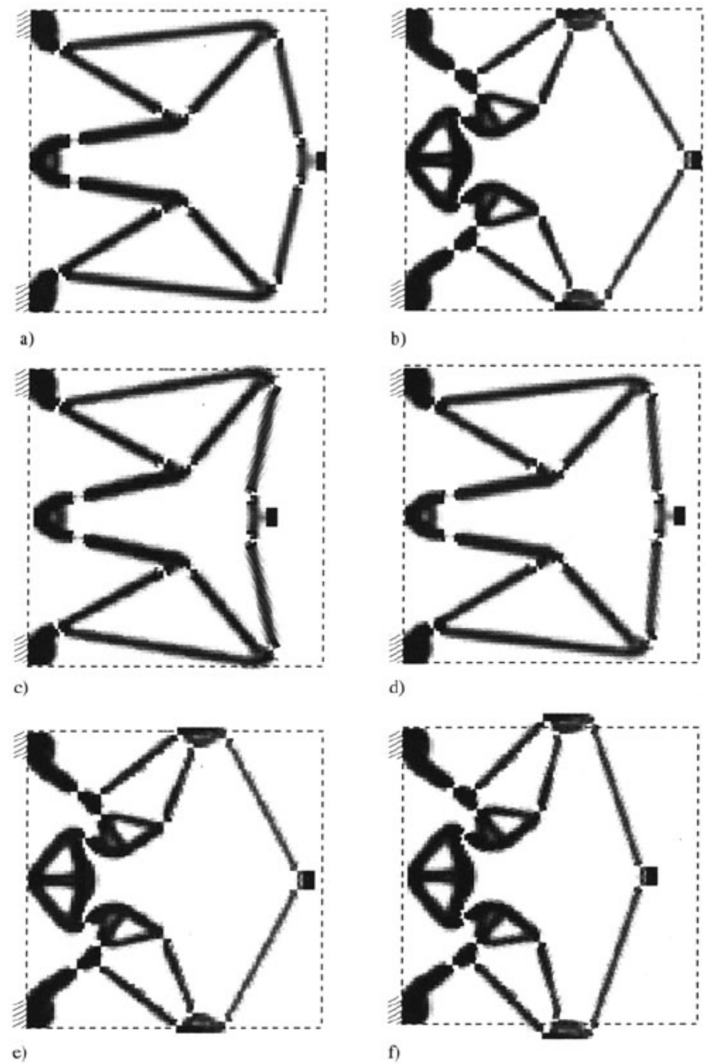
$$\frac{\partial u_{out}}{\partial \rho_e} = \boldsymbol{\lambda}^T \frac{\partial \mathbf{K}}{\partial \rho_e} \mathbf{u}$$

where (as for the non-linear case) $\boldsymbol{\lambda}^T$ is found as the solution to the adjoint load problem $\mathbf{K}\boldsymbol{\lambda} = -\mathbf{l}$.

Extensions and Applications – continue

6.5 Linear vs. non-linear modelling

The mechanism designs obtained using linear analysis typically behave differently when modelled using large displacement analysis. In the best of situations one merely has inaccurate results but in the worst cases the results are useless as large displacement mechanisms. Therefore, the use of geometrically non-linear finite element modelling is absolutely essential for mechanism synthesis. The inverter example below illustrates this. The goal in this synthesis problem is to maximize the work performed on a spring in the negative horizontal direction for an input force in the positive horizontal direction. The mechanism obtained for linear modelling is shown in (a). When modelled using small displacement theory it deflects as shown in (c). When modelled using large displacement theory it deflects as seen in (d). We see that linear theory ignores the locking of the two right-most bars when they reach the vertical position. The mechanism topology obtained using non-linear modelling (b) does not have this problem (f) and its output displacement is, in the large displacement setting, therefore more than two times higher than for the linearly optimized mechanism in (a).



