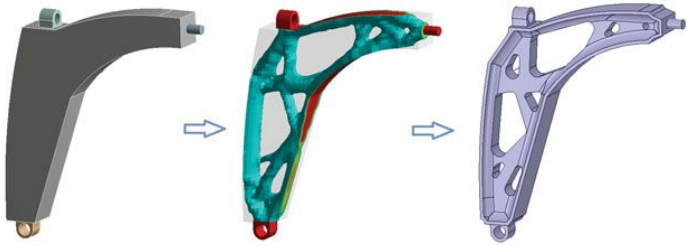
香港中文大學
The Chinese University of Hong Kong

MAEG5160: Design for Additive Manufacturing

Lecture 14: Design with anisotropic materials_2



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2. Optimized energy functionals

The introduction of composite materials as part of the design formulation signifies that the goal of the optimization is both to determine the optimal spatial distribution of material as well as the optimal local use of this material. If we allow the material variables to vary from point to point it seems reasonable to accentuate this local optimal choice of microstructure, and this perspective gives the inspiration for some alternative formulations of optimization problems involving composites.

2.1 Combining local optimization of material properties and spatial optimization of material distribution

In the following, we will consider the material distribution method for general anisotropic materials where an extra set of local variables (for example cell rotation and some geometric parameters) define the material tensor E of the problem. In turn the local variables *also* determine the pointwise density ρ of material (the bulk density), or rather, the density ρ determines the volume of material available for the pointwise (local) construction of E. Within this framework we can then write the minimum compliance design problems as

$$\max_{\substack{\text{density}\\ \rho(x),\, x\in\Omega,\\ \int_\Omega \rho \mathrm{d}\Omega \le V}} \; \max_{\substack{E \text{ for}\\ \text{microstructure}\\ \text{of density } \rho(x)}} \; \min_{u\in U} \left\{ \frac{1}{2}\int_\Omega E_{ijkl}(x)\varepsilon_{ij}(u)\varepsilon_{kl}(u)\mathrm{d}\Omega - l(u) \right\}$$

$$\min_{\substack{\text{density}\\ \rho(x),\, x\in\Omega,\\ \int_\Omega \rho \mathrm{d}\Omega \le V}} \; \min_{\substack{E \text{ for}\\ \text{microstructure}\\ \text{of density } \rho(x)}} \; \min_{\substack{\sigma\\ \mathrm{div}\sigma + f = 0\\ \sigma\cdot n = t}} \left\{ \frac{1}{2}\int_\Omega C_{ijkl}(x)\sigma_{ij}\sigma_{kl}\mathrm{d}\Omega \right\}$$

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The basic idea is then to interchange the optimization over the design of the microstructure and the optimization over stresses or displacement. This interchange gives valuable insight in problem structure and provides us with a basis for constructing some alternative solution procedures and computational schemes.

The interchange of min-min in the stress formulation results in an equivalent problem as the constraint sets for the two operators in the inf-inf problem are given entirely in terms of the variable over which each individual infimum is sought. Introduction of, for example, stress constraints at the outer design level of problem would destroy this feature. For the displacement formulation the interchange will in general not result in an equivalent problem. Nonetheless, as we have that

$$\sup_x \inf_y \phi(x,y) \leq \inf_y \sup_x \phi(x,y)$$

for any function of two parameters, the interchange will provide us with an upper bound on the optimal objective and thus a lower bound for the compliance of the optimal structure. In situations where the problem satisfies conditions for the existence of a saddle value (saddle point), the interchange will result in an equivalent problem *also* for the strain case - this holds if we work in the framework of layered materials, for a free parametrization of the tensor E and for laminated plates.

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The interchange of equilibrium analysis and optimization of local material properties results in a reformulated displacement based problem

$$\max_{\substack{\rho(x),\, x\in\Omega,\\ \int_\Omega \rho \mathrm{d}\Omega \le V}} \min_{u\in U} \left\{ \int_\Omega \overline{W}(\rho, \varepsilon_{ij}(u)) \mathrm{d}\Omega - l(u) \right\}$$

where $\overline{W}(\rho, \varepsilon)$ denotes the pointwise optimal strain energy density expression given by

$$\overline{W}(\rho, \varepsilon) = \max_{\substack{E \text{ for}\\ \text{microstructure}\\ \text{of density } \rho(x)}} \left\{ \frac{1}{2} E_{ijkl} \varepsilon_{ij} \varepsilon_{kl} \right\}$$

Here we have used that the optimization of microstructure is pointwise, so that one can move this extremization under the integration over the domain.

In the stress based case we have a problem form

$$\min_{\substack{\rho(x),\, x\in\Omega,\\ \int_\Omega \rho \mathrm{d}\Omega \le V}} \min_{\substack{\sigma\\ \mathrm{div}\sigma + f = 0\\ \sigma\cdot n = t}} \left\{ \int_\Omega \overline{\Pi}(\rho, \sigma_{ij}) \mathrm{d}\Omega \right\}$$

with an optimized complementary energy density

$$\overline{\Pi}(\rho, \sigma_{ij}) = \min_{\substack{E \text{ for}\\ \text{microstructure}\\ \text{of density } \rho(x)}} \left\{ \frac{1}{2} C_{ijkl} \sigma_{ij} \sigma_{kl} \right\} .$$

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For the optimization problems of above, we have two coupled optimization problems, which we label the *local anisotropy* and the *material distribution* optimization problems, respectively. The material distribution problems are the "master" problems (the outer problems) of this hierarchical formulation and they deal with the spatial distribution of resource/material (a global problem). The local anisotropy problems are the inner "slave" problems address the question of optimal choice of material (a local problem).

The local anisotropy problems correspond to finding the pointwise stiffest material for a given fixed strain or fixed stress field and a given density of material. This is a standard problem setting in the theory of variational bounds on effective moduli of anisotropic materials. It is of great importance in its own right and has been the subject of intense studies in material science.

