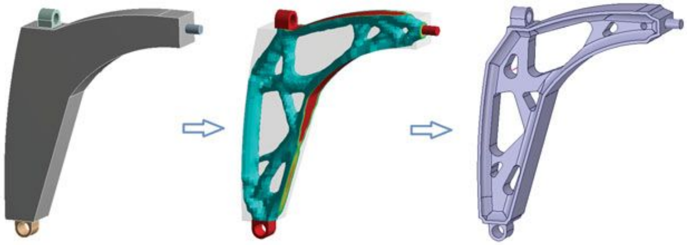
香港中文大學
The Chinese University of Hong Kong

MAEG5160: Design for Additive Manufacturing

Lecture 15: Design with anisotropic materials_3



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4. Design with a free parametrization of material

The goal of this section is to formulate a structural optimization problem in a form that encompasses the design of structural material in a *broad sense*, while also encompassing the provision of predicting the structural topologies and shapes associated with the optimum distribution of the optimized materials. This is accomplished by representing as design variables the material properties in the most general form possible for a (locally) linear elastic continuum namely as the *unrestricted* set of positive semi-definite constitutive tensors.

In the modelling of the optimization problem the parameters which describe the structure are, as in the preceding sections divided into two sets: *the parameters defining the local material tensor* and *those that describe the specific cost of the material*. In parallel with the developments for layered materials, it can be shown that the minimum compliance optimization of a structure with respect to these two sets of parameters can be performed independently. Furthermore, the optimization with respect to the local material tensor parameters can be performed analytically. This derivation is fairly simple for both the single load case and the multiple load problem and for any dimension of the spatial domain. Thus the more general problem statement is considerably simpler as compared to the homogenization topology problem.

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The very general framework of optimizing directly on a free parametrization of the material tensor results in developments which provide an attainable *global lower bound* on the performance of any structure designed for the same loads, boundary conditions and ground structure. At the same time, it provides an attainable *global upper variational bound* on the effective moduli of any elastic material, within the cost measures defined. Also, the considerable simplifications that can be demonstrated indicate that the broader form of a material design problem, as described and analyzed in this section, constitutes effective means for studying the global structural optimization problem involving sizing, shape, topology and material selection.

The results that we can obtain within the assumption of a locally unconstrained configuration of material are informative towards gaining insight into the nature of efficient local structures. This is useful for theoretical as well as practical purposes. As an example of the latter, recent work has thus employed the framework of free material design to generate procedures for tape-lay-up in composites. Also, the original theoretical work on the subject laid the seeds for the very successful use of topology design methodology for design of materials. Here one tries, for practical reasons, to understand how to match a particular local microstructure to the specific form of a elasticity tensor, for example the ones predicted here.

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4.1 Problem formulation for a free parametrization of design

Modelling considerations

In the homogenization method, the total volume of material, defined at the micro level, provides a natural cost function for the optimization problem. There is not at first glance a natural cost function for the general material design formulation we consider here, where we allow for all possible positive semi-definite constitutive tensors. Instead, *we use certain invariants of the stiffness tensor as the measure of cost*, thus ensuring that the optimal design solutions are independent of the choice of reference frame.

For physical reasons, the possible stiffness tensors in the design formulation are restricted to the set of positive semi-definite, symmetric tensors. Also, suitable cost functions must have the property of frame indifference. Since the goal is to optimize the local material properties as well as the global structural response, we choose to consider cost in terms of invariants of the constitutive tensor itself. Specifically, we choose for the developments in the following two invariants as examples of local cost:

$$\text{Case A}: \ \Psi_{\mathrm{A}}(E) = E_{ijij} \ , \qquad \text{Case B}: \ \Psi_{\mathrm{B}}(E) = [E_{ijkl}E_{ijkl}]^{\frac{1}{2}}$$

i.e., respectively, the trace and the Frobenius norm of the 4-tensor E. Note that these measures are homogeneous of degree one. Thus comparing to the conventional 2D problem for the design of material distribution in a sheet (where total cost is proportional to the volume of material), the above "cost measures" correspond in their role to the sheet thickness. More general considerations are also possible, combining several invariants of the tensor to provide for *generalized cost measures* which can be varied to cater for specific design goals, for example governed by available fiber composites.

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Problem statement

The problem we consider is the multiple load minimum compliance problem generalized to the situation where the material properties themselves appear in the role of design variables. This means that we consider a design parametrization (a definition of E_{ad}) in the form

$$E \succeq 0 \text{ in } \Omega \ , \quad E_{ijkl} \in L^{\infty}(\Omega), \text{ for all } ijkl \ , \quad \int_{\Omega} \Psi(E) \mathrm{d}\Omega \leq V$$

