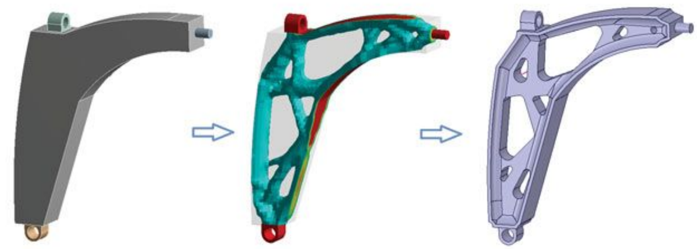
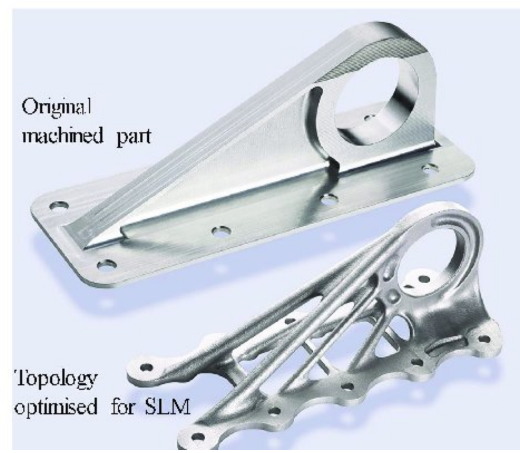
香港中文大學
The Chinese University of Hong Kong

MAEG5160: Design for Additive Manufacturing

Lecture 16: Design with anisotropic materials_4





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5. Plate design with composite materials

5.1 The homogenization approach for Kirchhoff plates

In analogy with the topology design problem treated so far, a relaxation of the Kirchhoff plate design problem[6] requires that one considers plates with infinitely many, infinitely thin integral stiffeners. This can be in the form of a rank-2 structure of stiffeners of height $h_{\max}$ on a solid plate of variable thickness *h,* i.e. a planar rank-2 layering of the weak tensor $\frac{h^3}{12}E^0_{ijkl}$ and the strong tensor $\frac{h^3_{\max}}{12}E^0_{ijkl}$

For the relaxed design problem we thus need to state the homogenization formulas for Kirchhoff plates, more specifically the effective material parameters for rib-stiffened plates. With these formulas at hand, the computational procedure for computing optimal designs is completely analogous to the procedure described previously. The optimality criteria for the densities are equivalent to those derived previously as well, with strains and stresses interpreted as curvatures and moments. However, extra care is required for use of the result on optimal rotations.

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In order to exemplify the difference to the plane stress situation, consider a constant thickness, perforated plate with an orthogonal rank-2 system of stiffeners. The effective bending stiffness is then

$$D = \frac{h_{\max}^3}{12}\tilde{D}$$

$$\tilde{D}_{1111} = \frac{\gamma E}{\mu\gamma(1-\nu^2)+(1-\mu)}, \quad \tilde{D}_{1122} = \mu\nu E_{1111},$$

$$\tilde{D}_{2222} = \mu E + \mu^2\nu^2 E_{1111}, \quad \tilde{D}_{1212} = (\gamma+\mu+\mu\gamma)\frac{E}{2(1+\nu)}$$

where the primary layering of density μ is in the 2-direction and the secondary layer has density γ. This material law satisfies that $D_{1111} + D_{2222} - 2D_{1122} - 4D_{1212} < 0$, meaning that the analysis of the minimum compliance plate problem is more tricky than the plane stress case. As an example, there may be regions in a plate where an optimal, orthogonal rank-2 layering is not aligned with the principal curvatures. We will not treat the plate problem in further detail, as this is a major subject.

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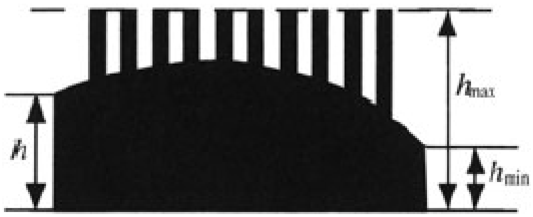
We do mention, however, that the optimal design of plates takes an extra twist when the analysis modelling is taken into account. The design problem and its associated relaxation can be viewed as a purely mathematical question of achieving well-posedness, but as any plate model is an approximate model, it is natural to question the validity of the relaxation in relation to the modelling restrictions / assumptions made to achieve the plate model under consideration. Thus the use of thin, high stiffeners in a Kirchhoff plate model is in fact a violation of the assumptions under which this model can be derived from 3-D elasticity. This means that the developments above should be seen in the framework of achieving regularization strictly within the Kirchhoff plate framework, ignoring eventual modelling restrictions. The modelling problem should by no means be dismissed but lies outside the scope of this presentation. The reader is referred to the literature for further information on this problem as well as to studies of optimal thickness design of Mindlin plates within the framework of the homogenization modelling.



Cross-section of the upper half of a rib-stiffened plate with one field of stiffeners running along the normal of the cutting plane.

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5.2 Minimum compliance design of laminated plates

This section is concerned with the optimal design of the lay-up of *laminated* plates for maximum stiffness. We consider optimization with respect to the ply thicknesses, fiber orientations and the stacking sequence of the laminates, keeping the ply material properties and the shape of the plate fixed. Instead of working directly with this mix of integer and real design parameters we employ a design parametrization through the so-called lamination parameters. These represent the effective, integrated properties of the laminate and are given as moments relative to the plate mid-plane of the trigonometric functions entering in the frame rotation formulas for stiffness matrices. In this way the properties related to the stiffness of the laminates are emphasized in the optimization model, while the realization of the optimal effective properties is postponed for subsequent post-processing.

