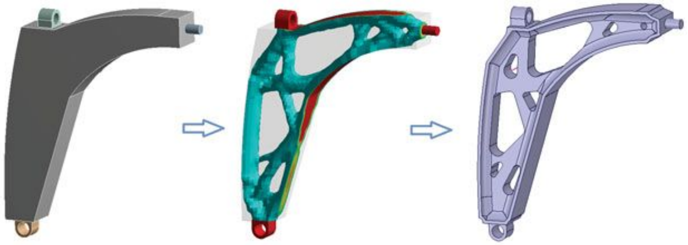
香港中文大學
The Chinese University of Hong Kong

MAEG5160: Design for Additive Manufacturing

Lecture 17: Topology design of truss structures



Prof SONG Xu
Department of Mechanical and Automation Engineering,
The Chinese University of Hong Kong.

Lecture 17: Topology design of truss structures

Topology optimization of trusses in the form of grid-like continua is a classical subject in structural design. The optimization of the geometry and topology of trusses can conveniently be formulated with the so-called *ground structure method*. In this approach the layout of a truss structure is found by allowing *a certain set of connections between a fixed set of nodal points* as potential structural or vanishing members. For the truss topology problem the geometry allows for using the continuously varying cross-sectional bar areas as design variables, including the possibility of zero bar areas. This implies that the truss topology problem can be viewed as a standard sizing problem. This sizing reformulation is possible for the simple reason that the truss as a continuum geometrically is described as one dimensional. Thus for both planar and space trusses there are extra dimensions in physical space that can describe the extension of the truss as a true physical element of space, simplifying the basic modelling for truss topology design as compared to topology design of three dimensional continuum structures.

Lecture 17: Topology design of truss structures

Truss topology design problems were in early work formulated in terms of member forces, ignoring kinematic compatibility to obtain a linear programming problem in member areas and forces. The resulting topology and force field are then often employed as a starting point for a more complicated design problem formulation. Alternatively, when displacement formulations are used, then (small) non zero lower bounds on the cross-sectional areas have been imposed in order to have a positive definite stiffness matrix. This means that standard techniques for optimal structural design can be used. Also, it allows for the use of optimality criteria methods for large scale design problems involving compliance, stress, displacement and eigenvalue objectives. In the simultaneous analysis and design approach, the design variables and state variables are not distinguished, so the full problem is solved by one unified numerical optimization procedure. However, unless specially developed numerical solution procedures are used, only very small problems can be treated. The use of simulated annealing and genetic algorithm techniques for the topology problems in their original formulation as discrete selection problems, has also been pursued but also these fairly general approaches are with the present technology restricted to fairly small scale problems.

Lecture 17: Topology design of truss structures

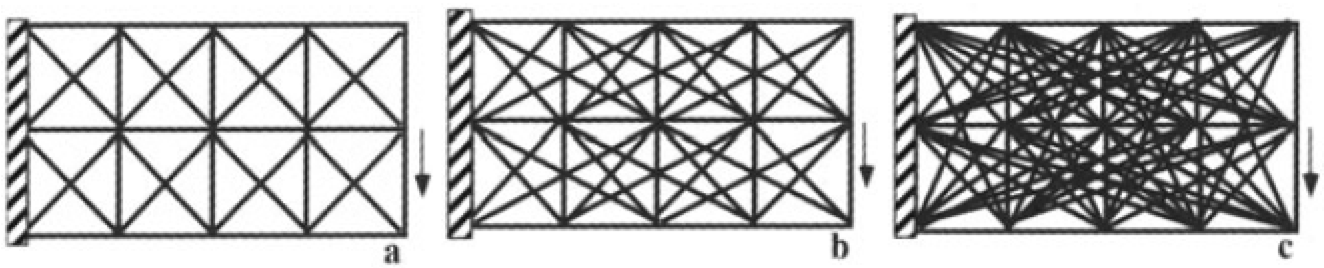
In this lecture we will investigate various formulations of truss topology design and outline some options for their numerical processing. We seek specifically to be able to handle problems with a very large number of potential structural elements, using the ground structure approach. For this reason we *consider primarily the simplest possible optimal design problem, namely the minimization of compliance (maximization of stiffness) for a given total mass of the structure where a very detailed examination of the properties of the problems is possible*. The analysis is general enough to encompass multiple load problems in the worst-case and weighted-average formulation, the case of self-weight loads and the problem of determining the optimal topology of the reinforcement of a structure as for example seen in fail-safe design. Also, variable thickness sheet, sandwich plate and free material problems are covered by the developments. In direct analogy with the continuum setting, these problems can be given in a number of equivalent problem statements, among them problems in the nodal displacements only or in the member forces only. With these reformulations at hand it is possible to devise very efficient algorithms that can handle large scale problems. Also, as we have seen in earlier chapters, the formulations can be obtained through duality principles and the resulting formulations in displacements or stresses correspond to equilibrium problems for an optimally global strain energy and an optimally global complementary energy, respectively.

Lecture 17: Topology design of truss structures

1. Problem formulation for minimum compliance truss design

1.1 The basic problem statements in displacements

In the ground structure approach for truss topology design a set of n chosen nodal points (N degrees of freedom) and m possible connections are given, and one seeks to find the optimal substructure of this structural universe. In some papers on the ground structure approach, the ground structure is always assumed to be the set of all possible connections between the chosen nodal points, but here we allow the ground structure to be any given set of connections. This approach may lead to designs that are not the best ones for the chosen set of nodal points, but the approach implicitly allows for restrictions on the possible spectrum of possible member lengths as well as for the study of the optimal subset of members of a given truss-layout.



Ground structures for transmitting a vertical force to a vertical line of supports. Truss ground structures of variable complexity in a rectangular domain with a regular 5 by 3 nodal layout. In c) all the connections between the nodal points are included.