Inverter synthesis. a) Optimized topology using linear modelling, b) optimized topology using non-linear modelling, c) and d) deflection of a) using linear and non-linear modelling, respectively and e) and f) deflection of b) using linear and non-linear modelling, respectively

Extensions and Applications – continue

6.6 Design of thermal actuators

In the applications of compliant mechanism design discussed so far the input load was design independent. However, when designing for example thermally dependent structures or thermal actuators the applied load depends on the design. Example design problems are optimization of thermal circuit breakers or minimization of displacements and stresses due to thermal mismatch. Here the temperature field is considered as uniform and the loads arise due to a uniform change in the temperature. The main difference in the design problem as compared to above is that the sensitivity analysis has to take the dependent loads into account. We will here just write the sensitivity expression for the linear case, were we have

$$\frac{\partial u_{out}}{\partial \rho_e} = \boldsymbol{\lambda}^T \left[\frac{\partial \mathbf{K}}{\partial \rho_e} \mathbf{u} - \frac{\partial \mathbf{f}}{\partial \rho_e} \right]$$

Here $\boldsymbol{\lambda}^T$ is again found as the solution to the adjoint load problem

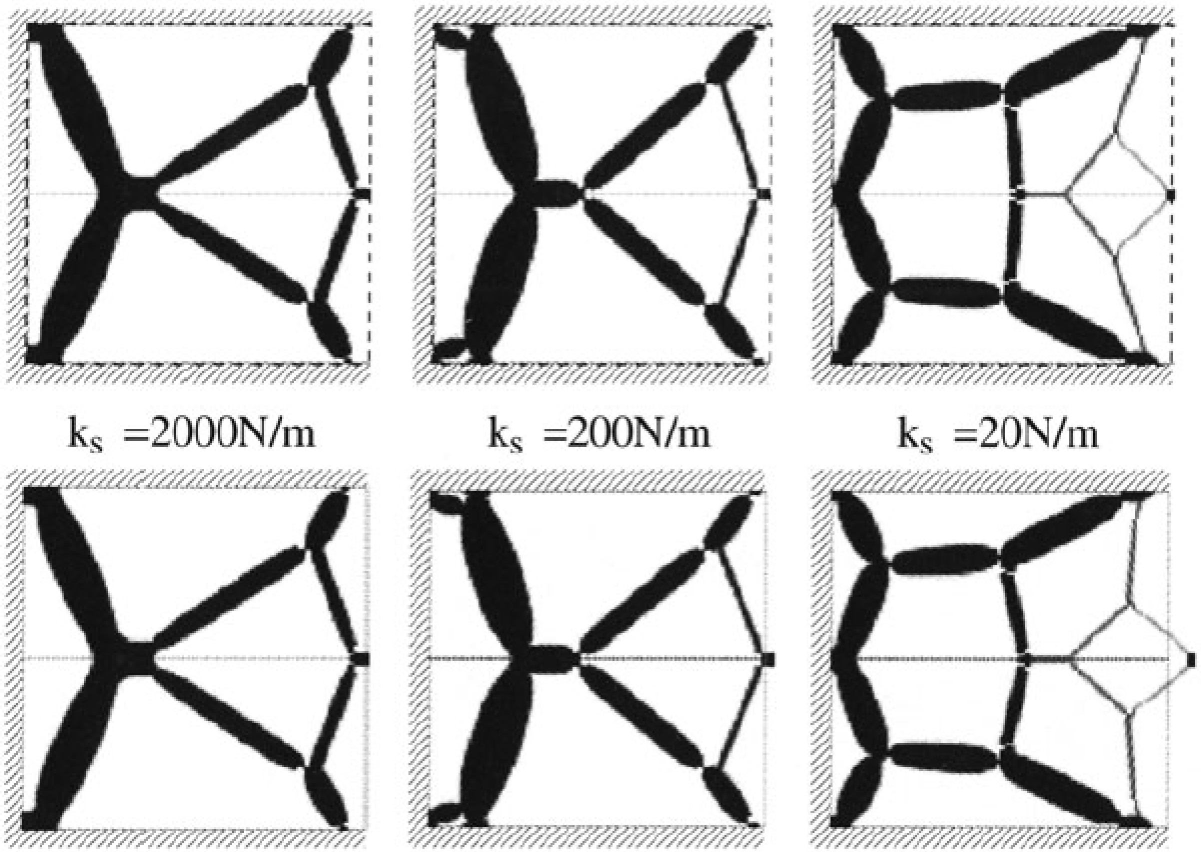
$$\mathbf{K}\boldsymbol{\lambda} = -\mathbf{l}.$$

while the load vector is found as

$$\mathbf{f} = \sum_e \int_{V^e} \mathbf{B}\mathbf{E}\boldsymbol{\alpha} \Delta T dV$$

where $\mathbf{B}$ is the finite element strain displacement matrix, $\mathbf{E}$ is the constitutive matrix, $\boldsymbol{\alpha}$ is the vector of thermal expansion coefficients and ΔT is the (uniform) temperature change.