The developments below are strongly related to the free material design and to the homogenization approach discussed earlier, and also here we can carry out an analytical derivation of the optimal local properties of material. Moreover, we choose to extend the design space to include "chattering“ designs, thereby allowing infinitely many small variations of the fiber orientation in each point through the thickness for each design domain of the plate. This corresponds to the introduction of periodic composites for topology design and the use of rib reinforced plates in plate design.

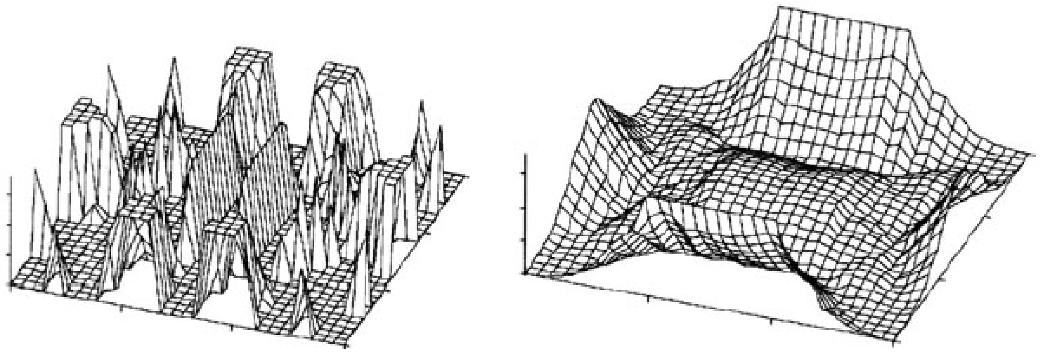

Plate design of a clamped Kirchhoff plate subject to uniform transverse load. Left: Optimal thickness design (ill-posed). Right: Optimal distribution of material with two fields of stiffeners. The design data is *hmin/hmax* = 5.0 and *hmax/hunif* = 2.84. In both illustrations only the variation over the minimum gauge *hmin* is shown.

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Parametrization by lamination parameters

Before defining the lamination parameters we first need to express the constitutive relations for a single ply of material in convenient form. Thus the elasticity tensor will for convenience be written as a matrix

$$\mathbf{E}_X = \begin{bmatrix} E_{1111} & E_{1122} & \sqrt{2}E_{1112} \\ E_{1122} & E_{2222} & \sqrt{2}E_{2212} \\ \sqrt{2}E_{1112} & \sqrt{2}E_{2212} & 2E_{1212} \end{bmatrix}_X$$

The index indicates that the constitutive parameters are given in the coordinate system X. In another coordinate system x rotated the angle φ positive anti-clockwise relative to the X-system, E_x is most easily expressed using the material parameters $E_{1\text{-}7}$. To ease the formulations later on, the constitutive matrix E_x is written in terms of five symmetric matrices containing the material parameters as:

$$\mathbf{E}_x = \boldsymbol{\Upsilon}_0 + \boldsymbol{\Upsilon}_1 \cos 2\psi + \boldsymbol{\Upsilon}_2 \cos 4\psi + \boldsymbol{\Upsilon}_3 \sin 2\psi + \boldsymbol{\Upsilon}_4 \sin 4\psi \tag{3.40}$$

$$\boldsymbol{\Upsilon}_0 = \begin{bmatrix} E_1 & E_4 & 0 \\ E_4 & E_1 & 0 \\ 0 & 0 & 2E_5 \end{bmatrix}, \quad \boldsymbol{\Upsilon}_1 = \begin{bmatrix} E_2 & 0 & \sqrt{2}E_6 \\ 0 & -E_2 & \sqrt{2}E_6 \\ \sqrt{2}E_6 & \sqrt{2}E_6 & 0 \end{bmatrix},$$

$$\boldsymbol{\Upsilon}_2 = \begin{bmatrix} E_3 & -E_3 & \sqrt{2}E_7 \\ -E_3 & E_3 & -\sqrt{2}E_7 \\ \sqrt{2}E_7 & -\sqrt{2}E_7 & -2E_3 \end{bmatrix}, \quad \boldsymbol{\Upsilon}_3 = \begin{bmatrix} 2E_6 & 0 & -\frac{1}{\sqrt{2}}E_2 \\ 0 & -2E_6 & -\frac{1}{\sqrt{2}}E_2 \\ -\frac{1}{\sqrt{2}}E_2 & -\frac{1}{\sqrt{2}}E_2 & 0 \end{bmatrix},$$

$$\boldsymbol{\Upsilon}_4 = \begin{bmatrix} E_7 & -E_7 & -\sqrt{2}E_3 \\ -E_7 & E_7 & \sqrt{2}E_3 \\ -\sqrt{2}E_3 & \sqrt{2}E_3 & -2E_7 \end{bmatrix},$$