The equilibrium problem seeks kinematically admissible equilibrium displacements for the locally optimum energy functional, for a given distribution of resource ρ, while the equilibrium problem seeks statically admissible equilibrium stress fields which minimize the locally optimum energy functional, again for a given distribution of resource ρ. It should be noted that, since the locally optimum energies depend on the displacement and stress fields in a complex fashion via the optimization problems, the inner equilibrium statements of the problems are in fact constitutively non-linear and non-smooth elasticity problems, except in very special cases. However, as we shall see in the coming sections, there are important cases of material modelling where these equilibrium problems become problems in linear elasticity or where the non-smoothness is isolated to unimportant strain/stress values. For the strain based problem, it is worth remarking that the equilibrium problem remains a convex problem after the optimization over local material properties. The optimal strain energy density $W(\rho, \varepsilon)$ is derived as a maximization of convex functions in the strains and is thus in itself convex in these variables.

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2.2 A hierarchical solution procedure

The problem separation described above naturally leads one to consider a different computational implementation as compared to the procedure described previously. Such an implementation can for example work with problem in the displacements and density only. We accordingly consider the solution as given, either through an analytical or a computational procedure. Then it has exactly the format of the compliance problem dealt with the SIMP model, that is, the compliance is a function of the density and is given by the solution of a minimum potential energy problem. This problem can then be solved for example by an optimality criterion method or by MMA. Here one needs sensitivity information of the compliance, i.e., derivative information for the equilibrium problem, which is given by the derivative of the optimized strain energy W with respect to the density. For an analytically derived optimal strain energy functional this derivative is straightforward to obtain, while for a computationally derived optimal strain energy functional this derivative is given simply as the Lagrange multiplier for the volume constraint of problem, i.e., the derivative is given directly from the computation of the optimal energy. The equilibrium problem is in general a non-linear problem, so the equilibrium problem requires an inner iteration loop at this point, but computational experience has shown that, as the optimization over the bulk density is in itself iterative, only one (or a few) equilibrium iterations need to be used for each design update.

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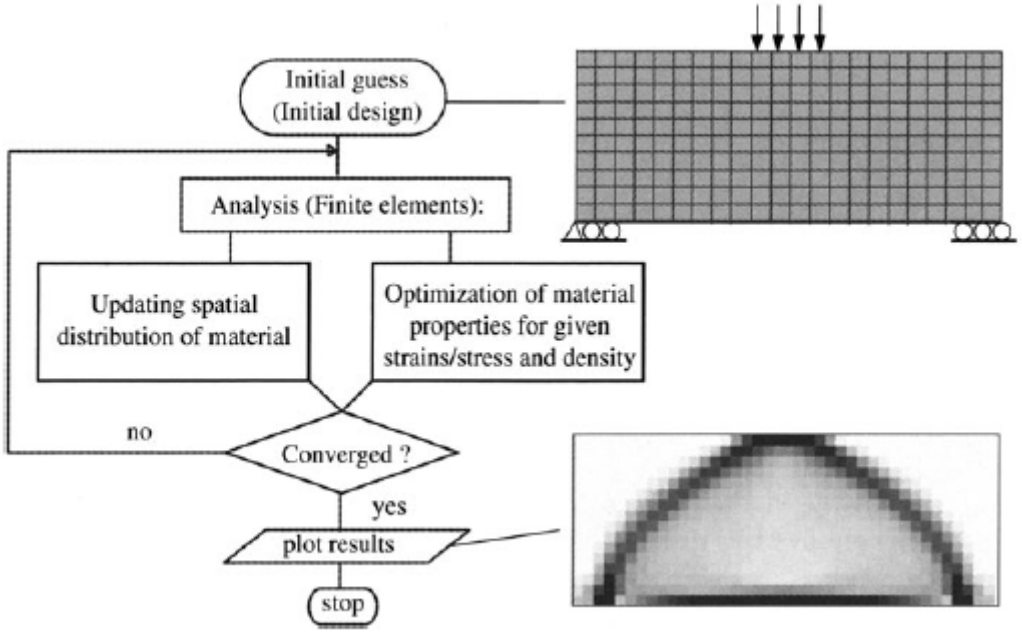
One of the advantages of the computational program just described is that the main flow of the procedure is *independent* of the modelling of the material used for the description of design. This latter information is added as an external module. This feature makes it possible to generate flexible procedures, where the material model can be changed easily.



Optimal design using a hierarchical approach. The resulting structure is here a low volume solution to the problem

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In many implementations the non-linear analysis iterations are avoided**.** Thus linear analysis is applied for the equilibrium problem with *fixed* material parameters and problem is used to generate the parameters of the optimal stiffness tensor for each displacement iteration. The direct coupling between the material parameters and the displacements is therefore ignored in the implementation of the *linear* equilibrium analysis. This computational procedure is especially attractive for multiple load problems where the use of the linear analysis also circumvents the coupling between the displacements for the different loads that is introduced.

The procedure described here has been implemented for a broad variety of models. It is particularly well suited for the parametrization by moments of the effective parameters for rank-N layered materials needed for multiple load cases. Here the inner problem becomes a convex problem that can be solved efficiently by computational means. In other situations, as we shall see in the following sections, this inner problem can actually be solved analytically. One can go one step further and solve the inner anisotropy problem by the material design method. In this situation one uses only microstructures which involve one length-scale, and the microstructure is designed by a topology design method. The computations involved in this approach are quite massive: the number of local topology design problems equals the number of finite elements in the mesh defining the material distribution ρ. However, all these local problems are independent and can be solved simultaneously using parallel processing methodologies.