Extensions and Applications – continue



Design of compliant thermal actuators with actuation caused by a uniform rise in temperature (linear modelling) . Top row: Optimized topologies for output spring stiffnesses of 2000, 200 and 20 N/m, respectively. Bottom row: Displacements patterns of the optimized actuators.

Extensions and Applications – continue

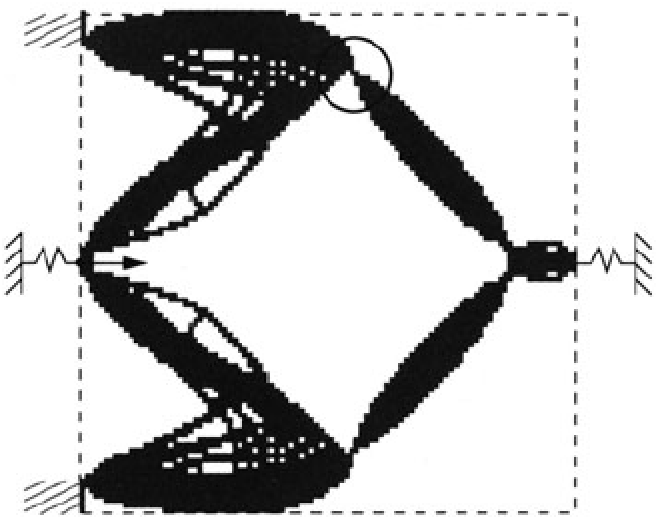
6.7 Computational issues

Mechanism design should, as we have seen, preferably be carried out within the framework of large displacement, non-linear analysis. Compared to stiffness optimization, the problems with excessive distortions of low density elements and ill-convergence of equilibrium iterations are even more pronounced for mechanism design. The methods of ignoring convergence in low density elements or entirely removing low density elements must therefore be implemented.

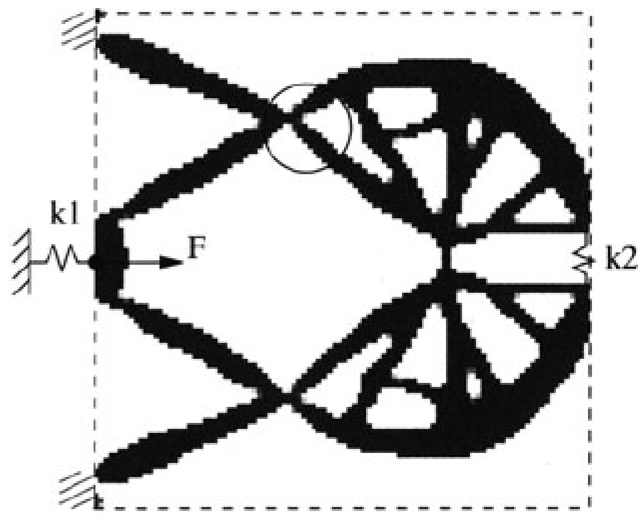
One-node connected hinges In the examples of compliant mechanism design shown so far, one notices that the resulting mechanisms are not truly compliant but rather tend to have what amounts to almost moment-free one-node connected hinges. This is especially the case for examples with large output displacements, i.e., small transfer of forces. In reality the stress in a sharp hinge would approach infinity and the structure would break, so techniques to avoid them are required. Like the checkerboard problem, one-node connected hinges are caused by bad computational modelling that the optimization procedure exploits. In the numerical model, the hinge is modelled by an artificially stiff corner to corner connection of two Q4 elements. Moreover, the stress variations are very badly modelled. The use of higher order elements only partly alleviates the problem, and local stress constraint should probably be added to the formulation . This is computationally prohibitive, so instead other methods have been devised.

Extensions and Applications – continue

Only some of the checkerboard and mesh-independency schemes described previously prevent the non-physical one-node connected hinges. For example, the *filter method* which has been applied in all the examples shown so far is based on a weighted averaging of neighbouring sensitivities. This means that if the gain (sensitivity) in building a hinge is big enough, it will dominate the average and cause hinges to appear. Also, the *perimeter constraint* will not prevent the hinge since only a global constraint is imposed on the design. The *local gradient control* will partially eliminate the problem but will result in "grey" (intermediate density) hinges.