Thus we take the minimization over all positive, semi-definite stiffness tensors E_{ijkl} (with the usual symmetry properties) and use the integral over the domain of some invariant $\Psi(E_{ijkl})$ of the stiffness tensor as the measure of cost. For the the sake of simplifying the derivation, we introduce the resource density functions, $\rho_{\mathrm{A}} = \Psi_{\mathrm{A}}(E)$ and $\rho_{\mathrm{B}} = \Psi_{\mathrm{B}}(E)$ and state the minimum compliance problem for a multiple load setting in terms of potential energy as

$$\max_{\substack{\text{density } \rho \\ 0 \leq \rho_{\min} \leq \rho \leq \rho_{\max} < \infty \\ \int_{\Omega} \rho \mathrm{d}\Omega \leq V}} \ \max_{\substack{\text{stiffness } E \succeq 0 \\ \Psi(E) \leq \rho}} \ \min_{\substack{\hat{u} = \{u^1, \ldots, u^M\} \\ u^k \in U,\, k=1,\ldots,M}} \int_{\Omega} W(E, \hat{u}) \mathrm{d}\Omega - l(\hat{u})$$

$$W(E, \hat{u}) = \frac{1}{2} \sum_{k=1}^{M} w^k E_{ijpq}(x) \varepsilon_{ij}(u^k) \varepsilon_{pq}(u^k) \ , \tag{3.29}$$

$$l(\hat{u}) = \sum_{k=1}^{M} w^k l^k(u^k), \qquad \hat{u} = \{u^1, \ldots, u^M\} \ ,$$

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with M load cases. Here we have, provided a separation between the properties of the tensor *E* that can be optimized locally (at each point in the structure) and those that must be treated as a distributed parameter problem over the full domain.

In the max-min problems above we have introduced an upper bound on the resource densities in order to ensure that the problem is well posed. A possible non-zero lower bound is also catered for. Note that the resource constraints are convex for both case A and B.

In the developments to follow, we show that an analytical optimization actually can reduce the number of free design variables from 6 in dimension two and 21 in dimension three to only *one* in both dimensions (in any dimension that is).

Splitting the problem into a series of sub-problems

we can rearrange problem (3.29) and split it into two coupled optimization subproblems (the *local anisotropy problem*, and the *material distribution problem*). The interchange of the min and max for the inner problems of (3.29) here gives an *equivalent* problem as (3.29) satisfies the conditions for existence of a saddle point: the objective function is

concave (linear) in E and convex in the displacements u^k, and the set $\{E|\Psi(E) \leq \rho,\ E \succeq 0\}$ is closed, convex and weak*-compact in $L^\infty(\Omega)$

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4.2 The solution to the optimum local anisotropy problems

In this section we study the solution to the local anisotropy optimization problem. To this end we define the positive semi-definite, symmetric 4-tensor

$$A_{ijpq} = \sum_{k=1}^{M} w^k A^k_{ijpq}, \quad A^k_{ijpq} = \varepsilon_{ij}(u^k)\varepsilon_{pq}(u^k)$$

and write the optimization of the energy

$$\max_{\substack{E \succeq 0 \\ \Psi(E) \leq \rho}} \frac{1}{2} E_{ijpq} A_{ijpq}$$

The Frobenius norm case.

For the norm resource measure, problem above corresponds to finding the tensor *E* of given norm that has the largest standard inner product with the given tensor *A*. The optimal stiffness tensor is thus proportional to *A* and because of the resource constraint it is (uniquely) given as

$$E^B_{ijpq} = \rho \frac{A_{ijpq}}{\sqrt{A_{mnrs} A_{mnrs}}}$$

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The corresponding extremal energy functional is

$$\overline{W}_B(\rho, \hat{u}) = \rho \breve{W}_B(\hat{u}) = \frac{\rho}{2}\sqrt{A_{ijpq}A_{ijpq}} = \frac{\rho}{2}\sqrt{\sum_{k,l=1}^{M} w^k w^l [\varepsilon_{ij}(u^k)\varepsilon_{ij}(u^l)]^2}$$

We have denoted the optimum energy density function per unit amount of resource ρ. Here and elsewhere we embellish with an upper inverted “hat” quantities per unit amount of resource. Note that the optimized material properties represented by E_{ijpq} do not possess any specific symmetry properties and the material is thus generally anisotropic for all but very special cases. The optimized material tensor can have zero eigenvalues, and this happens always if the number of load cases that we consider is one or two in dimension 2 or one to five in dimension 3. For more than this number of load cases, the material will generically be stable, with zero eigenvalues only appearing if the strain fields are linearly dependent.

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The trace case.