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Additional problem reduction

In the development above we could have performed one further interchange for the stress case[3], namely the interchange of the optimization over density and the extremum form of the equilibrium problem. Such an interchange results in the problem

$$\min_{\substack{\sigma \\ \mathrm{div}\sigma+f=0 \\ \sigma\cdot n=t}} \left\{\widehat{\Pi}(\sigma)\right\}, \qquad \widehat{\Pi}(\sigma) = \min_{\substack{\text{density} \\ \rho(x),\, x\in\Omega, \\ \int_\Omega \rho \mathrm{d}\Omega \le V}} \left[\int_\Omega \overline{\Pi}(\rho,\sigma_{ij})\mathrm{d}\Omega\right]$$

We have written problem in a form which underlines that this reduced problem should be interpreted as an equilibrium *only* problems for a globally optimized complementary energy expression. The optimized energy is non-smooth and couples all degrees of freedom through the volume constraint. This latter complication can be circumvented by considering the volume constraint of the original problem in the form of a penalization and not a constraint. With this interpretation problem becomes

$$\min_{\substack{\sigma \\ \mathrm{div}\sigma+f=0 \\ \sigma\cdot n=t}} \left\{\widehat{\Pi}_\Lambda(\sigma)\right\}, \quad \widehat{\Pi}_\Lambda(\sigma) = \min_{\substack{\text{density} \\ \rho(x),\, x\in\Omega}} \left[\overline{\Pi}(\rho,\sigma_{ij}) + \Lambda\rho\right] \mathrm{d}\Omega$$

where A is now a fixed penalty factor. For a computational procedure for problem above one could solve the inner problem by analytical or computational means and implement a non-smooth optimization method for solving the equilibrium problem. As outlined above, based on linear analysis, both the material properties *and* the density are updated based on the algebraic solution of the optimization of the complementary energy.

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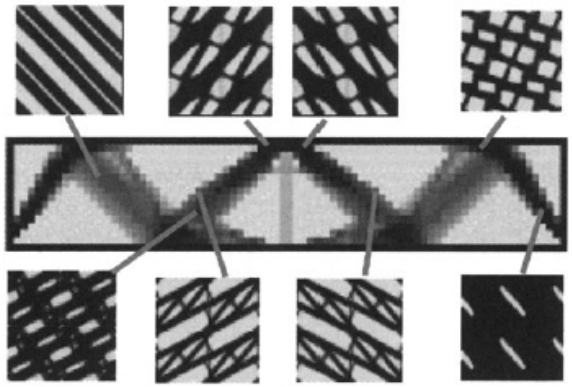

The MBB beam. The optimal distribution of material and associated microstructures obtained from a hierarchical approach. The local material anisotropy problem has here been solved numerically, using topology design of the unit cell of a composite. The cell is not rotated - the necessary rotation arises from the material design

3. Optimized energy functionals for the homogenization modelling

In the following we will compute the optimal strain and complementary energies for rank-2 layered materials in 2-D, corresponding to the *local anisotropy* optimization problem for single load minimum compliance design. That is, we will develop the solution to problems for the class of composites that are rank-2 layered materials.

We use here the parametrization of the stiffness of the rank-2 material by the two layer-thicknesses μ and γ. If the primary layerings of density μ are placed in the 2-direction of our reference frame, the effective material properties in plane stress are

$$E^H_{1111} = \frac{\gamma E}{\mu\gamma(1-\nu^2)+(1-\mu)}, \qquad E^H_{1122} = \mu\nu E^H_{1111},$$
$$E^H_{2222} = \mu E + \mu^2\nu^2 E^H_{1111}, \qquad E^H_{1212} = 0\ .$$

when the weak material is void, i.e. $E^- \to 0$. It is straightforward to verify that such a material is weak in shear, i.e. that the material parameters satisfy $E^H_{1111} + E^H_{2222} - 2E^H_{1122} - 4E^H_{1212} \geq 0$

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3.1 The stress based problem of optimal layered materials

The results on optimal rotation of orthotropic materials shows that for the minimum compliance problem with a material which is weak in shear, the axes of orthotropy should be aligned with the axes of principal stresses σ_I, σ_{II}. This gives a complementary energy of the form

$$\Pi = \frac{1}{2} C^H_{ijkl}\sigma_{ij}\sigma_{kl} = \frac{1}{2|D|}\left[E^H_{1111}\sigma^2_{II} + E^H_{2222}\sigma^2_I - 2E^H_{1122}\sigma_I\sigma_{II}\right] ,$$

with $|D| = E^H_{1111}E^H_{2222} - (E^H_{1122})^2$. Here, we have the well-known relations between principal stresses and stresses in an arbitrary frame:

$$\sigma_I = \frac{1}{2}\left(\sigma_{11} + \sigma_{22} + \sqrt{(\sigma_{11} - \sigma_{22})^2 + 4\sigma^2_{12}}\right) ,$$

$$\sigma_{II} = \frac{1}{2}\left(\sigma_{11} + \sigma_{22} - \sqrt{(\sigma_{11} - \sigma_{22})^2 + 4\sigma^2_{12}}\right) .$$

We see that the alignment of axes is consistent with the fact that $E^H_{1212} = 0$ for the layered material; the vanishing shear stiffness for the layered material plays no role as the material automatically rotates to a frame of zero shear.

Note that the material law described by the energy expression above represents a non-linear material, by virtue of the optimal rotation and the fact that $E^H_{1111} \neq E^H_{2222}$. Here and in the following we use the term "material law" to describe the characteristics of the optimized energy expressions. This should not be interpreted as properties of the layered materials in a physical sense, but expresses the peculiarity of the energy of a structure which automatically assigns the real material in accordance with the applied load (stress/strain field).

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We now fix the density ρ and express γ in terms of μ from the relation $\rho = \mu + \gamma - \mu\gamma$. Stationarity of the energy with respect to the layer density μ can now be found by standard but fairly lengthy calculations. We find the stationary layer density μ and corresponding layer density γ given as

$$\mu = \frac{\rho|\sigma_{II}|}{|\sigma_{II}| + (1-\rho)|\sigma_I|}, \quad \gamma = \frac{\rho|\sigma_I}{|\sigma_I| + |\sigma_{II}|} .$$