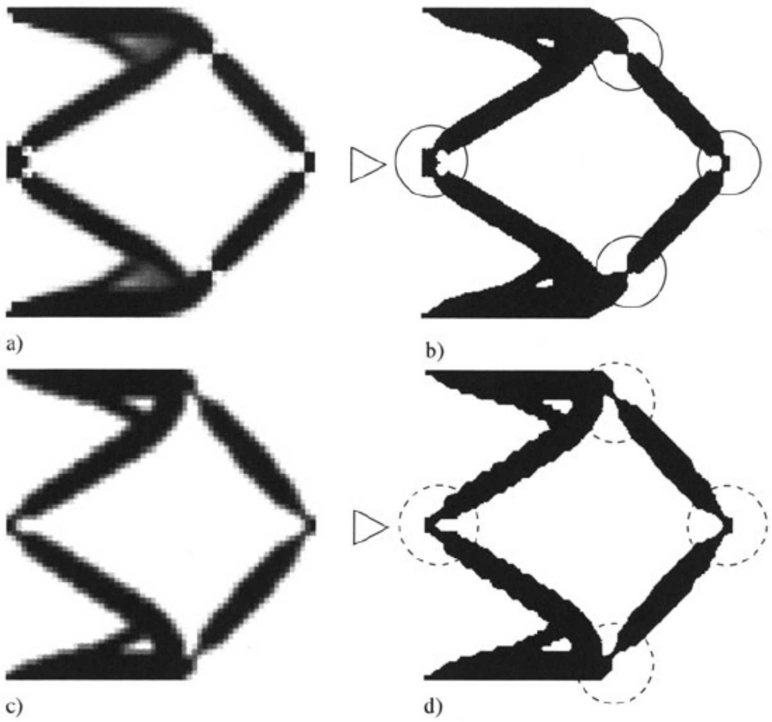
Hinge prevention by the No-Hinge constraint



Hinge prevention by MOLE constraint

Extensions and Applications – continue

The MOLE constraint as well as the checkerboard (No-Hinge) constraint described previously were developed precisely with the hinge problem in mind and they do actually prevents hinges. The former method furthermore imposes a minimum width of the hinge. An alternative, but somewhat questionable, solution is to perform a postprocessing of the resulting topology and substitute the one-node connected hinges with long slender compliant hinges. The post-processing may be based on a contour plot of the topology as seen below.



Post-processing of topology optimization results for the inverter problem. a) Optimized inverter topology obtained using conventional element based densities and c) optimized inverter topology obtained using the nodal based approach. b) and d) are 200 by 100 element structures based on an automatic (one level) contour plot of a) and c) , respectively. The originals have output displacements of -1.18 and -1.11, respectively. The contour based structures have output displacements of -1.09 and -1.12, respectively. Hinge stresses in the nodal based structure (d) are approximately 80% lower than for (b). Full circles indicate highly stressed hinges and dashed circles indicate better compliant and lowly stressed hinges.

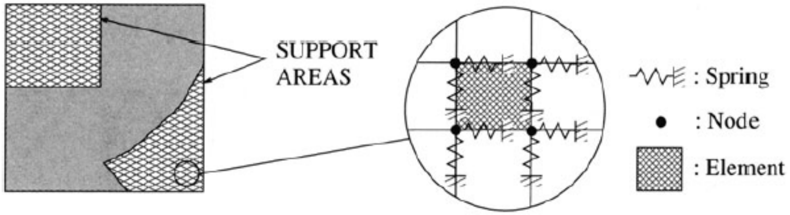
Extensions and Applications – continue

7. Design of supports

Hitherto, we have only considered optimum structural design by material distribution. However, the positions and amounts of supports in a structure also play a major role in structural optimization, and substantial gains from introducing design of supports is obtained for especially compliant mechanism design. If one can place supports anywhere in the design domain, the optimum position of supports in a compliance minimization problem would be directly under the load, causing zero compliance. Therefore, a judicious choice of the possible location of the supports and their cost is in place.