For the trace resource measure, problem corresponds to solving a linear programming problem, with objective given by the tensor *A***.**

In order to find the solution to this problem, introduce the spectral decompositions of E and A. Now let $0 \leq \eta_1 \leq \ldots \leq \eta_N$, $\sum_{i=1}^{N} \eta_i = \rho$, and $0 \leq \lambda_1 \leq \ldots \leq \lambda_N$ be the ordered eigenvalues of E and A, respectively ($N = 3$ in dimension 2 and $N = 6$ in dimension 3). From a result on the eigenvalues of positive symmetric matrices (Mirsky 1959), it follows that

$$E_{ijpq} A_{ijpq} \leq \sum_{i=1}^{N} \eta_i \lambda_i \leq \sum_{i=1}^{N} \eta_i \overline{\lambda} = \rho \overline{\lambda} \ ,$$

where $\overline{\lambda}$ denotes the largest eigenvalue of the tensor A, with orthonormal eigentensors $\varepsilon_{ij}^{\alpha}$, $\alpha = 1, \ldots, P$. We observe that the right hand side of these inequalities is achieved by any stiffness tensor E of the form

$$E_{ijpq}^{A} = \rho \sum_{\alpha=1}^{P} \mu^{\alpha} \varepsilon_{ij}^{\alpha} \varepsilon_{pq}^{\alpha}, \quad \text{with} \ \sum_{\alpha=1}^{P} \mu^{\alpha} = 1 \ ,$$

so we conclude that the optimal energy in the trace case is

$$\overline{W}_A(\rho, \hat{u}) = \rho \breve{W}_A(\hat{u}) = \frac{\rho}{2} \overline{\lambda} = \frac{\rho}{2} \max \operatorname{eig} \left\{ \sum_{k=1}^{M} w^k \varepsilon_{ij}(u^k) \varepsilon_{pq}(u^k) \right\} \ .$$

If $\overline{\lambda}$ is a simple eigenvalue, E^A is unique and it corresponds to an orthotropic material, but in the generic case the form of an optimal E^A is only determined when the full problem is solved (the parameters μ^{α} of the expansion of E^A is found from this "outer" problem).

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The single load case

For the case of a single load case (M=1), the optimal energy in the trace and norm case reduce to the same expression, namely

$$\rho \breve{W}_0(u^1) = \frac{1}{2}\rho \varepsilon_{ij}(u^1)\varepsilon_{ij}(u^1) = \frac{1}{2}\rho I_{ijkl}\varepsilon_{ij}(u^1)\varepsilon_{kl}(u^1)$$

corresponding the energy of an isotropic, zero-Poisson-ratio material, with stiffness tensor ρI, which is ρ times the identity tensor. This matrix has norm $\Psi_B(\rho I) = \sqrt{N}\rho$ and trace $\Psi_A(\rho I) = N\rho$ ($N = 3$ in dimension 2 and $N = 6$ in dimension 3). Note however, that the bound $\breve{W}_0$ is achieved with the (unique) tensor

$$E^*_{ijkl} = E^A_{ijkl} = E^B_{ijkl} = \rho \frac{\varepsilon_{ij}(u^1)\varepsilon_{kl}(u^1)}{\varepsilon_{pq}(u^1)\varepsilon_{pq}(u^1)} \ ,$$

which has norm as well as trace equal to ρ. The optimized material represented by E^* is orthotropic, with axes of orthotropy given by the axes of principal strains (and stresses) for the field $\varepsilon_{ij}(u^1)$

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For completeness of presentation, we write for dimension 2 the resulting optimal stiffnesses in terms of and in the frame of the principal strains ε_I, ε_{II} of the single strain field $\varepsilon_{ij}(u)$ (for convenience we have dropped the index "1" for this load case)

$$(E^*)_{\text{matrix}} = \frac{\rho}{\varepsilon_I^2 + \varepsilon_{II}^2} \begin{bmatrix} \varepsilon_I^2 & \varepsilon_I \varepsilon_{II} & 0 \\ \varepsilon_I \varepsilon_{II} & \varepsilon_{II}^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} .$$

Note again that the optimized material is indeed orthotropic, and that the material stiffness tensor has *two* zero eigenvalues. Thus, the extremization of the strain energy density results in a material which is at the utmost limit of feasibility for satisfying the positivity constraint, and the material can only carry strain fields which are direct scalings of the given strain field for which the optimization was undertaken. This underlines the true optimal nature of the material. Such behaviour of extremized materials was also seen in the homogenization method for topology design with one given material; in that case the optimized material has one zero eigenvalue corresponding to vanishing shear stiffness.

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For the single load case, we have for both resource measures obtained the reduced equivalent problem statement in the form

$$\max_{\substack{\text{density } \rho \\ 0 \le \rho_{\min} \le \rho \le \rho_{\max} \\ \int_\Omega \rho \mathrm{d}\Omega \le V}} \quad \min_{u \in U} \left\{ \frac{1}{2} \int_\Omega \rho \varepsilon_{ij}(u) \varepsilon_{ij}(u) \mathrm{d}\Omega - l(u) \right\} ,$$

which not only gives the optimal distribution of material, but also the displacements, strains, stresses and material properties of the optimal structure. For this problem we can return to the original form of the minimum compliance problem as stated in (1.6) taking the development "full circle"

$$\begin{aligned} &\min_{u, \rho} l(u) \\ &\text{s.t.} : \int_\Omega \rho \varepsilon_{ij}(u) \varepsilon_{ij}(v) \mathrm{d}\Omega = l(v) \quad \text{for all } v \in U , \\ &\qquad 0 \le \rho_{\min} \le \rho \le \rho_{\max}, \quad \int_\Omega \rho \mathrm{d}\Omega \le V . \end{aligned}$$

This reduced problem is exactly equivalent to the variable-thickness design problem for a sheet made of an isotropic zero-Poisson-ratio material, with the density ρ playing the role of the thickness of the sheet.