These values turn out to represent minimizing values if the value of μ satisfy the constraints $0 < \mu < \rho$. This implies that the stresses should satisfy $\sigma_I\sigma_{II} \neq 0$ and for such values of stress the optimal layering is a true rank-2 layering. If $\sigma_I\sigma_{II} = 0$ we have a region with an unidirectional, single layering or a solid region corresponding to $\mu = 0, \gamma = \rho$ or $\mu = \rho, \gamma = 0$. The numerical values of stresses in the formula above indicate that there for the rank-2 regions are two distinct types of layerings depending on the sign of the quantity $\sigma_I\sigma_{II}$. We denote the two types of stationary layerings as mode I ($\sigma_I\sigma_{II} < 0$) and mode II ($\sigma_I\sigma_{II} > 0$) materials, and the rank-1 materials as mode III materials. Note that the expressions above were derived under the assumption that the direction of the outer layer of the rank-2 layering (corresponding to μ) is aligned with σ_{II}, and that no restrictions where imposed on the relative sizes of σ_I and σ_{II}. The analysis shows that the optimization over layer densities automatically assures that the axis of maximal stiffness is aligned with the axis of the largest stress, in accordance with the result on optimal rotations. Also note that a second, equally optimal layering can be obtained by aligning the outer layerings with the stress σ_I; the formulas above now hold with σ_I and σ_{II} interchanged. The effective complementary energy for both optimal microstructures is given by the expressions

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$$\text{Mode I}: \quad \overline{\Pi} = \frac{1}{2E\rho}\left[\sigma_I^2 + \sigma_{II}^2 - 2(1-\rho+\rho\nu)\sigma_I\sigma_{II}\right] ,$$

$$\text{Mode II}: \quad \overline{\Pi} = \frac{1}{2E\rho}\left[\sigma_I^2 + \sigma_{II}^2 + 2(1-\rho-\rho\nu)\sigma_I\sigma_{II}\right] ,$$

$$\text{Mode III}: \quad \overline{\Pi} = \frac{\sigma_I^2}{2E\rho} \ \text{ if } \ \sigma_{II} = 0 , \quad \overline{\Pi} = \frac{\sigma_{II}^2}{2E\rho} \ \text{ if } \ \sigma_I = 0 .$$

The material properties of the now optimized microstructure are completely given in terms of the density and the principal stresses. Noting that

$$\sigma_I^2 + \sigma_{II}^2 = \sigma_{11}^2 + \sigma_{22}^2 + 2\sigma_{12}^2 , \quad \sigma_I\sigma_{II} = \sigma_{11}\sigma_{22} - \sigma_{12}^2 ,$$

we observe the surprising fact that the optimized energy corresponds to a material law which for the regions with two layerings is linearly elastic and quasi isotropic. For the single layering regions the material law is non-linear. Note that the isotropy of the optimized material law is natural in view of the rotation of the rank-2 material. The linearity and isotropy of this extremal material law can be understood in a broader context from the so-called translation method for obtaining optimal bounds on effective moduli of composite materials.

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The expression above is the solution to the problem for the single load case we consider. For the stress based problem a further reduction to a design-free problem is possible. To this end we should optimize with respect to the density of material as well. Taking the volume constraint into account for the inner problem, we minimize with respect to the bulk density ρ the expression $\overline{\Pi}+\Lambda\rho$, where $\Lambda \geq 0$ is a Lagrange multiplier for the volume constraint. By fairly straightforward algebraic manipulations, we get the following optimality condition for the bulk density ρ:

$$\rho = \frac{|\sigma_I| + |\sigma_{II}|}{\sqrt{2\Lambda E}} \quad \text{in all modes}$$

In (3.22) the absolute value operators indicate that we have different expressions for mode-I and mode-II. The corresponding densities γ and μ are

$$\mu = \frac{|\sigma_{II}|}{\sqrt{\Lambda E} - |\sigma_I|}, \quad \gamma = \frac{|\sigma_I|}{\sqrt{\Lambda E}},$$

and the optimal distribution of the bulk density should satisfy the volume constraint

$$\int_\Omega \rho \mathrm{d}\Omega = \int_\Omega \min\left\{\frac{|\sigma_I| + |\sigma_{II}|}{\sqrt{2\Lambda E}}, 1\right\} \mathrm{d}\Omega = V \ .$$

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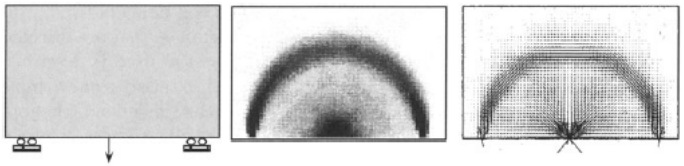
This constraint determines the value of the Lagrange multiplier A for any relevant volume constraint. Thus the volume constraint implies that we can consider Λ as a function of the principal stresses. Taking this feature into consideration, the optimal complementary energy density can be expressed in terms of *stresses* only, and we have reduced the stress based *design* problem to a *design independent* non-linear, non-smooth elasticity problem of the form.

$$\min_{\substack{\sigma \\ \mathrm{div}\sigma + f = 0 \\ \sigma \cdot n = t}} \left\{ \int_\Omega \hat{\Pi}(\sigma) \mathrm{d}\Omega \right\}$$

3.2 The strain based problem of optimal layered materials

The algebra involved in optimizing the microstructure for the strain based formulation is much more complicated than for the stress case and for simplification of presentation in this case, it turns out to be convenient to impose the choice $|\varepsilon_I| \geq |\varepsilon_{II}|$ for the principal strain directions. The steps of the analysis are all analogous to the procedure for the stress case, but the algebraic manipulations now become very involved, and the use of symbolic manipulations is recommended.