The support design formulation consists in assigning rigid or no supports to each element in a support design domain which may be a subset of the normal (material) design domain. As in material distribution problems we convert this integer type problem into a continuous problem by introducing an element support design variable. The model of the variable support of an element in the FE mesh is sketched below. All the nodes of the element are supported by variable stiffness springs and for high spring stiffnesses this corresponds to fixing the element (as also used in the penalization approach for imposing prescribed boundary conditions). We may then introduce a diagonal element support stiffness matrix

Each node is supported by a horizontal and a vertical spring.
Thus a 4-node element is supported by 8 springs



Extensions and Applications – continue

$$\mathbf{K}_s(\xi_e) = \xi_e^q \mathbf{K}_{s,e}, \quad \xi_e \in [\xi_{min}; 1]$$

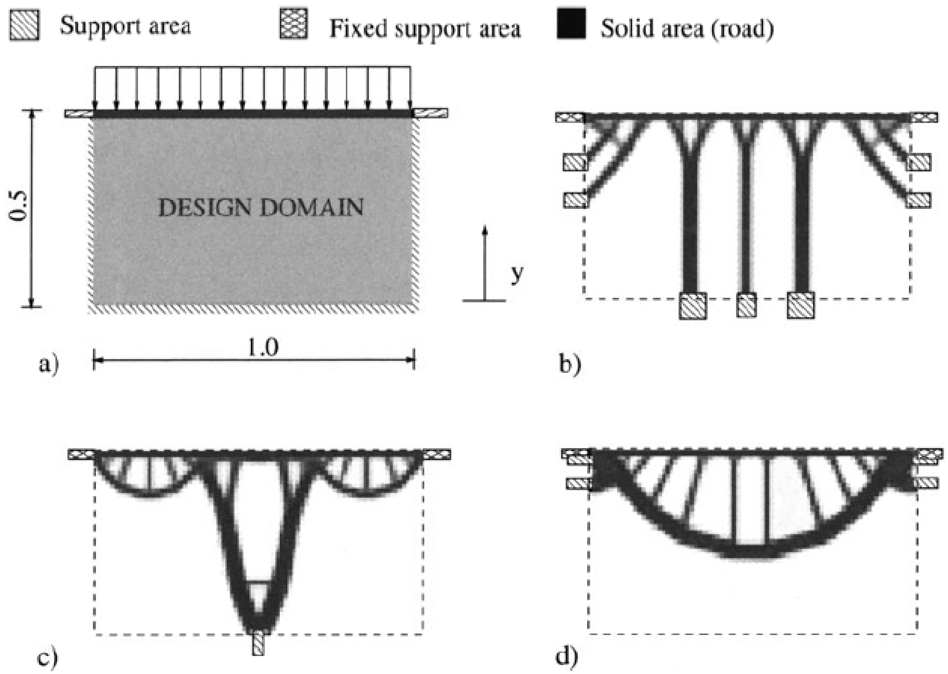
where $\mathbf{K}_{s,e}$ is a diagonal matrix with "high" values compared to the diagonals of the original stiffness matrix and q is a penalization factor corresponding to the power p for stiffness variables in the SIMP approach. The global stiffness matrix may thus be assembled as

$$\mathbf{K} = \sum_{e=1}^{N} \rho_e^p \mathbf{K}_e + \sum_{e=1}^{N} \xi_e^q \mathbf{K}_{s,e}$$

To reduce the possibility of the design being forced into a local optimum, a small lower bound ξ_{min} is imposed on the support design variables. This assures that the sensitivities always are non-zero making a re-introduction of supports possible[5].

Extensions and Applications – continue

As for the material distribution part of the topology design problem, we introduce a bound on the total support area S. For mechanism design this bound is not very important but for stiffness problems the objective function will obviously be improved if more supports are added. In order to encourage or discourage the forming of supports in certain areas or along certain boundaries, we introduce an element support cost factor f_e. The constraint on support area thus becomes $\sum_{e=1}^{N} f_e \xi_e \leq S$. If all $f_e = 1$, the cost of supports is uniform, whereas if some support cost factors are set to higher values (e.g. $f_e = 10$), supports appearing in these elements will be discouraged.