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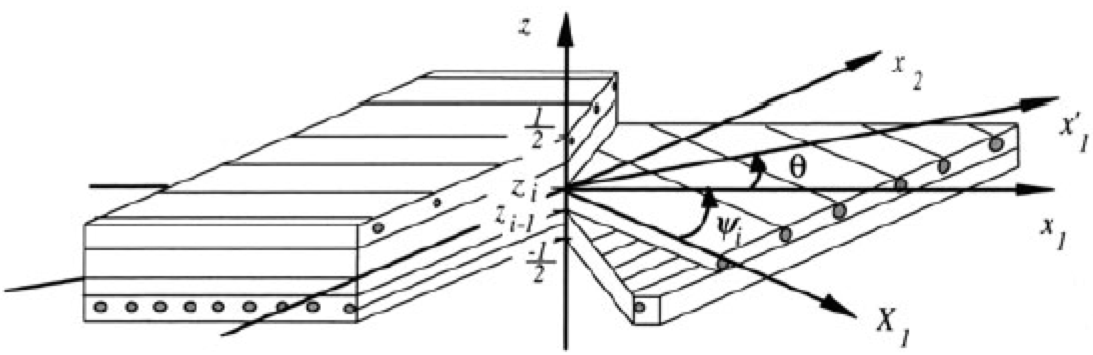
where the material parameters $E_{1\text{-}7}$ are expressed as

$$E_1 = \tfrac{1}{2}(E_{1111} + E_{2222})_X - E_3 \ , \ E_2 = \tfrac{1}{2}(E_{1111} - E_{2222})_X \ ,$$
$$E_3 = \tfrac{1}{8}(E_{1111} + E_{2222} - 2E_{1122} - 4E_{1212})_X \ ,$$
$$E_4 = (E_{1122})_X + E_3 \ , \ E_5 = (E_{1212})_X + E_3 = \tfrac{1}{2}(E_1 - E_4) \ ,$$
$$E_6 = \tfrac{1}{2}(E_{1112} + E_{2212})_X \ , \ E_7 = \tfrac{1}{2}(E_{1112} - E_{2212})_X \ .$$

If the material is orthotropic in the X-system $E_6 = E_7 = 0$ and in the case of an isotropic material $E_2 = E_3 = 0$ as well.

We consider a laminate of the *fixed* thickness *h* made from several plies. Here the orientation of the i'th ply with respect to a suitable, fixed frame of reference is specified by φ_i and z_i gives the location (dimensionless) of the interface between ply *i* and *i*+1, see below. All the plies consist of the same anisotropic material.



Sketch of a laminate with the global coordinate systems x and x', a material system *X* and orientations of the plies shown.

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In the classical plate theory the global relation between the membrane forces and moments per unit length $\{N\}$, $\{M\}$ and the mid-plane strains $\{\varepsilon^0\}$ and curvatures $\{\kappa\}$ is

$$\begin{bmatrix} \{N\} \\ \{M\} \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B} & \mathbf{D} \end{bmatrix} \begin{bmatrix} \{\varepsilon^0\} \\ \{\kappa\} \end{bmatrix}$$

where a $\sqrt{2}$-notation is used ($\{N\} = \{N_{11},\ N_{22},\ \sqrt{2}N_{12}\}^T$

The stiffness matrices for the whole laminate can be expressed in a very similar way as the constitutive matrix in equation (3.40). The symmetric membrane, coupling and bending stiffness matrices **A**, **B** and **D**, respectively are in terms of the material parameters E_{1-7} and the lamination parameters $\xi_{1-4}^{A,B,D}$ given as

$$\begin{aligned}
\mathbf{A} &= h\left(\boldsymbol{\Upsilon}_0 + \boldsymbol{\Upsilon}_1\xi_1^A + \boldsymbol{\Upsilon}_2\xi_2^A + \boldsymbol{\Upsilon}_3\xi_3^A + \boldsymbol{\Upsilon}_4\xi_4^A\right) \ , \\
\mathbf{B} &= h^2\left(\boldsymbol{\Upsilon}_1\xi_1^B + \boldsymbol{\Upsilon}_2\xi_2^B + \boldsymbol{\Upsilon}_3\xi_3^B + \boldsymbol{\Upsilon}_4\xi_4^B\right) \ , \\
\mathbf{D} &= h^3\left(\tfrac{1}{12}\boldsymbol{\Upsilon}_0 + \boldsymbol{\Upsilon}_1\xi_1^D + \boldsymbol{\Upsilon}_2\xi_2^D + \boldsymbol{\Upsilon}_3\xi_3^D + \boldsymbol{\Upsilon}_4\xi_4^D\right) \ .
\end{aligned}$$

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The *lamination parameters* in a global coordinate-system *x* are defined as the weighted trigonometric integrals over the thickness (compare with the definition of the moments used for parametrization of the stiffness rank-N layered materials):

$$\xi_{[1,2,3,4]}^{A,B,D} = \int_{-\frac{1}{2}}^{\frac{1}{2}} z^{0,1,2} \left[\cos 2\psi(z), \cos 4\psi(z), \sin 2\psi(z), \sin 4\psi(z)\right] \mathrm{d}z \ .$$

This compact notation implies for instance that ξ_3^B is given as

$$\xi_3^B = \int_{-\frac{1}{2}}^{\frac{1}{2}} z \sin 2\psi(z) \, \mathrm{d}z \ .$$

Generalized lamination parameters

In the following we will consider lamination parameters arising from any arbitrary variations of the ply angles through the thickness of the plate, including limits of rapidly varying oscillations. We thus extend the definition of the lamination parameters to

$$\xi_{[1,2,3,4]}^{A,B,D} = \int_{-\frac{1}{2}}^{\frac{1}{2}} z^{0,1,2} P_{[1,2,3,4]}(z) \, \mathrm{d}z$$

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where P is the vector

$$P_{[1,2,3,4]}(z) = \int_0^\pi [\cos 2\psi, \cos 4\psi, \sin 2\psi, \sin 4\psi] \, d\theta_z(\psi)$$

corresponding to a microscopic lay-up defined by a probability measure $\theta_z(\psi)$ with support in $[0, \pi]$.