The optimal density μ and corresponding density γ are again given by different expressions, depending on the relative values of the principal strains ε_I, ε_{II} as well as the size of the bulk density ρ. We again denote the different expressions as Mode-I, Mode-II and Mode-III regions (there is a one-to-one correspondence with the stress energy modes). The optimal values are



Optimal design using an optimized rank-2 material strain energy density. Optimized designs computed using element wise constant density function and a 8-node displacement model. Center: the optimal density distribution, and Right: the associated principal stress distribution for a volume fraction of 20%. Note that grey area is not limited to biaxial response. The bicycle wheel like design has an area with radial uniaxial stress as well as a rim of circumferential uniaxial stress

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$$\text{Mode I}: \left\{\begin{array}{l}\mu = \frac{\varepsilon_I(1+\nu\rho-\rho)+\varepsilon_{II}}{\nu\varepsilon_I+(2-\rho-\nu+\nu\rho)\varepsilon_{II}} \\ \gamma = \frac{\varepsilon_I+\varepsilon_{II}(1+\nu\rho-\rho)}{(1-\nu)(\varepsilon_I-\varepsilon_{II})}\end{array}\right\} \quad \text{if} \quad \frac{\varepsilon_I+\varepsilon_{II}}{(1-\nu)\varepsilon_I} < \rho < 1\ ,$$

$$\text{Mode II}: \left\{\begin{array}{l}\mu = \frac{\varepsilon_I(\nu\rho+\rho-1)+\varepsilon_{II}}{\nu\varepsilon_I+(2-\rho+\nu-\nu\rho)\varepsilon_{II}} \\ \gamma = \frac{\varepsilon_I+\varepsilon_{II}(\nu\rho+\rho-1)}{(1+\nu)(\varepsilon_I+\varepsilon_{II})}\end{array}\right\} \quad \text{if} \quad \frac{\varepsilon_I-\varepsilon_{II}}{(1+\nu)\varepsilon_I} < \rho < 1\ ,$$

$$\text{Mode III}: \left\{\begin{array}{l}\mu = 0 \\ \gamma = \rho\end{array}\right\} \quad \text{if} \quad 0 \le \rho \le \frac{\varepsilon_I-\varepsilon_{II}}{(1+\nu)\varepsilon_I} \quad \text{or} \quad 0 \le \rho \le \frac{\varepsilon_I+\varepsilon_{II}}{(1-\nu)\varepsilon_I}\ .$$

The effective strain energy corresponding to either optimal layering is given by the expressions

$$\text{I}: \ \overline{W}(\rho,\varepsilon) = \frac{E}{2(1-\nu)(2-\rho+\nu\rho)}\left[\varepsilon_I^2+\varepsilon_{II}^2+2(1-\rho+\rho\nu)\varepsilon_I\varepsilon_{II}\right]\ ,$$

$$\text{II}: \ \overline{W}(\rho,\varepsilon) = \frac{E}{2(1+\nu)(2-\rho-\nu\rho)}\left[\varepsilon_I^2+\varepsilon_{II}^2-2(1-\rho-\rho\nu)\varepsilon_I\varepsilon_{II}\right]\ ,$$

$$\text{III}: \ \overline{W}(\rho,\varepsilon) = \frac{\rho E\varepsilon_I^2}{2} \text{ if } |\varepsilon_I| \ge |\varepsilon_{II}|, \quad \overline{W}(\rho,\varepsilon) = \frac{\rho E\varepsilon_{II}^2}{2} \text{ if } |\varepsilon_I| \le |\varepsilon_{II}|\ .$$

In the Mode-III regions with single layers, the material law is non-linear and, as for the stress based analysis, the rank-2 layered regions of Modes I and II correspond to a linearly elastic material law which has the same stiffness matrix as the optimal material obtained in the stress case. This is consistent with a duality principle for the optimized strain and complementary energies

$$\min_{\substack{\text{div}\sigma+f=0 \\ \sigma\cdot n=t}} \left\{\int_\Omega \overline{\Pi}(\rho,\sigma)\mathrm{d}\Omega\right\} = \max_{u\in U}\left\{l(u)-\int_\Omega \overline{W}(\rho,\varepsilon(u))\mathrm{d}\Omega\right\}\ ,$$

that holds when the bulk density ρ is kept fixed

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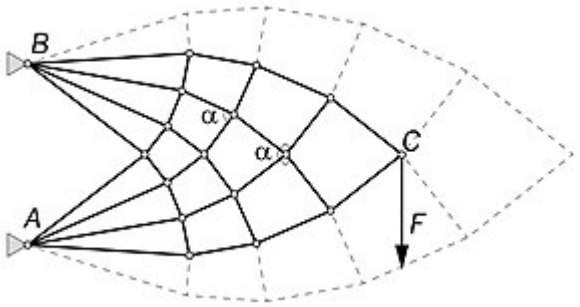
The optimization of the strain energy with respect to layer directions as well as layer densities results in an optimized strain energy $\overline{W}$ which is convex in the density ρ; this is readily checked by examining the second derivative of the energy for the different modes. This excludes the possibility of interchanging min and the max in the reduced problem

$$\max_{\substack{\text{density } \rho \\ \int_\Omega \rho \mathrm{d}\Omega = V}} \min_{u \in U} \left\{ \int_\Omega \overline{W}(\rho, \varepsilon(u)) \mathrm{d}\Omega - l(u) \right\}$$

and this is thus the final reduced form of the strain based formulation

3.3 The limiting case of Michell's structural continua

The lay-out theory of Michell frames and its extensions to flexural systems is the classical approach to topology and lay-out design of structures. It has been illustrated earlier that the material distribution method predicts structures that resemble truss-type lay-outs and Michell continua type layouts, when constrained to small volumes of available material. We show here that this limiting process can be formalized through an asymptotic expansion of the problem under rescaling of the geometric and load data.



discrete optimum trusses

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A Michell frame is a continuum in dimension two consisting of two mutually orthogonal fields of tension/compression only members that are directed along the principal strain. The total amount of material used is described by two independent densities of material, constrained to satisfy some volume constraint. The problem is a continuum analogue to the single-load truss optimal design problem, and there are a number of equivalent stress or strain based problem statements. The frame is described by a specific strain energy of the form

$$W = \frac{E}{2}\left[\alpha\varepsilon_I^2 + \beta\varepsilon_{II}^2\right]$$

where α, ε_I and β, ε_{II} are the densities and corresponding principal strains in the two directions of the continua, and the optimization problem is the one of minimizing compliance for a given volume of material, or equivalently, maximizing of compliance for given constraints on the strains in each bar, cf., Hemp (1973), Bendsøe, Ben-Tal & Zowe (1994). Lay-out theory for grid-type structures in general, as treated by Prager and Rozvany, deals with problems with a wider scope of objectives and constraints, but with basically the same energy definition as above.