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Let us briefly for the single load case consider the *stress based formulation* for the design parametrization used here. This problem can be stated as

$$\inf_{\substack{E \succ 0 \\ \int_\Omega \Psi(E) \mathrm{d}\Omega \leq V}} \; \min_{\substack{\mathrm{div}\sigma + f = 0 \\ \sigma \cdot n = t \text{ on } \Gamma_T}} \left\{ \frac{1}{2} \int_\Omega E^{-1}_{ijkl} \sigma_{ij} \sigma_{kl} \mathrm{d}\Omega \right\}$$

where we take the infimum with respect to all *positive definite* stiffness tensors, in order to give meaning to $E^{-1}_{ijkl}\sigma_{ij}\sigma_{kl}$ Interchanging the equilibrium minimization with the local minimization of complementary energy and using that we from a spectral decomposition can derive that

$$\inf_{E \succ 0,\, \Psi(E) = \rho} E^{-1}_{ijkl} \sigma_{ij} \sigma_{kl} = \frac{1}{\rho} \sigma_{ij} \sigma_{ij}$$

for both of our resource measures, we see that the stress based case has a reduced formulation

$$\inf_{\substack{\text{density } \rho \\ 0 < \rho \leq \rho_{\max} \\ \int_\Omega \rho \mathrm{d}\Omega \leq V}} \; \min_{\substack{\mathrm{div}\sigma + f = 0 \text{ in } \Omega \\ \sigma \cdot n = t \text{ on } \Gamma_T}} \left\{ \frac{1}{2} \int_\Omega \frac{1}{\rho} \sigma_{ij} \sigma_{kl} \mathrm{d}\Omega \right\}$$

as expected in light of the form of the displacements based formulation above

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4.3 Analysis of the reduced problems

The equilibrium problem for the optimized energy

The solution to the local anisotropy problems has shown that the equilibrium problem with the optimized strain energy functions for both cases we consider can be written as

$$\min_{\substack{\hat{u}=\{u^1,\ldots,u^M\}\\ u^k\in U,\, k=1,\ldots,M}} \left\{ \int_\Omega \rho \breve{W}(\hat{u}) \mathrm{d}\Omega - l(\hat{u}) \right\}$$

This is a coupled, non-linear problem for all the load cases at once, the coupling arising through the optimized strain energy functional.

We note here that the function $\breve{W}(u^1,\ldots,u^M)$ of the displacements is homogeneous of degree two, that is, under proportional loading the optimized material behaves as a linearly elastic material. Moreover, $\breve{W}$ is a convex function. This follows from the fact that $\breve{W}$ is given as a maximization of convex functions of the displacements. For the Frobenius norm resource measure, we note that $\breve{W}_B$ is a smooth function, except at the origin $(u^1,\ldots,u^M) = (0,\ldots,0)$ when all displacements are zero. For the trace resource measure the optimized strain energy functional involves an eigenvalue problem, which implies that the functional $\breve{W}_A$ is only differentiable at sets of displacements for which the maximal eigenvalue of the tensor A is not repeated, and it is non-differentiable at displacements for which the maximal eigenvalue is multiple. This includes the origin $(u^1,\ldots,u^M) = (0,\ldots,0)$ where all displacements are zero. Remark that for the single load case, the equilibrium problem (3.33) is just a single linear equilibrium problem for a structure made of a zero-Poisson-ratio material with varying Young moduli, as described through the variable ρ.

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The optimization problem in resource density

The reduced optimization problem is as described earlier

$$\max_{\substack{\text{density } \rho \\ 0 \leq \rho_{\min} \leq \rho \leq \rho_{\max} \\ \int_\Omega \rho \mathrm{d}\Omega \leq V}} \left[\Phi(\rho) = \min_{\substack{\hat{u}=\{u^1,\ldots,u^M\} \\ u^k \in U,\, k=1,\ldots,M}} \left\{ \int_\Omega \rho \breve{W}(\hat{u}) \mathrm{d}\Omega - l(\hat{u}) \right\} \right] \tag{3.34}$$

This is of the form of a variable thickness sheet problem for a sheet made of a non-linear elastic material. Here the function $\Phi(\rho)$ of the density distribution ρ is defined through the non-linear equilibrium problem discussed in the previous section. Since $\Phi(\rho)$ is given as a minimization of concave (linear) functions in ρ, $\Phi(\rho)$ is in itself concave. Thus (3.34) is a convex minimization problem in the density variable ρ, where the condition of optimality is that the energy $W(u^1, \ldots, u^M)$ is constant in the region of intermediate density.