Lecture 17: Topology design of truss structures

Let a_i, l_i denote the cross-sectional area and length of bar number i, respectively, and we assume that all bars are made of linear elastic materials, with Young's moduli E_i. The volume of the truss is $V = \sum_{i=1}^{m} a_i l_i$. In order to simplify the notation at a later stage, we introduce the bar volumes $t_i = a_i l_i$, $i = 1, \ldots, m$, as the fundamental design variables. Static equilibrium is expressed as

$$\mathbf{B}\mathbf{q} = \mathbf{f} \ ,$$

where $\mathbf{q}$ is the member force vector and $\mathbf{f}$ is the nodal force vector of the free degrees of freedom. The ground structure is chosen so that the compatibility matrix $\mathbf{B}$ has full rank and so that $m \geq N$, excluding mechanisms and rigid body motions. The stiffness matrix of the truss is written as

$$\mathbf{K}(\mathbf{t}) = \sum_{i=1}^{m} t_i \mathbf{K}_i \ ,$$

where $t_i \mathbf{K}_i$ is the element stiffness matrix for bar number i, written in global coordinates. Note that $\mathbf{K}_i = \frac{E_i}{l_i^2} \mathbf{b}_i \mathbf{b}_i^T$ where $\mathbf{b}_i$ is the i'th column of $\mathbf{B}$.

The problem of finding the minimum compliance truss for a given volume of material (the stiffest truss) has the well-known formulation (cf., the continuum setting Sect. 1.1)

$$\begin{aligned} &\min_{\mathbf{u},\mathbf{t}} \ \mathbf{f}^T \mathbf{u} \\ &\text{s.t.}: \sum_{i=1}^{m} t_i \mathbf{K}_i \mathbf{u} = \mathbf{f}, \quad \sum_{i=1}^{m} t_i = V, \quad t_i \geq 0, \ i = 1, \ldots, m \ . \end{aligned} \tag{4.1}$$

Lecture 17: Topology design of truss structures

Problem 4.1 is well studied in the case of an imposed non-negative lower bound on the volumes t_i. In this case the stiffness matrix K(t) is positive definite for all feasible t and the displacements can be removed from the problem. The resulting problem in bar volumes turns out to be convex and existence of solutions is assured. Allowing for zero lower bounds complicates the analysis, but it also provides valuable insight. The zero lower bound on the variables t_i thus means that bars of the ground structure can be removed and the problem statement thus covers topology design. Problem 4.1 can result in an optimal topology that is a mechanism; this mechanism is in equilibrium under the given load, and infinitesimal bars can be added to obtain a stable structure. Also, if the optimal topology has straight bars with inner nodal points, these nodal points should be ignored. The resulting truss maintains the stiffness and the equilibrium of the original optimal topology.

The zero lower bound in problem 4.1 implies that the stiffness matrix is not necessarily positive definite and the state vector U cannot be removed by solving K(t)u = f. Removing u from the formulation is not very important for the size of the problem, as, typically, the number m of bars is much greater than the number of degrees of freedom. In the complete ground structure we connect all nodes, having m = n(n - 1)/2, while the degrees of freedom are only of the order 2n or 3n (for planar and 3-D trusses). For the complete ground structure we also have a fully populated stiffness matrix lacking any sparsity and bandedness.

Lecture 17: Topology design of truss structures

Our aim here is to develop methods which can be applied to large scale truss topology problems and for this reason we employ the simplest possible design formulation as stated in problem (4.1). More general problem statements are briefly covered in Sect. 4.4. Nonetheless, a number of extended design settings can be covered within the framework of (4.1). In the case of *multiple loads*, we formulate also for trusses the problem of minimizing a weighted average of the compliances. For a set of M different load cases $\mathbf{f}^k$, $k = 1, \ldots, M$, and weights w^k, $k = 1, \ldots, M$, the multiple load problem reads

$$
\begin{aligned}
&\min_{\mathbf{u},\mathbf{t}} \sum_{k=1}^{M} w^k {\mathbf{f}^k}^T \mathbf{u}^k \\
&\text{s.t.}: \sum_{i=1}^{m} t_i \mathbf{K}_i \mathbf{u}^k = \mathbf{f}^k, \quad k = 1, \ldots, M \;, \qquad (4.2) \\
&\qquad \sum_{i=1}^{m} t_i = V, \quad t_i \geq 0,\; i = 1, \ldots, m \;.
\end{aligned}
$$

Note that in problem (4.2) it is possible to refer each load case to a distinct ground sub-structure, and that it thus is possible to cover ***fail-safe design***

Lecture 17: Topology design of truss structures

If we introduce an extended displacement vector

$$\hat{\mathbf{u}} = (\mathbf{u}^1, \ldots, \mathbf{u}^M) \ ,$$

of all the displacement vectors $\mathbf{u}^k$, $k = 1, \ldots, M$, an extended force vector

$$\hat{\mathbf{f}} = (w^1 \mathbf{f}^1, \ldots, w^M \mathbf{f}^M) \ ,$$

of the weighted force vectors $w^k \mathbf{f}^k$, $k = 1, \ldots, M$, and the extended element stiffness matrices as the block diagonal matrices

$$\hat{\mathbf{K}}_i = \begin{pmatrix} w^1 \mathbf{K}_i & & \\ & \ddots & \\ & & w^M \mathbf{K}_i \end{pmatrix} \ .$$

Then problem (4.2) can be written as

$$\begin{aligned} &\min_{\hat{\mathbf{u}}, \mathbf{t}} \hat{\mathbf{f}}^T \hat{\mathbf{u}} \\ &\text{s.t.} : \sum_{i=1}^{m} t_i \hat{\mathbf{K}}_i \hat{\mathbf{u}} = \hat{\mathbf{f}}, \quad \sum_{i=1}^{m} t_i = V, \quad t_i \geq 0, \ i = 1, \ldots, m \ , \end{aligned} \tag{4.3}$$

which is precisely of the same form as problem (4.1).