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The Michell frame is usually understood as a limiting case for low densities of material, where the interaction of thin members in a planar frame can be ignored. Thus, we are concerned with the limiting situation where the layers in a layered material become "thin" relative to the cell size of the problem. This can be modelled by letting the density of material tend to zero in an asymptotic expansion. Taking the limit of zero density of material requires a complementary rescaling of the loads and tractions to make the energy limit well posed. We thus introduce a scaling parameter ξ which reduces the layer densities by rescaling the dimensions of the microstructure relative to the unit cell The rescaled densities are

$$\hat{\mu} = \xi^2 \mu, \quad \hat{\gamma} = \xi^2 \gamma, \quad \hat{\rho} = \xi^2 \rho \, .$$

We now use the rescaled densities together with an expansion of the stresses and strains in the expressions for the optimized energies described above, using only the terms of zero order in ξ and requiring that the energies remain finite in the limit of $\xi \to 0$. For the stress based case the stress expansion reads

$$\hat{\sigma}_{ij} = \ldots + \xi^{-2}\sigma_{ij}^{-2} + \xi^{-1}\sigma_{ij}^{-1} + \sigma_{ij}^{0} + \xi\sigma_{ij} + \xi^2\sigma_{ij}^{2} + \ldots \quad .$$

For the energy to remain finite in the limit, the expansion in stresses must be of order greater than or equal to 1. The zero-order part of the optimized complementary energy $\overline{\Pi}$ (see Sect. 3.3.1) then becomes (for all modes)

$$\Pi_M = \frac{1}{2E\rho}(|\sigma_I| + |\sigma_{II}|)^2 \, ,$$

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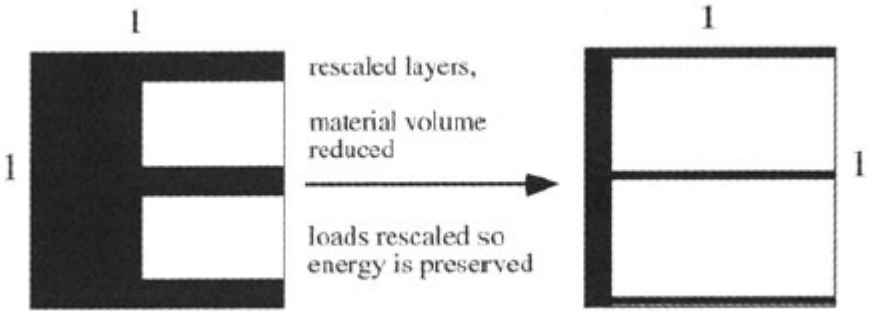

The rescaling of the layerings that leads to the Michell frame limit

corresponding to a rescaling of stress given by $\xi\sigma_{ij}$. This is expected from equilibrium considerations for the unit cell.

The rescaling at the limit of $\xi \to 0$ implies that the upper constraint on bulk density ρ is not active. Thus the optimization over ρ under the volume constraint results in the stress based problem (3.17) reducing to the form

$$\min_{\substack{\sigma \\ \mathrm{div}\sigma+f=0 \\ \sigma\cdot n=t}} \left\{ \int_\Omega (|\sigma_I| + |\sigma_{II}|) \mathrm{d}\Omega \right\} .$$

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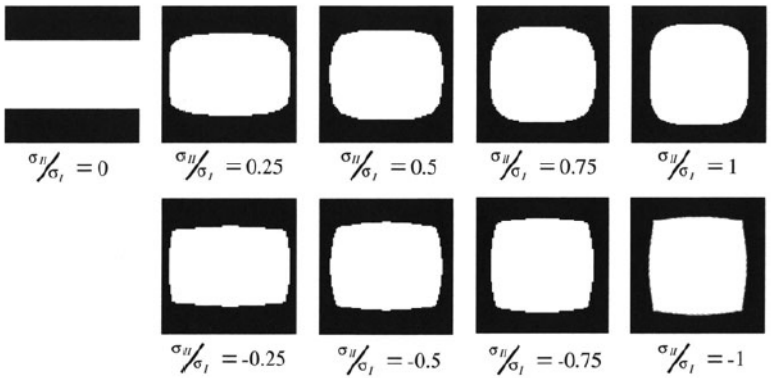
This is the classical Michell problem formulated in stresses. Here the specific reference to the volume constraint is not present, as the Lagrange multiplier for this constraint only enters as a scaling parameter which has no influence on the form of the optimal solution. The problem corresponds to a lay-out problem, where the cost of carrying the principal stresses is minimized over all statically admissible stress fields. This corresponds directly to the classical stress-based truss optimization problem stated as

$$\min_{\mathbf{q}^+,\mathbf{q}^-} \sum_{i=1}^{m} \frac{l_i}{\overline{\sigma}_i}(q_i^+ + q_i^-)$$

$$\text{s.t.} : \mathbf{B}(\mathbf{q}^+ - \mathbf{q}^-) = \mathbf{f}, \quad q_i^+ \geq 0,\ q_i^- \geq 0,\ i = 1,\ldots,m\ ,$$

which is a problem in plastic design. Here, q_i^+, q_i^- are the truss bar member forces in compression and tension, respectively, $\mathbf{B}$ is the compatibility matrix, l_i the lengths of the bars and $\overline{\sigma}_i$ the yield limit for bar number i . This problem is equivalent to the problem of fixed volume, minimum compliance design of an elastic truss structure with Young's moduli $E_i = \overline{\sigma}_i^2$ and a volume equal to the optimal volume for the plastic problem, thus taking the development "full circle".