The reduced problem (3.34) is also a saddle point problem in the resource density ρ and displacements $\{u^k\}$. The existence of a saddle point is also here assured and we can thus find an optimal solution of the optimization problem (3.34) by solving

$$\min_{\substack{\hat{u}=\{u^1,\ldots,u^M\} \\ u^k \in U,\, k=1,\ldots,M}} \left\{ \widehat{W}(\hat{u}) - l(\hat{u}) \right\}, \quad \widehat{W}(\hat{u}) = \max_{\substack{\text{density } \rho \\ 0 \leq \rho_{\min} \leq \rho \leq \rho_{\max} \\ \int_\Omega \rho \mathrm{d}\Omega \leq V}} \int_\Omega \rho \breve{W}(\hat{u}) \mathrm{d}\Omega$$

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Using a Lagrange multiplier Λ for the resource constraint, the globally optimized weighted strain energy functional $\widehat{W}(\hat{u})$ can then be expressed as

$$\widehat{W}(\hat{u}) = \min_{\Lambda \geq 0} \left\{ \int_\Omega \max \left\{ \rho_{\min}[\breve{W}(\hat{u}) - \Lambda],\, \rho_{\max}[\breve{W}(\hat{u}) - \Lambda] \right\} \mathrm{d}\Omega + \Lambda V \right\} .$$

This implies that the design variables can be removed entirely from the problem, and the resulting problem becomes a non-linear and non-smooth, convex, analysis-only problem. Similar results are also developed for truss design

An extension to contact problems

It is clear from the analysis above that all steps can be performed without restriction for problems that include design independent, convex displacement constraints in the equilibrium statement. Thus design problems including unilateral contact can be treated by a similar analysis.

Now let Γ_C denote the boundary of potential contact and let $u \cdot n \geq 0$ on Γ_C be the unilateral contact condition; this is a convex constraint. Then the design problem for minimum compliance under multiple loads can be stated

$$\max_{\substack{\text{stiffness } E \succeq 0 \\ \int_\Omega \Psi(E)\mathrm{d}\Omega \leq V}} \quad \min_{\substack{\hat{u}=\{u^1,\ldots,u^M\} \\ u^k \in U,\, u^k \cdot n \geq 0 \text{ on } \Gamma_C}} \left\{ \int_\Omega W(E,\hat{u})\mathrm{d}\Omega - l(\hat{u}) \right\}$$

where the inner problem is the minimum potential energy principle expressed for a contact problem. For both resource measures this problem can be reduced to the forms seen earlier, the only change being the addition of the contact condition on the admissible displacements. Also, the optimal materials are given by the same expressions.

Materials with piecewise linear elastic behaviour

The general framework of free material optimization can also be extended to cover the design of a structure and associated material properties for a system composed of a generic form of *nonlinear softening material*. Here the optimal distribution of material properties depends on the magnitude of load, in contrast to the case with linear material. The relevant mechanics is now represented in terms of a generalized complementary energy principle and the design objective is likewise based on complementary energy. Net material properties of the softening medium reflect a superposition of properties associated with each of a number of material constituents, and the collection of these properties, expressed through the stiffness tensors for each of these constituents, provides the problem with a set of design parameters. It is the availability of an extremum problem formulation for the analysis part of the problem that makes it possible to treat the design of nonlinear materials conveniently. The formulation used amounts to a generalized form of the complementary energy principle, and is stated here in stresses alone. With the superposition of M softening components and one purely elastic basis component to make up the total stress, the analysis problem has the form:

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$$\max_{\alpha,\sigma^k,\gamma} \ \alpha$$

$$\text{s.t.} : \operatorname{div}(\gamma_{ij} + \sum_{k=1}^{M} \sigma_{ij}^k) + \alpha f = 0 \ ,$$

$$(\gamma_{ij} + \sum_{k=1}^{M} \sigma_{ij}^k) \cdot n = \alpha t \text{ on } \Gamma_T \ ,$$

$$\sigma^k \in \mathcal{K}_k \ , k = 1, \dots, M \ ,$$

$$\frac{1}{2} \int_{\Omega} (D_{ijrs}\gamma_{ij}\gamma_{rs} + \sum_{k=1}^{M} C_{ijrs}^k \sigma_{ij}^k \sigma_{rs}^k) \mathrm{d}\Omega \leq \Pi \ .$$

Here $C_{ijrs}^k = [E_{ijrs}^k]^{-1}$ are the compliance tensors for the M softening components and $D_{ijrs} = [F_{ijrs}]^{-1}$ is the compliance tensor for the basis component. The stresses for the softening components are denoted σ^k and the stress of the basis component is γ. The convex sets of admissible stresses for the softening components are denoted by $\mathcal{K}_k$. This problem statement is a parametrized complementary energy formulation for the general softening material. The solution to this problem predicts a bound to the equilibrium load within the limit Π on total complementary energy.