The set of lamination parameters D constitutes a convex and compact set in R^{12} and this property is expressed in the representation given by equations; convexity, for example, follows from the possible use of a chattering design with a density of two laminate angles (see also below).

The advantage of expressing laminate plate design in terms of the lamination parameters is that one obtains a reduction in the number of variables to twelve (per point or per design area), irrespective of the number of plies. Moreover, one avoids a troublesome optimization over periodic functions of the rotation angles, as well as working with a discrete number of plies. In the sense of topology design of choosing between plies, the lamination parameters thus constitute the basis for an interpolation model. Also, the convexity of the set of lamination parameters together with the linear dependence of the stiffnesses on these parameters also leads to further simplifications as seen also for the free material optimization problem.

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Minimum compliance of laminated plates

The problem of minimizing the compliance of a laminated plate can now be analyzed along the lines used previously for free material design and for layered media. The design variables are the *lamination parameters* varying from point to point throughout the plate and we will base the developments on the multiple load case. As the stiffness matrices A, Band D are all linear in the lamination parameters, the strain energy is linear (and thus concave) in the lamination parameters. As seen for the free material case the problem thus satisfies the conditions for existence of a saddle point and we can perform our analysis by solving the local anisotropy problem together with the equilibrium problem. Here there is no material distribution problem, unless one chooses also to consider a variable thickness *h* as a design variable.

The local anisotropy problem for laminates

The local anisotropy problem of finding the pointwise best use of material is for laminates of the form

$$\max_{\xi \in \mathbf{D}} \sum_{k=1}^{M} w^k W_k \ , \qquad \text{with}$$

$$W_k = \frac{1}{2h}\left(\left\{\varepsilon^0\right\}_k^T \mathbf{A}\left\{\varepsilon^0\right\}_k + 2\left\{\varepsilon^0\right\}_k^T \mathbf{B}\left\{\kappa\right\}_k + \left\{\kappa\right\}_k^T \mathbf{B}\left\{\kappa\right\}_k\right)$$

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Here the displacement field at equilibrium u^k for the load case *k* enters via the strain energy density W_k (w^k is the weight factor for this load case). We note here that the objective function of problem is linear and that the constraint set is convex and compact. There thus exists a solution among the extreme points of the convex set *D*.

The energy density can also be written directly in terms of the total strains $\{\varepsilon(z)\}_k = \{\varepsilon^0\}_k + zh\{\kappa\}_k$ as (setting $P_0 = 1$ and using the matrix definitions in equation (3.40))

$$W_k = \frac{1}{2}\int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\sum_{i=0}^{4} \{\varepsilon(z)\}_k^T \boldsymbol{\Upsilon}_i \{\varepsilon(z)\}_k P_i(z) \right) \mathrm{d}z$$

As we also will allow for any variation of the ply lay-up through the thickness of the plate, we see that in order to solve (3.45) we have for each position z through the thickness to maximize the expression

$$\sum_{l=1}^{L} w^k \left(\sum_{i=0}^{4} \{\varepsilon(z)\}_k^T \boldsymbol{\Upsilon}_i \{\varepsilon(z)\}_k P_i(z) \right)$$

over the parameters P_i. Thus the optimal lay-up of the laminate (for maximum stiffness) for each position (x_1, x_2, z) in the plate domain can be found by solving the problem

$$\max_{\mathbf{y}\in\mathcal{M}} \sum_{l=1}^{L} w^k \left(\sum_{i=1}^{4} \{\varepsilon(z)\}_k^T \boldsymbol{\Upsilon}_i \{\varepsilon(z)\}_k y_i \right) \qquad (3.48)$$

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The pure membrane case

Let us now consider the situation of designing the lay-up for a situation of only in-plane loading, i.e. the pure membrane case. In that setting the strain energy density of the plate reduces to

$$W_k = \frac{1}{2}\int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\sum_{i=0}^{4} \{\varepsilon^0\}_k^T \boldsymbol{\Upsilon}_i \{\varepsilon^0\}_k P_i(z) \right) \mathrm{d}z$$

Thus the optimization over the variables P_i gives the same result at any cross-sectional position *z* of the plate. Together with the fact that the stacking sequence is of no consequence for the membrane stiffness and the fact that any element can be constructed as a convex combination of at most three points on the curve, $(\cos 2\psi, \cos 4\psi, \sin 2\psi, \sin 4\psi),\ \psi \in [0, \pi]$ this implies that the optimal plate can be constructed from at most *three* plies. This holds for the single as well as the multiple load case. In the single load case this can be reduced to at most two plies, as will be shown below.