Examples of design of supports combined with compliance minimization. a) Design domain with possible support areas at the all edges except the top edge. b) Optimized topology for equal support cost in the design domain (c=1.12 $*10^{-4}$) c) Optimized topology for support cost varying linearly from 1.0 at the top edge to 10.0 at the bottom edge (c=2.38 $*10^{-4}$) d) Optimized topology for support cost varying linearly from 1.0 at the top edge to 20.0 at the bottom edge (c=3.79 $*10^{-4}$)

Extensions and Applications – continue

For compliance minimization (the linear case) , the optimization problem can now be written as

$$\min_{\boldsymbol{\rho}} \; \{c(\boldsymbol{\rho}) = \mathbf{f}^T\mathbf{u}\}$$

$$\text{s.t.}: \left(\sum_{e=1}^{N} \rho_e^p \mathbf{K}_e + \sum_{e=1}^{N} \xi_e^q \mathbf{K}_{s,e}\right)\mathbf{u} = \mathbf{f}$$

$$\sum_{e=1}^{N} v_e\rho_e \leq V, \qquad 0 < \rho_{min} \leq \rho_e \leq 1, \quad e = 1, \ldots, N \; ,$$

$$\sum_{e=1}^{N} f_e\xi_e \leq S, \qquad 0 < \xi_{min} \leq \xi_e \leq 1, \quad e = 1, \ldots, N \; .$$

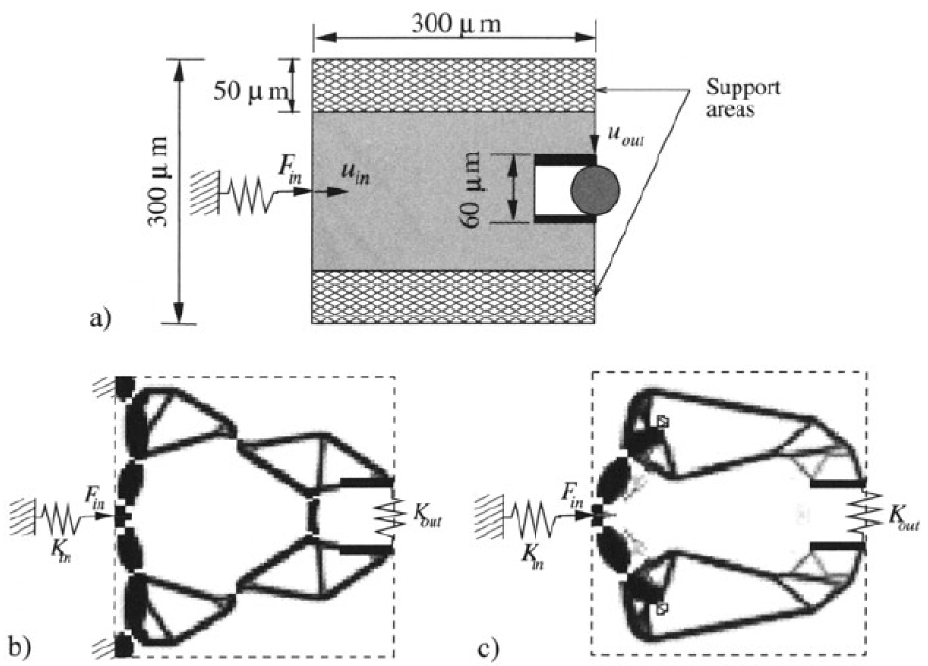
Here, the sensitivity of the compliance with respect to the support design variable is simply

$$\frac{dc}{d\xi_e} = -q\xi_e^{q-1}\mathbf{u}^T\mathbf{K}_{s,e}\mathbf{u}$$

As an example of compliance minimization including costs of supports, we consider the design of the bridge structure sketched in (a). Gradually making the cost of supports more expensive at the bottom of the design domain results in bridge structures with three columns (b), two columns (c) and no columns (d). Correspondingly, the compliances of the three structures increase.

Extensions and Applications – continue

An example of the possible gains in using variable supports in compliant mechanism design is shown below. The goal is to design a gripping mechanism that maximizes the gripping motion for a given input actuation. A limited amount of support may be located in the top and bottom parts of the design domain. (b) shows the optimized gripper obtained with fixed supports at the left edge and (c) shows the optimized gripper including support design. The output displacement of the latter is 77% higher than for the former, demonstrating the importance of including support design in mechanism synthesis problems.



Design of a micro-gripper including design of supports. b) Optimized topology without support design *(uout=10.8µm)* and c) Optimized topology including support design *(uout=19.1µm).* The gain in output displacement is 77%

Thank you for your attention