The shape of single inclusions of void in a cell of a homogenized, periodic medium minimizing complementary energy (Vigdergauz-like structures for ν=1/3 and a density ρ = 0.5) . Results for a range of principal stress ratios of a macroscopic stress field

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3.4 Comparing optimal energies

A key question for understanding the nature of the results that can be obtained from optimization of material distribution is a comparison of the stiffness parameters of various microstructures at hand. For compliance design the local anisotropy problems give the relevant measures to consider, i.e., one works in terms of strain or complementary energies. It is known from work in the theoretical materials science that the optimal complementary energy for rank-2 layered materials constitutes the attainable *lower bound* on the complementary energy of *any* composite constructed from void and an isotropic, linearly elastic material with Young's modulus E and Poisson's ratio ν. This means that any elasticity tensor E_{ijkl} related to the given material satisfies that

$$\left[E^H_{ijkl}\right]^{-1}\sigma_I\sigma_{II} \geq \begin{cases} \frac{1}{2E\rho}\left[\sigma_I^2+\sigma_{II}^2-2(1-\rho+\rho\nu)\sigma_I\sigma_{II}\right] \text{ if } \sigma_I\sigma_{II}\leq 0\,, \\ \frac{1}{2E\rho}\left[\sigma_I^2+\sigma_{II}^2+2(1-\rho-\rho\nu)\sigma_I\sigma_{II}\right] \text{ if } \sigma_I\sigma_{II}\geq 0\,, \end{cases}$$

for any stress tensor σ with principal stresses σ_I, σ_{II}. We have seen that this upper bound on the stiffness of a composite can also be expressed in terms of strain energy. As we have seen, the bound can be attained by a rank-2 layering that have *two* length scales. For stresses with $\sigma_I\sigma_{II}\geq 0$, *single scale*, single inclusion microstructures which attain the bounds have been presented before. For illustration, previous figure shows a range of single inclusion Vigdergauz-like microstructures for a range of positive as well as negative values of $\sigma_I/\ \sigma_{II}$; these structures have been computed by the inverse homogenization methodology. Note, however, that for ($\sigma_I\sigma_{II}\ \leq 0$ no single scale periodic composite can obtain the bounds, and any composite obtaining the bound (in 2-D) must be degenerate.

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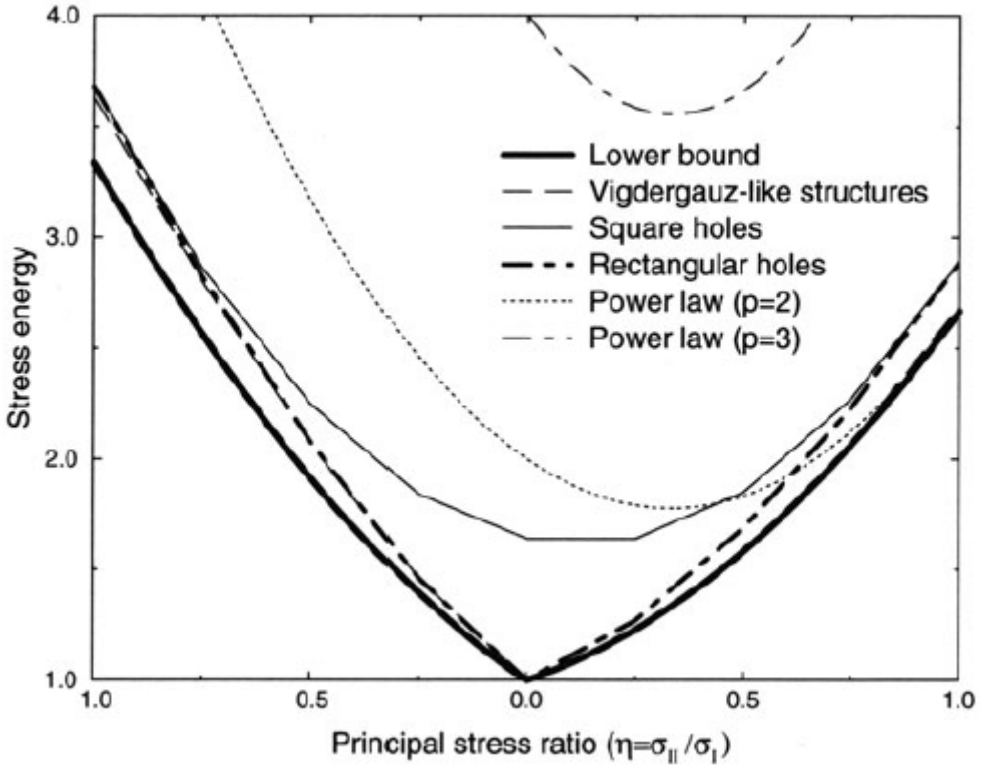

Comparison of the optimal (minimal) complementary energy as a function of the ratio of the principal stresses, for a density ρ = 0.5, and for various types of microstructures and interpolation schemes (material and void mixtures). The Vigdergauz-like structures are shown in previous figure.

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For their use in optimal topology design it is useful to compare energies attainable by other microstructures and interpolation schemes with the bound. Previous figure thus shows (for p=0.5) a comparison of the optimal bound, achievable by the ranked layered materials, with the range of *minimal* complementary energies which can be obtained by the SIMP interpolation, by microstructures with square holes, by microstructures with rectangular holes, and by the Vigdergauz microstructures. What is noticeable is how close the various energies are for stress fields close to pure dilation, while shearing stress fields demonstrates a considerable difference. In the latter case, the microstructural based models are considerably stiffer than the SIMP model. Moreover, the microstructure with square holes is notably less stiff for uni-axial stresses compared to the other microstructures, since the imposed symmetry of this microstructure here hinders an efficient use of material. The plots of the complementary energy explains many features of computational experience with the various interpolation schemes. For compliance optimization, the complementary energy should be minimized. As ranked laminates are efficient also at intermediate densities, optimal design with this material model leads to designs with typically rather large areas of intermediate density. This is also the case when using the microstructures with rectangular holes and the Vigdergauz microstructures. Thus if such materials are used for obtaining black-and-white designs, some other form of penalization of intermediate density has to be introduced, as discussed earlier. On the other hand, the SIMP model and the microstructure with square holes usually lead to designs with very little "grey", as intermediate values of density tend to give poor performance in comparison with cost.