Lecture 17: Topology design of truss structures

The problem of *worst case* minimum compliance design for multiple loads $\mathbf{f}^k$, $k = 1, \ldots, M$, reads

$$\min_{\mathbf{u}^k, \mathbf{t}} \left\{ \max_{k=1,\ldots,M} \mathbf{f}^{k^T} \mathbf{u}^k = \max_{\substack{\lambda^k \geq 0,\, k=1,\ldots,M \\ \sum_{i=1}^m \lambda^k = 1}} \sum_{k=1}^{M} \lambda^k \mathbf{f}^{k^T} \mathbf{u}^k \right\}$$

$$\text{s.t.} : \sum_{i=1}^{m} t_i \mathbf{K}_i \mathbf{u}^k = \mathbf{f}^k, \quad k = 1, \ldots, M \ , \tag{4.4}$$

where $\mathbf{u}^k$, $k = 1, \ldots, M$, are again the displacements corresponding to the different load cases. Note how the discrete optimization over the compliance values can be converted into a smooth maximization by introducing a convex combination of weighting parameters λ^k, $k = 1, \ldots, M$.

In analogy to the continuum problems treated earlier in Chaps. 1 and 3, it is also in this discretized case convenient to rewrite the problem statements in terms of a minimum potential energy formulation of the equilibrium constraint. Thus problem (4.5) can be rewritten as a max-min problem in the form

$$\max_{\substack{\mathbf{t} \geq \mathbf{0} \\ \sum_{i=1}^m t_i = V}} \min_{\mathbf{u}} \left\{ \frac{1}{2} \mathbf{u}^T \left(\sum_{i=1}^{m} t_i \hat{\mathbf{K}}_i \right) \mathbf{u} - \mathbf{f}^T \mathbf{u} \right\} . \tag{4.6}$$

This is a saddle point problem for a concave-convex problem, and we shall also for the truss problem in the following use that the max and min operator in (4.6) can be interchanged.

Lecture 17: Topology design of truss structures

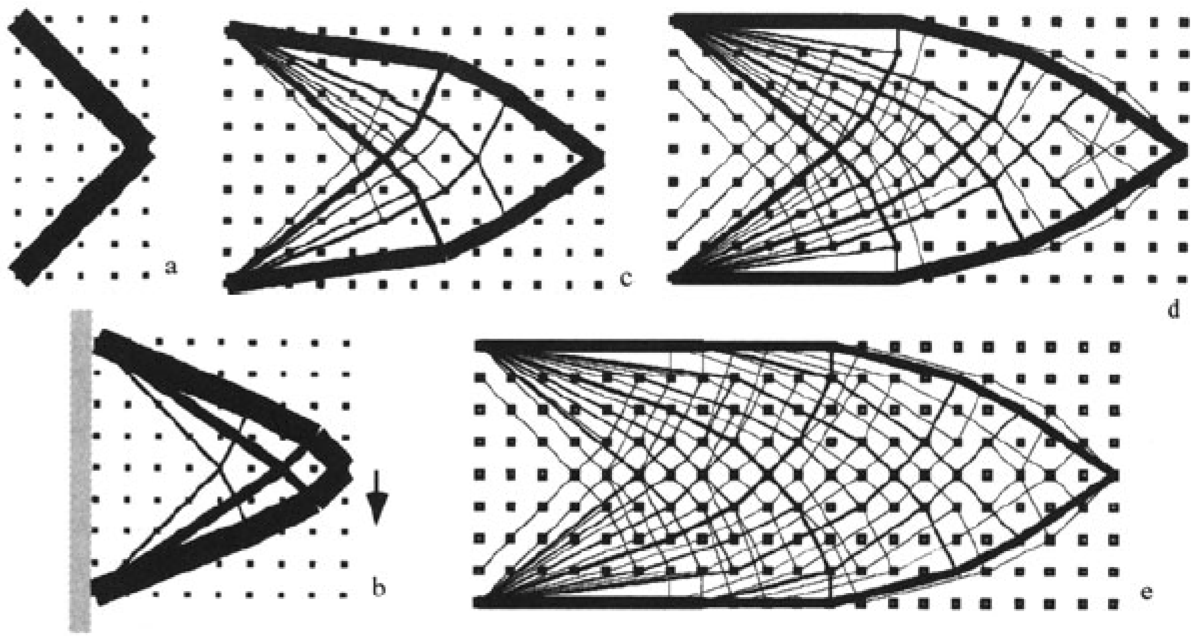
1.2 The basic problem statements in member forces

For the continuum formulations of topology design we formulated a stress based minimum compliance problem using the minimum complementary energy principle. Writing here for the single load truss problem we have the problem

$$\begin{aligned} &\inf_{\mathbf{t}} \min_{\mathbf{q}} \frac{1}{2} \sum_{i=1}^{m} \frac{l_i^2}{E_i} \frac{(q_i)^2}{t_i} \\ &\text{s.t.}: \mathbf{B}\mathbf{q} = \mathbf{f}, \quad \sum_{i=1}^{m} t_i = V, \quad t_i > 0,\ i = 1, \dots, m\ , \end{aligned} \tag{4.7}$$

where we have to take the infimum over all positive bar volumes in order to have a well-posed problem. In (4.7), $\mathbf{q}$ is the vector of member forces. For a given $\mathbf{t}$ the solution $\mathbf{q}^*$ to the inner problem of (4.7) satisfies $q_i^* = \frac{E_i}{l_i^2} t_i \mathbf{b}_i^T \mathbf{u}^*$, where $\mathbf{u}^*$ is the displacement of the truss, i.e., $\mathbf{u}^*$ is the solution to the inner problem of (4.6). Note that problem (4.7) is a problem which is *simultaneous convex* in member forces and member volumes.