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Let us now for exemplification of the use of lamination parameters solve (3.48) for the case of a single load case and an orthotropic material. For simplicity, we use the directions of principal strains, $\varepsilon_I, \varepsilon_{II}$, as a local frame of reference, while for the ply material we assume that the directions of orthotropy are ordered so $E_2 \geq 0$, i.e. so $E_{1111} \geq E_{2222}$. Problem (3.48) then reduces to

$$\max_{\mathbf{y} \in \mathcal{M}} \left(E_2 \left(\varepsilon_I^2 - \varepsilon_{II}^2 \right) y_1 + E_3 \left(\varepsilon_I - \varepsilon_{II} \right)^2 y_2 \right)$$

Furthermore, as the third and fourth lamination parameters do not enter we can reduce the constraint set to the range of the trigonometric averages P_1, P_2, that is, to the set

$$\widehat{\mathcal{M}} = \left\{ \mathbf{y} \in \mathbf{R}^2 \,\middle|\, -1 \leq y_1 \leq 1, -1 \leq y_2 \leq 1, 2y_1^2 (1 - y_2) + y_2^2 \leq 1, \right\}$$

So that it reduces to

$$\max_{(y_1, y_2) \in \widehat{\mathcal{M}}} \left(E_2 \left(\varepsilon_I^2 - \varepsilon_{II}^2 \right) y_1 + E_3 \left(\varepsilon_I - \varepsilon_{II} \right)^2 y_2 \right)$$

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Assume first that $E_3 \geq 0$; this is a material which has low shear stiffness,

As $E_3(\varepsilon_I - \varepsilon_{II})^2 \geq 0, E_2 \geq 0$, the optimal

energies will depend on the sign of $(\varepsilon_I^2 - \varepsilon_{II}^2)$ and will be given by the energies obtained from the lamination parameters $(y_1, y_2]) = (1, 1)$ (if $(\varepsilon_I^2 - \varepsilon_{II}^2) \geq 0$) and $(y_1, y_2) = (-1, 1)$ (if $(\varepsilon_I^2 - \varepsilon_{II}^2) \leq 0$). If $(\varepsilon_I^2 - \varepsilon_{II}^2) \neq 0$, the design is unique and corresponds to a single ply rotated so the numerically largest principal strain is aligned with the material axis corresponding to E_{1111} (we have assumed $E_{1111} \geq E_{2222}$). The optimal energy

$$\Phi(\{\varepsilon\}) = \max \left\{ \begin{array}{l} E_2\left(\varepsilon_I^2 - \varepsilon_{II}^2\right) + E_3\left(\varepsilon_I - \varepsilon_{II}\right)^2, \\ -E_2\left(\varepsilon_I^2 - \varepsilon_{II}^2\right) + E_3\left(\varepsilon_I - \varepsilon_{II}\right)^2 \end{array} \right\}$$

which is non-smooth at strains which satisfy $\varepsilon_1^2 = \varepsilon_{II}^2$, i.e. uniform dilation or pure shear (in terms of strains). The resulting reduced minimum potential energy problem (the reduced equilibrium problem, cf., (3.15)) is then a non-smooth, convex problem, for which the necessary conditions of optimality at points with $\varepsilon_1^2 = \varepsilon_{II}^2$ will involve a convex combination of the gradients of the two smooth branches of Φ. This implies that at points where the strains of the optimal plate satisfy $\varepsilon_1^2 = \varepsilon_{II}^2$, the optimal design can consist of some cross-ply consisting of two plies rotated at 0 and 90 degrees relative to the principal strain axes, with thicknesses decided through the conditions of equilibrium.[7] The relative thicknesses of the two plies can actually be determined by considering the complementary energy formulation of the compliance problem. In terms of principal *membrane forces* N_I, N_{II} (with $|N_I| \geq |N_{II}|$) one gets (see Hammer (1999)) that the optimal $[0^\circ_t/90^\circ_{1-t}]$ laminate has a relative thickness t ($t \in [\frac{1}{2}, 1]$) of the zero degree ply given as:

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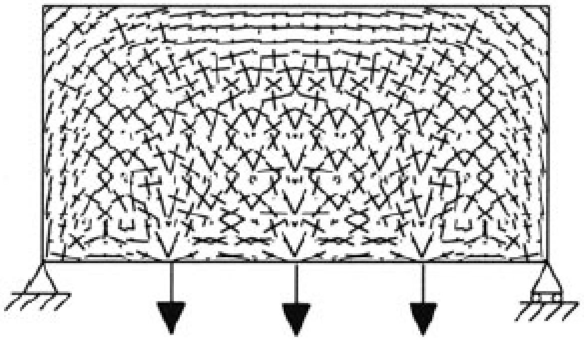
$$2t-1=\begin{cases}\frac{N_I+N_{II}}{N_I-N_{II}}\frac{E_{1111}+E_{2222}-2E_{1122}}{E_{1111}-E_{2222}} & \text{for } \frac{N_{II}}{N_I}\le -\frac{E_{2222}-E_{1122}}{E_{1111}-E_{1122}}\ ,\\ \frac{N_I-N_{II}}{N_I+N_{II}}\frac{E_{1111}+E_{2222}+2E_{1122}}{E_{1111}-E_{2222}} & \text{for } \frac{N_{II}}{N_I}\ge \frac{E_{2222}+E_{1122}}{E_{1111}+E_{1122}}\ ,\\ 1 & \text{otherwise}\ .\end{cases}$$

Then consider the case $E_3 \le 0$ (a material with high shear stiffness). Here the algebra becomes somewhat messier. In this case we get unique solutions to (3.51) if $\varepsilon_1 \neq \varepsilon_{II}$, with the solution corresponding to a single ply rotated at an angle ψ given by $\cos 2\psi = -\frac{E_2}{4E_3}\frac{\varepsilon_1+\varepsilon_{II}}{\varepsilon_1-\varepsilon_{II}}$ if $\left|\frac{E_2}{4E_3}\frac{\varepsilon_1+\varepsilon_{II}}{\varepsilon_1-\varepsilon_{II}}\right| \le 1$ and given as $\psi = 0$ if $\left|\frac{E_2}{4E_3}\frac{\varepsilon_1+\varepsilon_{II}}{\varepsilon_1-\varepsilon_{II}}\right| \ge 1$. In the case $\varepsilon_I = \varepsilon_{II}$ we have a non-unique solution and the optimal energy becomes non-smooth, resulting in an optimal design which also in this case must consist of some cross-ply at 0 and 90 degrees relative to the principal axes



Optimal three ply laminate for a plate with three *independent* single loads applied.