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The formulation above leads one naturally to consider the design of the nonlinear material for maximization of load carrying capacity within the framework of free material design. Up to a rescaling factor on the load this problem is equivalent to the convex problem:

$$\inf_{E^k,F}\min_{\sigma^k,\gamma}\frac{1}{2}\int_\Omega([F_{ijrs}]^{-1}\gamma_{ij}\gamma_{rs}+\sum_{k=1}^{M}[E^k_{ijrs}]^{-1}\sigma^k_{ij}\sigma^k_{rs})\mathrm{d}\Omega$$

$$\text{s.t. : div}(\gamma_{ij}+\sum_{k=1}^{M}\sigma^k_{ij})+\bar{\alpha}f=0\ ,$$

$$(\gamma_{ij}+\sum_{k=1}^{M}\sigma^k_{ij})\cdot n=\bar{\alpha}t\text{ on }\Gamma_T\ ,$$

$$\sigma^k\in\mathcal{K}_k\ ,k=1,\ldots,M\ ,$$

$$F\succ 0\ \ E^k\succ 0\ ,k=1,\ldots,M\ ,$$

$$\int_\Omega\Psi(F)\mathrm{d}\Omega\leq V_0\ ,\int_\Omega\Psi(E^k\mathrm{d}\Omega\leq V_k\ ,k=1,\ldots,M$$

where each phase has a limited total amount of resource. This is a generalized complementary energy formulation of the design of structures with piecewise linear behaviour.

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In the formulation above it is assumed that the softening constraints for the softening components σ^k of total stress are design independent. Thus the solution predicts the optimal distribution of stiffnesses within these specified softening limits. With this assumption we can now perform the minimization with respect to the pointwise variation of the stiffness tensors, using the result (3.32). With the introduction of these optimal local energy expression, the problem can be reduced to the convex problem:

$$\inf_{\rho_k,\rho_0}\ \min_{\sigma^k,\gamma} \tfrac{1}{2}\int_\Omega \Big(\frac{1}{\rho_0}\gamma_{ij}\gamma_{rs} + \sum_{k=1}^{M}\frac{1}{\rho_k}\sigma^k_{ij}\sigma^k_{rs}\Big)\mathrm{d}\Omega$$

$$\text{s.t. : } \operatorname{div}\Big(\gamma_{ij} + \sum_{k=1}^{M}\sigma^k_{ij}\Big) + \bar{\alpha} f = 0\ ,$$

$$\Big(\gamma_{ij} + \sum_{k=1}^{M}\sigma^k_{ij}\Big)\cdot n = \bar{\alpha} t \text{ on } \Gamma_T\ ,$$

$$\sigma^k \in \mathcal{K}_k\ , k = 1,\ldots,M\ ,$$

$$F \succ 0\ \ E^k \succ 0\ , k = 1,\ldots,M\ ,$$

$$\int_\Omega \rho_0 \mathrm{d}\Omega \le V_0\ , \int_\Omega \rho_k \mathrm{d}\Omega \le V_k\ , k = 1,\ldots,M\ ,$$

where the energy measure for each constituent corresponds to the complementary energy of a linear elastic, zero-Poisson-ratio material of density equal to the locally assigned resource value.

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4.4 Numerical implementation and examples

Computational procedure for the single load case

For the single load case, both the trace and Frobenius norm resource measures lead to the same reduced problem of what amounts to a variable thickness sheet problem for a sheet made of a zero-Poisson-ratio material. In this case we have a design problem that shares important features with minimum compliance problems for trusses, and the problem can be efficiently solved using one of the algorithms on truss topology optimization. This is based on the format of the problem formulation, which in discretized FE form can be rewritten as a smooth and convex optimization problem in displacements only (with the notation of):

$$\begin{aligned} &\min_{\mathbf{u},\Lambda,\tau} \; \{\tau - \mathbf{f}^T\mathbf{u} + \Lambda V\} \\ &\text{s.t.}: \rho_{\min}[\tfrac{1}{2}\mathbf{u}^T\mathbf{K}_e\mathbf{u} - \Lambda] \le \tau \quad e = 1,\dots,N \, , \\ &\qquad \rho_{\max}[\tfrac{1}{2}\mathbf{u}^T\mathbf{K}_e\mathbf{u} - \Lambda] \le \tau \quad e = 1,\dots,N \, . \end{aligned}$$

This format is well-suited for solution by the so-called PBM interior point. Note that above only involves the displacement variables (and two auxiliary variables), that it is a linear optimization problem with quadratic constraints, and that the Lagrange multipliers for the constraints determines the values of the density ρ

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Computational procedure for the general case

The presence of multiple load cases introduces significant complications if the reduced energy expressions are applied. These complications arise because the locally optimal material couples deformations associated with the different load cases in a complex way that, as we have already seen, involves non-linear, non-smooth energy functionals which depend on all the load cases simultaneously. This stands in sharp contrast with the solution of the problem of design for a single load case. Early numerical work for the multiple load scenario employed an iterative secant method for solving the inner non-linear equilibrium problem and an optimality criteria method for the density optimization. This can be applied for the Frobenius norm case, but experience has shown that the complicated non-smoothness for the trace resource case prevents the use of this approach.