Lecture 17: Topology design of truss structures



The influence of the ground structure geometry on the optimal topology. Optimal truss topologies for transmitting a single vertical force to a vertical line of supports. The ground structures consist of all possible non-overlapping connections between the nodal points of a regular mesh in rectangles of varying aspect ratios $R = a/b$. a): 632 potential bars for 5 by 9 nodes in a rectangle with $R = 0.5$. Optimal non-dimensional compliance $\Phi = 4.000$. b): 2040 potential bars, 9 by 9 nodes, $R = 1.0$, $\Phi = 5.975$. c): 4216 potential bars, 13 by 9 nodes, $R = 1.5$, $\Phi = 9.1676$. d): 7180 potential bars, 17 by 9 nodes, $R = 2.0$, $\Phi = 12.5756$. e): 10940 potential bars, 21 by 9 nodes, $R = 2.5$, $\Phi = 16.4929$

Lecture 17: Topology design of truss structures

The traditional formulation of truss topology design in terms of member forces is for single load , *plastic* design. This problem is normally stated as a minimum weight design problem, for all trusses that satisfy static equilibrium within certain constraints on the stresses in the individual bars. With the same stress constraint value for both tension and compression, the formulation is in the form of a linear programming problem

$$\min_{\mathbf{q},\mathbf{t}} \sum_{i=1}^{m} t_i$$

$$\text{s.t.}: \mathbf{Bq} = \mathbf{f}\,; \quad -\bar{\sigma}_i t_i \leq l_i q_i \leq \bar{\sigma}_i t_i, \quad i = 1, \dots, m\,,$$

$$t_i \geq 0, \quad i = 1, \dots, m\,.$$

Notice that in problem above the stress constraints are written in terms of *member forces*. This turns out to be important in order to give a consistent formulation. For some truss problems, the stress in a number of members will converge to a finite non-zero level as the member areas converge to zero, but the member forces will converge to zero. This fact should be observed for any truss design problem involving stress constraints. *Problem above is a formulation purely in terms of statics, with no kinematic compatibility included in the formulation.* However, a basic solution to this LP problem will automatically satisfy kinematic compatibility, a rather puzzling fact. We note here that (4.8) can be extended to cover cost of supports and to problems involving local stability constraints (buckling etc.) while maintaining the basic properties.

Lecture 17: Topology design of truss structures

With the change of variables $t_i = \frac{l_i}{\bar{\sigma}_i}(q_i^+ + q_i^-)$, $\mathbf{q} = (\mathbf{q}^+ - \mathbf{q}^-)$ we can write (4.8) in standard LP form, as

$$\begin{aligned} &\min_{\mathbf{q}^+,\mathbf{q}^-} \sum_{i=1}^{m} \frac{l_i}{\bar{\sigma}_i}(q_i^+ + q_i^-) \\ &\text{s.t. : } \mathbf{B}(\mathbf{q}^+ - \mathbf{q}^-) = \mathbf{f} \ ; \quad q_i^+ \geq 0,\ q_i^- \geq 0, \quad i = 1, \ldots, m \ . \end{aligned} \tag{4.9}$$

Here q_i^+, q_i^- can be interpreted as the member forces tension and compression, respectively. It is easy to see from the necessary conditions of optimality that the problem (4.8) gives rise to fully stressed designs, i.e. designs for which all bars with non-zero bar area have stresses at the maximum allowed level $\bar{\sigma}_i$. Thus one can often in the literature find (4.9) stated directly without reference to (4.8), as for a fully stressed design, the objective function of (4.9) is precisely the weight of the structure. We shall in a later section revert to these formulations and will show that (4.8), (4.7), and (4.1) are all equivalent in a certain sense.

LP form = Linear Programming, Linear Optimization

Lecture 17: Topology design of truss structures

The plastic design formulation can easily be extended to a multiple load situation as

$$\min_{\mathbf{q}^k,\mathbf{t}} \sum_{i=1}^{m} t_i$$

$$\text{s.t.} : \mathbf{B}\mathbf{q}^k = \mathbf{f}^k + \sum_{i=1}^{m} t_i \mathbf{g}_i, \quad k = 1, \ldots, M \ ,$$

$$-\bar{\sigma}_i t_i \leq l_i q_i^k \leq \bar{\sigma}_i t_i, \quad i = 1, \ldots, m, \quad k = 1, \ldots, M \ ,$$

$$t_i \geq 0, \quad i = 1, \ldots, m \ ,$$

where self-weight loads in the form $\sum_{i=1}^{m} t_i \mathbf{g}_i$ are also considered (see below for details on notation). This problem is also a linear programming problem. However for this case the precise relation between this problem and the minimum compliance problem is not known.

Lecture 17: Topology design of truss structures

1.3 Problem statements including self-weight and reinforcement

The formulations given above lend themselves to natural extensions, such as to the problem of finding the optimal topology of the reinforcement of a given structure and the optimal topology problem with self-weight taken into consideration.