The analysis above is consistent with the result on the optimal rotation of an orthotropic material derived by different means. It was remarked there that this problem does not have existence of solutions in general and that some type of additional microstructure is necessary. Here we use lamination parameters, and find that just *two* plies are required as part of the optimal solution that we know exists. This simpler situation is possible as we work with effective material parameters given directly as a summation of stiffness via the out-of-plane stacking of the different plies.

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6. Optimal topology design with a damage related criterion

Here we discuss an attempt at introducing damage related criteria in topology design of continuum structures. We use an interpretation of continuum damage models, where a variational statement is adopted to replace the standard internal variable representation. The model is represented in the form of an optimal remodeling problem, where a damaged material of reduced stiffness is distributed in a healthy structure so as to maximize the compliance, i.e., to minimize overall stiffness, for a given set of damage loads. Thus the treatment of the damage model is in itself a study of optimal structural design. Evolution would be described as a timeseries of such static remodeling models, but we accept here the limitation of only considering the onset of damage.

6.1 A damage model of maximizing compliance

The damage model takes the form of a design problem involving the layout of a structure made from two materials, a stiff material and a flexible, damaged material. The structure we consider is made of a linearly elastic material with elasticity tensor E^+. Under the action of the damage loads the material is damaged in some parts of the structure, leaving there a more compliant material with elasticity tensor E^-.

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By more flexible, it is meant here simply that we in terms of tensors have that $E^+ - E^- > 0$, i.e., for any strains, the specific strain energy of the flexible material is strictly less than that of the stiffer material. When damage occurs, energy is released, and we denote *by K the energy release per unit volume*. Interpreting the distribution of damage as a material distribution design problem, we impose that for a certain load, the damage is distributed so that the compliance of the structure is *maximized*, making the structure as *flexible* as possible among all distributions of damage.

The damage problem is thus formulated as a "design" problem as

$$\min_{E \in \mathsf{E}_{ad}} \min_{u \in U} \left\{ \frac{1}{2} \int_\Omega E_{ijkl}(x) \varepsilon_{ij}(u) \varepsilon_{kl}(u) \mathrm{d}\Omega - l(u) + K \int_\Omega 1_{\Omega^-} \mathrm{d}\Omega \right\} \tag{3.54}$$

where Ω^- is the damaged zone, and where the set of admissible tensors E_{ad} is given by the relations:

$$E_{ijkl} = 1_{\Omega^-} E^-_{ijkl} + (1 - 1_{\Omega^-}) E^+_{ijkl} = \begin{cases} E^+_{ijkl} & \text{if } x \in \Omega \setminus \Omega^- , \\ E^-_{ijkl} & \text{if } x \in \Omega^- . \end{cases}$$

We note that (3.54) could alternatively be written as a fixed volume *maximum* compliance problem, where K is the Lagrange multiplier for the volume constraint. Thus our problem is just a variant of the 0-1 topology design problem, now with an objective of minimizing stiffness.

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Here, we will not go into details on the relation between current model and other types of models used in continuum damage mechanics. However, the close relationship is evident if one solves in current model for the minimization over stiffness tensors. One then obtains a minimum potential energy principle for the damaged structure in the form

$$\min_{u \in U} \left\{ \frac{1}{2} \int_\Omega \Phi(u) \mathrm{d}\Omega - l(u) \right\} \qquad (3.56)$$

where the specific strain energy $\frac{1}{2}\Phi(u)$ in the damaged structure is given by

$$\Phi = \begin{cases} E^+_{ijkl}\varepsilon_{ij}(u)\varepsilon_{kl}(u) & \text{if } \Delta_{ijkl}\varepsilon_{ij}(u)\varepsilon_{kl}(u) \le 2K \ , \\ E^-_{ijkl}\varepsilon_{ij}(u)\varepsilon_{kl}(u) + 2K & \text{if } \Delta_{ijkl}\varepsilon_{ij}(u)\varepsilon_{kl}(u) > 2K \ , \end{cases}$$

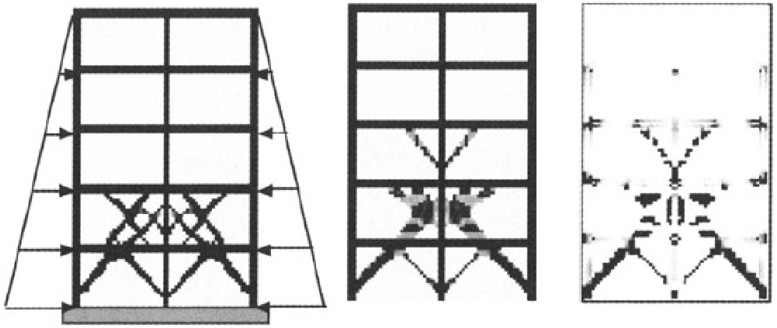
where $\Delta = E^+ - E^-$. In an internal variable representation of damage, in the spirit of the equivalent stress principle of Kachanov (Krajcinovic 1996, Lemaitre 1996), one represents the reduced stiffness of a partially damaged material as

$$E_{ijkl}(t) = (1-t)E^+_{ijkl} + tE^-_{ijkl} = E^+_{ijkl} - t\Delta_{ijkl}, \quad 0 \le t \le 1 \ .$$