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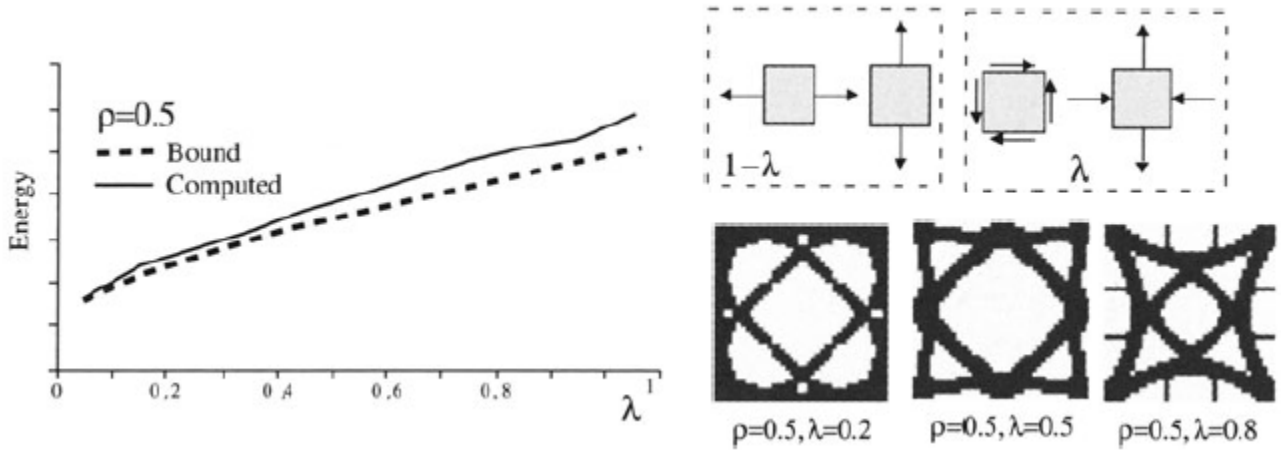
The multiload case

For the local anisotropy problems for multiple loads, one works with a weighted average of strain or complementary energies. Also here the optimal bound (i.e., the lowest average complementary energy) can be found by using rank-N layered materials. One can here benefit by working with the moment-based parametrization of stiffness by moments. As above, it is instructive for this situation also to compare this optimal energy with computational results (inverse homogenization) that approximate the energy bounds by use of single scale microstructures. The example below considers four load cases. The same weight factor is used for each pair of load cases, where the first pair gives tension and the second pair gives shear. The weighting factors on the energies are written as $w_1 = \lambda$ and $w_2 = (1-\lambda)$ where λ varies from zero to one, where zero corresponds to the tension load situations and λ equal to one corresponds to shear.



Comparison of the complementary energy of optimized base cells for a multiload situation with ρ=0.5 (see text). The single scale composites are obtained with square base cells

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3.5 Optimal energies and the checkerboard problem

Previously we showed by example that a checkerboard of material in a uniform grid of square Q4 elements has a stiffness which is comparable to the stiffness of a ρ=1/2 *variable thickness* sheet. Let us here formalize this in light of the energy considerations carried out so far. For example, consider the optimal design of a planar, infinite and periodic medium with an average density of material equal to 1/2 and subject to an average, macroscopic strain $\bar{\varepsilon}$. The minimization of compliance then corresponds to the problem:

$$\max_{\rho,<\rho>=0.5} \min_{u,\ \text{periodic}} \Psi(\rho,u)\ ,$$

$$\text{with } \Psi(E,\rho,u) = \int_\Omega \rho^p E^0_{ijkl} \left(\bar{\varepsilon} - \varepsilon(u)\right)_{ij} \left(\bar{\varepsilon} - \varepsilon(u)\right)_{kl} \mathrm{d}\Omega$$

where we use a SIMP interpolation. Assuming now that the displacement is restricted to the space of Q4 discretizations for a square mesh we first note that if p is distributed in a 0-1 checkerboard pattern in this mesh, then

$$\min_{u\in \text{Q4},\ \text{periodic}} \Psi(\rho_P,u) = \frac{1}{2} E^0_{ijkl} \bar{\varepsilon}_{ij} \bar{\varepsilon}_{kl}$$

This can also be understood as follows: the Q4-homogenized properties of a checkerboard pattern is $\frac{1}{2}E^0$. By Q4-homogenized we mean the homogenized properties that one obtains if the displacement fields are restricted to Q4 discretizations at the level of the checkerboard.

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For the above design problem we also have:

$$\min_{u\in Q4,\ \text{periodic}} \Psi(\rho,u) \le \int_\Omega \rho^p E^0_{ijkl}\bar{\varepsilon}_{ij}\bar{\varepsilon}_{kl}\mathrm{d}\Omega$$
$$= E^0_{ijkl}\bar{\varepsilon}_{ij}\bar{\varepsilon}_{kl}\int_\Omega \rho^p \mathrm{d}\Omega \le \frac{1}{2}E^0_{ijkl}\bar{\varepsilon}_{ij}\bar{\varepsilon}_{kl}\ .$$

Thus the checkerboard pattern is an optimal design, for the model with Q4-displacements. This is unphysical for several reasons. First, the *true* homogenized material parameters for a checkerboard of material and void is actually zero. Second, the stiffest material that can be constructed is the rank-2 layered composite, which has a strain energy W. Comparing, we obtain that

$$\overline{W}(\rho=0.5,\bar{\varepsilon}) \le \frac{1}{2}E^0_{ijkl}\bar{\varepsilon}_{ij}\bar{\varepsilon}_{kl}$$

where equality *only* holds if the principal strains satisfy $\frac{\bar{\varepsilon}_I}{\bar{\varepsilon}_{II}} = -\nu$ (with the convention $|\bar{\varepsilon}_I| \ge |\bar{\varepsilon}_{II}|$). This means that a Q4-checkerboard grossly overestimates the stiffness, to the extend that it is "stiffer" than the stiffest lay-out of material (the stiffness corresponds to the Voigt bound, which cannot be realized by a composite).

Thank you for your attention