For the reinforcement problem, see, e.g. Olhoff & Taylor (1983), using the ground structure approach, we divide a given ground structure into the set S of bars of fixed size and the set R of possible reinforcing bars. Typically S and R will be chosen as disjoint. We prefer here to allow R to contain (a part of) S as a subset; in this way non-zero lower bounds on the design variables can easily be included in the general problem analysis. The bars (elements) of the given structure have given bar volumes s_i, $i \in S$, and the optimal reinforcement t_i, $i \in R$, is the solution of the minimum compliance problem

$$\min_{\mathbf{u},\mathbf{t}} \ \mathbf{f}^T\mathbf{u}$$
$$\text{s.t.} : \left[\sum_{i\in R} t_i\hat{\mathbf{K}}_i + \sum_{i\in S} s_i\hat{\mathbf{K}}_i\right]\mathbf{u} = \mathbf{f}, \quad \sum_{i\in R} t_i = V, \quad t_i \geq 0,\ i \in R\ . \tag{4.11}$$

This problem can be solved by analogous means as can be used for the other topology design problems formulated above. Note that a reinforcement formulation in connection with a multiple load formulation with distinct sub-ground structures of a common ground structure will allow for a very general fail-safe design formulation.

Lecture 17: Topology design of truss structures

For the important case of optimization where loads due to the weight of the structure are taken into account, we employ the standard assumption that the weight of a bar is carried equally by the joints at its ends, thus neglecting bending effects. With g_i denoting the specific nodal gravitational force vector due to the self-weight of bar number i, the problem of finding the optimal topology with self-weight loads and external loads takes the form

$$\min_{\mathbf{u},\mathbf{t}} \left[\mathbf{f} + \sum_{i=1}^{m} t_i \mathbf{g}_i\right]^T \mathbf{u}$$

$$\text{s.t.}: \sum_{i=1}^{m} t_i \hat{\mathbf{K}}_i \mathbf{u} = \mathbf{f} + \sum_{i=1}^{m} t_i \mathbf{g}_i; \ \sum_{i=1}^{m} t_i = V, \ t_i \geq 0, \ i = 1, \ldots, m \ .$$

Note that for the problem with self-weight, any feasible truss design for which the self-weight load equilibrates the external load is an optimal design with compliance zero and zero displacement field (compliance is non-negative in all cases). Thus to avoid trivial situations, it is natural to assume that such designs are not possible:

$$\left\{ \mathbf{t} \,\middle|\, \sum_{i=1}^{m} t_i = V, \quad t_i \geq 0, \quad i = 1, \ldots, m, \quad \mathbf{f} + \sum_{i=1}^{m} t_i \mathbf{g}_i = 0 \right\} = \emptyset \ .$$

Lecture 17: Topology design of truss structures

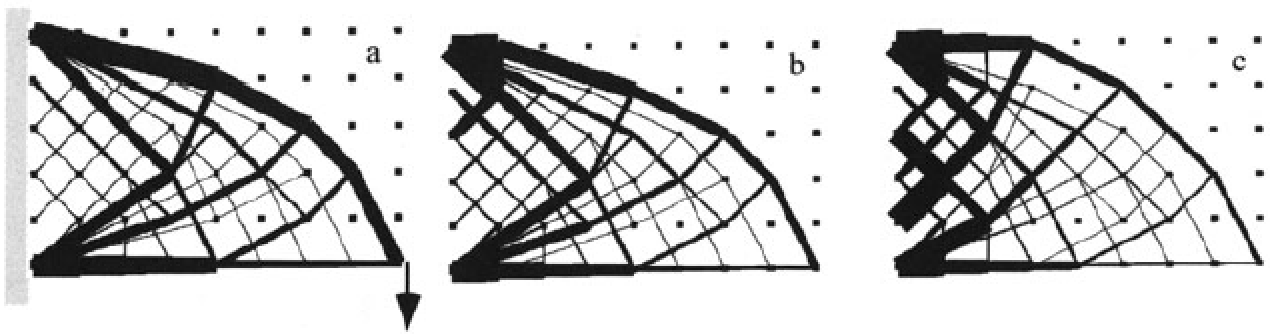
We complete this exposition of problem statements by stating the reinforcement problem, with self-weight loads, and general stiffness matrices and loads, so that all cases above are covered as special cases

$$\min_{\mathbf{u},\mathbf{t}} \left[\mathbf{f} + \sum_{i\in R} t_i \mathbf{g}_i + \sum_{i\in S} s_i \mathbf{g}_i\right]^T \mathbf{u}$$

$$\text{s.t.} : \sum_{i\in R} t_i \hat{\mathbf{K}}_i \mathbf{u} + \sum_{i\in S} s_i \hat{\mathbf{K}}_i \mathbf{u} = \mathbf{f} + \sum_{i\in R} t_i \mathbf{g}_i + \sum_{i\in S} s_i \mathbf{g}_i \ ,$$

$$\sum_{i\in R} t_i = V, \quad t_i \geq 0, \quad i \in R \ .$$

Here a max-min formulation as in (4.6) can also be formulated, maintaining the concave-convex nature of the basic problem (4.6).



The effect of self-weight loads. Optimal truss topologies for transmitting a single vertical force to a vertical line of supports. The figures show the variation for increasing specific self-weight loads, corresponding to increasing real lengths of the structures. The self-weight is in b) increased by 2 times compared to the design a). These designs are obtained for a 9 by 6 equidistant nodal lay-out in a rectangular domain of aspect ratio 1.6, and all 919 possible *non-overlapping* connections. If all 1431 possible connections are used, the design b) is modified to the design c) .

Lecture 17: Topology design of truss structures

2. Problem equivalence and globally optimized energy functionals

2.1 Conditions or optimality

For the sake of completeness of the presentation and as one gains extra information in the truss case, we will in this section derive the optimality conditions also for the minimum compliance truss topology problem (in the formulation with self-weight). As for the continuum problems treated earlier, these conditions constitute the basis for the well-known computational scheme named the *optimality criteria method*; we will describe this under a general discussion on computational procedures in a later section.