Here t is interpreted as a volume fraction of the damaged state. The reduced problem (3.56) then also appears from a damage model in the form

$$\min_{\substack{t(x) \\ 0 \le t(x) \le 1}} \ \min_{u \in U} \left\{ \frac{1}{2} \int_\Omega E_{ijkl}(t)\varepsilon_{ij}(u)\varepsilon_{kl}(u)\mathrm{d}\Omega - l(u) + K \int_\Omega t(x)\mathrm{d}\Omega \right\} ,$$

when one solves for the internal parameter t.



Optimal reinforcement of a frame. Left: The frame and loads, etc., with the standard minimum compliance solution. Mid: the solution of the design problem with damage taken into consideration, Right: the corresponding distribution of damage

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Relaxed problem statement

Just as for the minimum compliance topology design problem, the damage models above are not well-posed and we have to relax the problem by introducing composite mixtures of the two phases in a development that is completely parallel to that for the minimum compliance problem. Thus rank-2 materials also provide for a realization of the most flexible composites, and the corresponding "optimal" energy can be derived.

We will write these expressions for a case where the healthy material and the damaged material have the same Poisson's ratio ν, and only the Young moduli E^+, E^- are distinct. For given principal strains $\varepsilon_I, \varepsilon_{II}$ (ordered so that $|\varepsilon_I| \geq |\varepsilon_{II}|$) and a local volume fraction of damage ϑ, the relaxed specific strain energy $\frac{1}{2}\hat{\Phi}(u,\vartheta)$, including here the released energy due to damage, is given through the formulas

$$\hat{\Phi}(u,\vartheta) = \begin{cases} \hat{\Phi}_1(u,\vartheta) + 2K\vartheta & \text{if } \varepsilon_I\varepsilon_{II} \leq 0 \,, \\ \hat{\Phi}_2(u,\vartheta) + 2K\vartheta & \text{if } \varepsilon_I\varepsilon_{II} \geq 0 \text{ and } \vartheta \leq \bar{\vartheta} \,, \\ \hat{\Phi}_3(u,\vartheta) + 2K\vartheta & \text{if } \varepsilon_I\varepsilon_{II} \geq 0 \text{ and } \vartheta \geq \bar{\vartheta} \,, \end{cases}$$

$$\hat{\Phi}_1 = \frac{1}{1-\nu^2}\frac{E^+E^-}{(1-\vartheta)E^- + \vartheta E^+}(\varepsilon_I^2 + \varepsilon_{II}^2 + 2\nu\varepsilon_I\varepsilon_{II}) \,,$$

$$\hat{\Phi}_2 = \frac{(1-\vartheta)E^+ + \vartheta E^-}{1-\nu^2}(\varepsilon_I^2 + \varepsilon_{II}^2 + 2\nu\varepsilon_I\varepsilon_{II}) - \frac{\vartheta(1-\vartheta)}{(1-\vartheta)E^- + \vartheta E^+}\frac{(E^+ - \vartheta E^-)^2}{2(1-\nu^2)} \times$$
$$\times \left((1+\nu^2)(\varepsilon_I^2 + \varepsilon_{II}^2) + 4\nu\varepsilon_I\varepsilon_{II} + (1-\nu^2)|\varepsilon_I - \varepsilon_{II}|\right) \,,$$

$$\hat{\Phi}_3(u,\vartheta) = \frac{E^-}{2(1+\nu)}(\varepsilon_I - \varepsilon_{II})^2 + \frac{E^-}{2(1-\nu)}\frac{2E^+ - \vartheta(1-\nu)(E^+ - E^-)}{2E^- + \vartheta(1+\nu)(E^+ - E^-)}(\varepsilon_I + \varepsilon_{II})^2 \,,$$

where

$$\bar{\vartheta} = \frac{2E^-}{(1-\nu)(E^+ - E^-)}\left(\frac{|\varepsilon_I + \varepsilon_{II}|}{|\varepsilon_I - \varepsilon_{II}|} - 1\right)^{-1} .$$

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The first and second regimes of the strain energy correspond to the use of an optimal rank 1 material consisting of one system of layers of the two materials, at one microscale. The third regime corresponds to an orthogonal rank-2 layering, consisting of two systems of layerings, at two scales. The inner layering consists of the stiff and damaged material, the outer layering of this material layered with the damaged material. Thus in the rank-2 structure the stiff material is surrounded by flexible material, shielding this material in order to weaken the material as much as possible (for the fixed volume fraction). The functional above constitutes the relaxed form for the problem, Here we can go one step further and compute the final relaxed specific strain energy for the damaged structure, i.e., the relaxed form of the energy.

minimizing $\hat{\Phi}$ with respect to the volume fraction ϑ. This can be solved analytically (use of symbolic manipulation software is strongly recommended). Its solution provides the optimum value of ϑ for the given strain field, along with details of the optimum material orientation, and provides an analytical form of the effective, relaxed strain energy density $\tilde{\Phi}(\varepsilon)$ of the damaged structure, with no reference to ϑ. Thus (3.54) becomes the non-linear and non-smooth problem

$$\min_{u \in U} \left\{ \frac{1}{2} \int_{\Omega} \tilde{\Phi}(\epsilon(u)) \mathrm{d}\Omega - l(u) \right\} . \tag{3.58}$$

Its solution provides the displacements and indirectly the *spatial* distribution of ϑ over the domain Ω.