An efficient alternative is to apply the formulation also in the multiple load case. Limiting ourselves to the *trace* case, a reformulation in the spirit is also possible, but it now involves constraints stating that certain matrices are positive definite; in the trace case the optimal specific energy *W* is the largest eigenvalue of the tensor *A* and this can be expressed as

$$\breve{W}_A = \inf_{\tau I_{ijpq} - A_{ijpq} \succeq 0} \tau$$

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This also means that W_A is bounded by a constant *k* if and only if $kI_{ijpq} - A_{ijpq} \succeq 0$. This can, as A is the sum of dyadic products, be rewritten as a condition that a certain matrix, which is linear in the strains $\varepsilon(u^k)$ is positive semidefinite. Based on which it is thus possible to write a FE discretized version of the problem as a *semidefinite program* in the displacements only where also contact conditions are treated). The advantage of this reformulation is that such problems can be solved very efficiently by modern mathematical programming methods.

4.5 Free material design and composite structures

The result of the free parametrization of material is in a sense the ultimately best physically attainable material and it is natural to utilize the full information obtained in the results in an attempt to design an attainable advanced material. This obviously depends on the type of the advanced material available and on the manufacturing technology.

Realization by tape-lay-up

First we consider a procedure that relies on the free material optimization for design of composite materials manufactured by the so-called tape-laying technology. In a post-processing phase one can here generate curves which indicate how to lay the tapes and how to organize the thickness of the tapes. This gives a good initial approximation for an optimization procedure that also takes into consideration all the technological restrictions of the tape-laying process.

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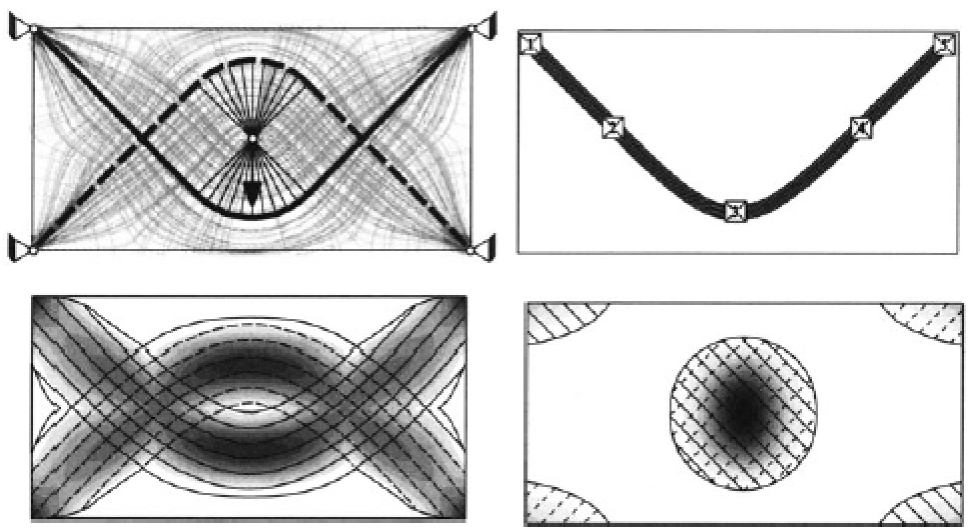
The post-processing uses that the optimal material for the single load free material design is orthotropic and that the axes of orthotropy correspond point-wise to the orthogonal directions of principal strains or stresses. This allows an interpretation where this governs the direction of fibres in a (weak) resin material. To get an impression of the lay-out of these fibres and the thickness, a graphical post-processing tool can be employed that plots the vector fields of principal strain direction by means of *smooth* curves. The optimal load path is interpreted as that of a fibre reinforced material, for example in the form of pre-pregs of Carbon Fibre Reinforced Plastic (CFRP) tapes. Tape-laying is thus a way to bridge the gap between free material design and the preliminary design phases for structures constructed from such tapes.



Tape-laying example. Top left is the stress directions from the free material optimization and a super-posed Michell solution. Top right shows the laying of the first tape. Bottom pictures show two tape layers obtained by postprocessing of the top left design.

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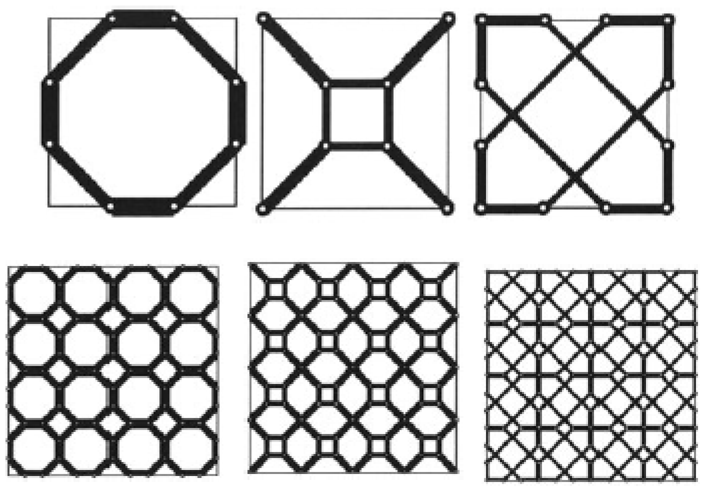
Realization by materials with microstructure