In order to obtain the necessary conditions for optimality for problem (4.12) we introduce Lagrange multipliers $\tilde{\mathbf{u}}, \Lambda, \mu_i$, $i = 1, \ldots, m$, for the equilibrium constraint, the volume constraint and the zero lower bound constraints, respectively. The necessary conditions are thus found as the conditions of stationarity of the Lagrangian

$$\mathcal{L} = (\mathbf{f} + \sum_{i=1}^{m} t_i \mathbf{g}_i)^T \mathbf{u} - \tilde{\mathbf{u}}(\sum_{i=1}^{m} t_i \hat{\mathbf{K}}_i \mathbf{u} - \mathbf{f} - \sum_{i=1}^{m} t_i \mathbf{g}_i)$$

$$+ \Lambda(\sum_{i=1}^{m} t_i - V) + \sum_{i=0}^{m} \mu_i(-t_i) \ .$$

By differentiation we obtain the necessary conditions

$$\sum_{i=1}^{m} t_i \hat{\mathbf{K}}_i \tilde{\mathbf{u}} = \mathbf{f} + \sum_{i=1}^{m} t_i \mathbf{g}_i \ , \quad \tilde{\mathbf{u}}^T(\hat{\mathbf{K}}_i \mathbf{u} - 2\mathbf{g}_i) = \Lambda - \mu_i \ ,$$

$$\mu_i \geq 0 \ , \quad \mu_i t_i = 0 \ , \quad i = 1, \ldots, m \ .$$

Lecture 17: Topology design of truss structures

If we impose a small non-negative lower bound on the areas, the stiffness matrix $\hat{\mathbf{K}}$ is positive definite and thus $\mathbf{u}$ is the unique Lagrange multiplier for the equilibrium constraint, but the situation without a lower bound is not so straightforward.

Now let $\Lambda^*(\mathbf{u})$ denote the maximum of the specific energies $\mathbf{u}^T(\hat{\mathbf{K}}_i\mathbf{u}-2\mathbf{g}_i)$ (with self-weight) of the individual bars, i.e.

$$\Lambda^*(\mathbf{u}) = \max\left\{\mathbf{u}^T(\hat{\mathbf{K}}_i\mathbf{u} - 2\mathbf{g}_i) \mid i = 1, \ldots, m\right\} ,$$

and let $J(\mathbf{u})$ denote the set of bars for which the specific energy attains this maximum level

$$J(\mathbf{u}) = \left\{i \mid \mathbf{u}^T(\hat{\mathbf{K}}_i\mathbf{u} - 2\mathbf{g}_i) = \Lambda^*(\mathbf{u})\right\} .$$

We also define non-dimensional element volumes $\tilde{t}_i = t_i/V$. Then the necessary conditions are satisfied with

$$\begin{aligned} &\tilde{\mathbf{u}} = \mathbf{u};\; t_i = \tilde{t}_i V,\, i \in J(\mathbf{u});\; t_i = 0,\, i \notin J(\mathbf{u});\; \Lambda = \Lambda^*(\mathbf{u}); \\ &\mu_i = 0,\, i \in J(\mathbf{u});\; \mu_i = \Lambda^*(\mathbf{u}) - \mathbf{u}^T(\hat{\mathbf{K}}_i\mathbf{u} - 2\mathbf{g}_i),\, i \notin J(\mathbf{u}) , \end{aligned} \tag{4.14}$$

provided that there exist a displacement field $\mathbf{u}$ with corresponding set $J(\mathbf{u})$ and non-dimensional element volumes $\tilde{t}_i$, $i \in J(\mathbf{u})$, such that

$$V \sum_{i\in J(\mathbf{u})} \tilde{t}_i\hat{\mathbf{K}}_i\mathbf{u} = \mathbf{f} + V \sum_{i\in J(\mathbf{u})} \tilde{t}_i\mathbf{g}_i \qquad \sum_{i\in J(\mathbf{u})} \tilde{t}_i = 1 . \tag{4.15}$$

The optimality condition (4.15) states that a convex combination of the gradients of the quadratic functions $V(\frac{1}{2}\mathbf{u}^T\hat{\mathbf{K}}_i\mathbf{u} - \mathbf{g}_i^T\mathbf{u})$, $i \in J(\mathbf{u})$, equals the load vector $\mathbf{f}$.

Lecture 17: Topology design of truss structures

It can be shown below that there does indeed exist a pair (u, t) which is a solution to the reduced optimality conditions. This implies that there exists an optimal truss that has bars with constant specific energies and the set J(u) is the set of these active bars. Note that a pair (u, t) satisfying the necessary conditions for problem is automatically a minimizer for the *non-convex* form of the minimum compliance problem.

For any design $\tilde{s}_i$, $i = 1, \ldots, m$, satisfying the volume constraint and with corresponding displacement field $\mathbf{v}$, we have that

$$
\begin{aligned}
(\mathbf{f} + \sum_{i=1}^{m} t_i g_i)^T \mathbf{u} &= 2\mathbf{f}^T\mathbf{u} - \sum_{i=1}^{m} t_i \mathbf{u}^T(\hat{\mathbf{K}}_i \mathbf{u} - 2g_i) \\
&= 2\mathbf{f}^T\mathbf{u} - \sum_{i=1}^{m} t_i \Lambda^*(\mathbf{u}) \\
&= 2\mathbf{f}^T\mathbf{u} - V\Lambda^*(\mathbf{u}) = 2\mathbf{f}^T\mathbf{u} - \sum_{i=1}^{m} \tilde{s}_i \Lambda^*(\mathbf{u}) \\
&\le 2\mathbf{f}^T\mathbf{u} - \sum_{i=1}^{m} \tilde{s}_i \mathbf{u}^T(\hat{\mathbf{K}}_i \mathbf{u} - 2g_i) \\
&\le 2\max_{w} \left\{ (\mathbf{f} + \sum_{i=1}^{m} \tilde{s}_i g_i)^T w - \frac{1}{2}\sum_{i=1}^{m} \tilde{s}_i w^T \hat{\mathbf{K}}_i w \right\} \\
&= 2(\mathbf{f} + \sum_{i=1}^{m} \tilde{s}_i g_i)^T \mathbf{u} - \sum_{i=1}^{m} \tilde{s}_i \mathbf{v}^T \hat{\mathbf{K}}_i \mathbf{v} = (\mathbf{f} + \sum_{i=1}^{m} \tilde{s}_i g_i)^T \mathbf{v} \ ,
\end{aligned}
$$