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6.2 Design problems

In the topology design problem that we consider, the design is parametrized by the SIMP model. This is assumed for both the healthy as well as the damaged phase. The minimum potential energy principles governing a fully healthy structure and a damaged structure are thus written as

$$\min_{u_S \in U} \left\{ \frac{1}{2} \int_\Omega \rho^p E^+_{ijkl}(x) \varepsilon_{ij}(u_S) \varepsilon_{kl}(u_S) \mathrm{d}\Omega - l_S(u_S) \right\} , \qquad (3.59)$$

$$\min_{u_D \in U} \left\{ \frac{1}{2} \int_\Omega \rho^p \tilde{\Phi}(\varepsilon(u_D)) \mathrm{d}\Omega - l_D(u_D) \right\} .$$

Topology design of reinforcement to reduce damage effects In the first problem we consider the design of the reinforcement of a structure so as to minimize the effect of damage on the reinforced structure. The existing structure is part of the ground structure and occupies the domain Ω_G, where we have $\rho(x) = 1,\ x \in \Omega_G$, and the reinforcement can be placed in any part of the remainder $\Omega \setminus \Omega_G$ of the reference domain. The effect of the damage is measured by the compliance of the damaged structure under some damage loads f_D. The design problem thus takes the form

$$\max_{\substack{\rho \\ \rho(x)=1, x \in \Omega_G \\ \int_{\Omega \setminus \Omega_G} \rho \mathrm{d}\Omega \le Vol \\ 0 < \rho_{\min} \le \rho \le 1}} \quad \min_{u_D \in U} \left\{ \frac{1}{2} \int_\Omega \rho^p \tilde{\Phi}(\varepsilon(u_D)) \mathrm{d}\Omega - \int_\Omega f_D u_D \right\}$$

where we maximize the potential energy of the damaged structure, for improved overall stiffness.

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Design of structural topology with a constraint on damage effect
In the second problem we seek designs that make use of a prescribed amount of material and are of minimum compliance under a set of service loads p_S. In addition, we require that the structure retain a certain amount of stiffness under the action of a separate set of damage loads f_D. The effect of damage is measured by the compliance of the damaged structure under the damage loads p_D. The optimization problem thus reads

$$\min_{\substack{\rho \\ \int_\Omega \rho \mathrm{d}\Omega \leq V \\ 0<\rho_{\min}\leq\rho\leq 1}} \quad \int_\Omega f_S u_S \mathrm{d}\Omega$$

$$\text{s.t.} : -2 \min_{u_D \in U} \left\{ \frac{1}{2} \int_\Omega \rho^p \tilde{\Phi}(\varepsilon(u_D)) \mathrm{d}\Omega - \int_\Omega f_D u_D \right\} \leq c_{\max}$$

where u_S is a solution to (3.59) with load f_S. Here the objective function states that the structure (for a given amount V of material) should be as stiff as possible under the service loads, while the constraint is a restriction on the amount of damage allowed under the damage loads, measured in terms of compliance (under the damage load).

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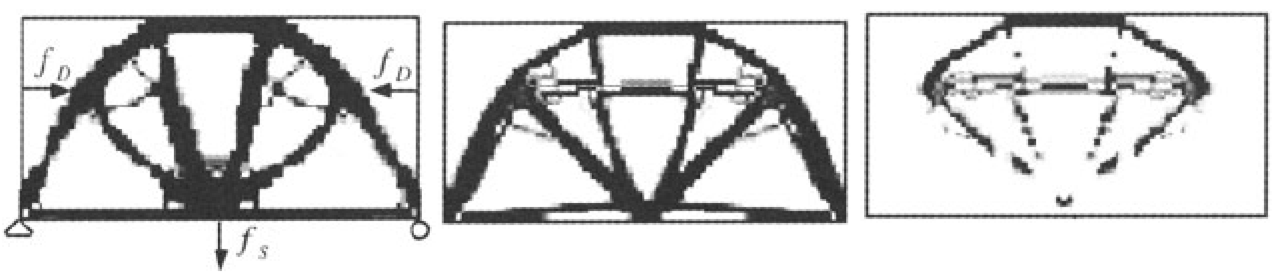
Implementation

The topology design problems can be solved along the lines described previously (all objectives and constraints are compliance values, simplifying derivative calculations). The extent of damage υ(x) associated with the damage loads is determined by solving a non-linear finite element problem so the process is iterative in the displacements. At each iteration step, the given displacements and associated strains are used to find, within each finite element, υ and the details of the local orthotropy and orientation of the rank-2 material that is consistent with the extremal energy expression. The update of the displacements then consists of a linear finite element analysis with this material data. A similar scheme was described for the stress-based minimum compliance problem.



Minimum compliance design. Left: In the absence of damage loads, Mid: With a constraint on the effect of damage loads, and Right: the distribution of damage in this optimal design

Thank you for your attention