Alternatively, skeletal bar structures could be used to generate microstructures that mimic the behaviour of the optimized material tensors, see figures below. These results are obtained numerically by an inverse homogenization operation that works with unit cells constructed from truss elements. The results substantiates the theoretical finding that any stiffness tensor can be constructed from layered materials made from an infinitely strong phase and an infinitely weak phase.



$$E^{*} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Minimum weight 2-D microstructures (upper row shows the unit cells, lower row is an assemblage of cells) for obtaining materials with the indicated stiffness in the axis of the cell, corresponding the optimal material for a single strain field $\varepsilon=(1,1,0)$ This is an isotropic material with Poisson's ratio 1.0. The three designs all have the same weight and are obtained using a 4 by 4 equidistant nodal lay-out in a square cell. All 120 possible connections between the nodal points are considered as potential members. Members not shown for the optimum cell (and structure) are at the minimum gauge which is 10^5 times smaller than the maximum gauge. The different designs are obtained by penalization of the lengths of the bars.

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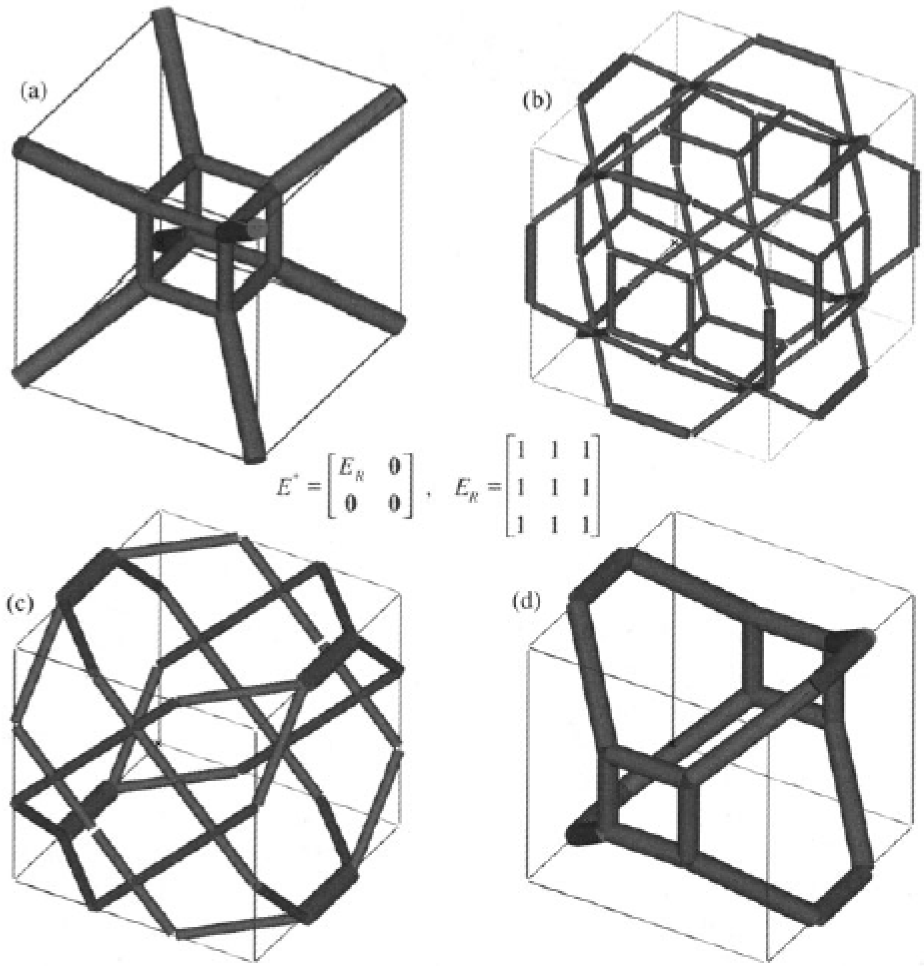

Minimum weight microstructures in dimension 3 for obtaining materials which corresponds to the optimal material for a single strain field $\varepsilon = (1,1,1,0,0,0)$. The three designs all have the same weight and are obtained using a 4 by 4 by 4 equidistant nodal lay-out in a cubic cell. All 2016 possible connection between the nodal points are considered as potential members. Members not shown for the optimum cell (and structure) are at the minimum gauge. The different designs are obtained by penalization of members with certain lengths. The topologies in a) and b) have full cubic symmetry. The topology in c) has bars on the surface of the cell only and is not cubic symmetric, even though the effective parameters are isotropic. Notice the similarity between the 3-D microstructures and the 2-D microstructures shown in previous figure.

Thank you for your attention