where we have invoked the extremum principle for equilibrium. Note that the existence of solutions to the optimality conditions shows that *there always exists an optimal solution with no more active bars than the degrees of freedom (dimension of* $\boldsymbol{u}$*) plus 1;* this follows from Caratheodory's theorem on convex combinations. Finally we also remark that such a design only has active bars that attain the maximum energy level, in accordance with what we have seen for the continuum problems as well.

2.2 Reduction to problem statements in bar volumes only

It was noted earlier that the truss topology problem is an unusual structural optimization problem, as the acceptance of zero bar volumes implies that the stiffness matrix of the problem can be singular. Thus the standard gradient/ adjoint methods of structural optimization which view the problems as optimization problems in the design variables only cannot be invoked directly. However, if we accept to consider the topology optimization problem as a limes inferior problem for a series of optimal design problems with decreasing positive lower bounds on the design variables we can remove the displacements from the formulation (this has consistently been the approach for the continuum structures).

Lecture 17: Topology design of truss structures

Rewriting (4.5) and imposing positive element volumes as a perturbation of the original problem, we can remove the displacement variables by solving for the now unique displacements

$$\inf_{\substack{t_i > 0 \\ \sum_{i=1}^m t_i = V}} \left\{ \Phi(\mathbf{t}) \equiv \mathbf{f}^T \hat{\mathbf{K}}(\mathbf{t})^{-1} \mathbf{f} = \max_{\mathbf{u}} \left[2\mathbf{f}^T \mathbf{u} - \mathbf{u}^T \sum_{i=1}^m t_i \hat{\mathbf{K}}_i \mathbf{u} \right] \right\} . \qquad (4.16)$$

Note that we have exchanged the min-operator with the inf-operator as well as changing the constraint $t_i \geq 0$ to $t_i > 0$. Problem (4.16) is a formulation which, as discussed before, is more standard in structural optimization. Nonetheless, it is still unusual from a computational point of view, as the stiffness matrix in *truss* topology optimization typically will be dense.

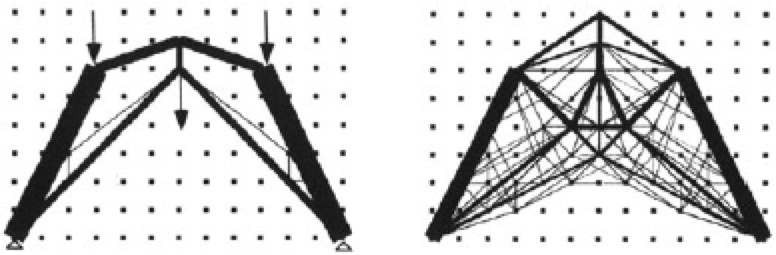
It was earlier pointed out that problem (4.16) is convex (see also Svanberg (1984)). This follows from the fact that the compliance function $\Phi(t)$, as a function of the design variables by the second expression in (4.16) is expressed as the supremum (maximum) over a family of convex (linear in this case) functions (cf., Sect. 1.5.2). Note that the gradient of the objective function $\Phi(t)$ is easy to compute and is given as (as seen in Sect. 1.2.3):

$$\frac{\partial \Phi}{\partial t_i} = -\mathbf{u}^T \hat{\mathbf{K}}_i \mathbf{u}, \text{ with } \sum_{i=1}^m t_i \hat{\mathbf{K}}_i \mathbf{u} = \mathbf{f} .$$

Note that we for the multiple load problem in its worst-case setting also have a convex formulation in bar volumes only

$$\inf_{\substack{t_i > 0,\, i=1,\dots,m \\ \sum_{i=1}^m t_i = V}} \max_{k=1,\dots,M} \mathbf{f}^{k^T} \mathbf{K}(\mathbf{t})^{-1} \mathbf{f}^k . \qquad (4.17)$$

Lecture 17: Topology design of truss structures



The difference between multiple load and single load case problems. Optimal truss topologies for transmitting three vertical forces to two fixed supports. The truss is optimized with the loads treated as a single load (left) as well as three individual load cases for a min-max, worst case design situation (right). The ground structures consist of all 8744 possible non-overlapping connections between the nodal points of a regular 13 by 13 mesh in a square domain. We do not show the uppermost rows of nodes, as these are not part of the optimal structure

Semidefinite programs

The structure of the minimum compliance *truss* problem also allows for yet another formulation in the displacements only. These fall in a class of mathematical programming problems named *semidefinite programs*, with the acronym SDP. If we rewrite problem in terms of a bound variable Φ, which is the value of the compliance, we can write the optimization problem as

$$\min_{\mathbf{u},\mathbf{t},\Phi} \Phi$$

$$\text{s.t.} : \max_{\mathbf{u}} \left\{ 2\mathbf{f}^T\mathbf{u} - \mathbf{u}^T \sum_{i=1}^{m} t_i \hat{\mathbf{K}}_i \mathbf{u} \right\} \leq \Phi \,,$$

$$\sum_{i=1}^{m} t_i = V, \quad t_i \geq 0,\ i = 1, \ldots, m \,.$$

Here the constraint on the potential energy is actually equivalent to a condition that the symmetric matrix

$$\widetilde{\mathbf{A}}(\mathbf{t},\Phi) = \begin{bmatrix} \Phi & \mathbf{f}^T \\ \mathbf{f} & \hat{\mathbf{K}} \end{bmatrix} = \begin{bmatrix} \Phi & \mathbf{f}^T \\ \mathbf{f} & 0 \end{bmatrix} + \sum_{i=1}^{m} t_i \begin{bmatrix} 0 & 0 \\ 0 & \hat{\mathbf{K}}_i \end{bmatrix}$$

is positive semidefinite, which we write as $\widetilde{A}\geq 0$. The matrix $\widetilde{A}$ is linear in the variables t_i and the inequality $\widetilde{A}\geq 0$ is then referred to as a *linear matrix inequality*, an LMI. The linearity implies that $\widetilde{A}\geq 0$ is a convex constraint, which is most directly seen by the definition: $\widetilde{A}\geq 0$ if and only if $\widetilde{A}$ is symmetric and $u^T \widetilde{A}\, u \geq 0$ for all u (this characterization can also be used to show that the condition $\widetilde{A}\geq 0$ is equivalent to the condition $2f^Tu - u^TKu \leq \Phi$ for all **u**.

Lecture 17: Topology design of truss structures

Our optimization problem can thus be rewritten as a convex problem in the variables t, Φ only (this has now become a standard test problem in mathematical programming):

$$\min_{\mathbf{t},\Phi} \Phi$$

$$\text{s.t.} : \widetilde{\mathbf{A}}(\mathbf{t},\Phi) \succeq \mathbf{0}, \quad \sum_{i=1}^{m} t_i = V, \quad t_i \geq 0,\ i = 1,\ldots,m\ .$$

This reformulation may seem as adding to the complexity of the problem, as initially a constraint like $\widetilde{A} \geq 0$ seems difficult to handle. However, this is not the case. Modern interior point algorithms solve SDP's in polynomial time. As the single load case problem is actually equivalent to an LP problem this feature is more interesting for the multiple load case which does not have an equivalent LP format. For the worst case situation we actually write the SDP form as:

$$\min_{\mathbf{t},\Phi} \Phi$$

$$\text{s.t.} : \begin{bmatrix} \Phi & \mathbf{f}^{k\,T} \\ \mathbf{f}^k & \hat{\mathbf{K}} \end{bmatrix} \succeq \mathbf{0},\ k = 1,\ldots,M\ ,$$

$$\sum_{i=1}^{m} t_i = V, \quad t_i \geq 0,\ i = 1,\ldots,m\ .$$

Also remark that the SDP form can be employed for truss design with bounds $0 \leq t_{\min} \leq t_i \leq t_{\max}$ on the truss volumes; in this situation an LP form is not known either.

Lecture 17: Topology design of truss structures

If we alternatively consider the compliance Φ as given and consider the minimum weight design for an upper bound on compliance, an alternative SDP problem can be written as:

$$\min_{\mathbf{t}} \sum_{i=1}^{m} t_i$$
$$\text{s.t.}: \tilde{\Phi} \sum_{i=1}^{m} t_i \mathbf{K}_i - \mathbf{f}^k {\mathbf{f}^k}^T \succeq \mathbf{0},\ k = 1, \ldots, M\ ,$$
$$t_i \geq 0,\ i = 1, \ldots, m\ .$$

2.3 Reduction to problem statements in displacements only

We will now use the max-min formulation (4.6) of the truss topology design problem to derive a globally optimal strain energy functional that describes the energy of the optimal truss. This leads to an alternative, equivalent convex formulation of the problem for which a number of computationally very efficient algorithm can be devised. The derivation is in concept and results similar to the derivation for simultaneous design of structure and material as described, but the dyadic nature of the stiffness matrix for trusses means that one can go somewhat further, as will be shown in the coming sections.

We will for notational simplicity not cater for the reinforcement situation, so that the version of problem (4.13) that we consider has the form

$$\max_{\substack{\mathbf{t} \geq \mathbf{0} \\ \sum_{i=1}^{m} t_i = V}} \min_{\mathbf{u}} \left\{ \frac{1}{2} \sum_{i=1}^{m} t_i \mathbf{u}^T \hat{\mathbf{K}}_i \mathbf{u} - (\mathbf{f} + \sum_{i=1}^{m} t_i \mathbf{g}_i)^T \mathbf{u} \right\} \tag{4.23}$$

Lecture 17: Topology design of truss structures

and this problem is linear in the design variable and convex in the displacement variable. Thus the problem is concave-convex (with a convex and compact constraint set in t) and we can interchange the max and the min operators, to obtain

$$\min_{\mathbf{u}} \max_{\substack{\mathbf{t} \geq 0 \\ \sum_{i=1}^m t_i = V}} \left\{ \frac{1}{2} \sum_{i=1}^m t_i \mathbf{u}^T \hat{\mathbf{K}}_i \mathbf{u} - (\mathbf{f} + \sum_{i=1}^m t_i \mathbf{g}_i)^T \mathbf{u} \right\}$$

The inner problem is now a linear programming problem in the $\mathbf{t}$ variable. To solve this problem, note that with $t_i \geq 0$, $\sum_{i=1}^m t_i = V$, we have the inequality

$$\sum_{i=1}^m t_i (\mathbf{u}^T \hat{\mathbf{K}}_i \mathbf{u} - 2 \mathbf{g}_i{}^T \mathbf{u}) \leq V \max_{i=1,\ldots,m} \left\{ \mathbf{u}^T \hat{\mathbf{K}}_i \mathbf{u} - 2 \mathbf{g}_i{}^T \mathbf{u} \right\} .$$

Here the equality holds if all material is assigned to a bar with maximum specific energy $\mathbf{u}^T \hat{\mathbf{K}}_i \mathbf{u} - 2\mathbf{g}_i^T \mathbf{u}$. Thus we see that the problem (4.23) can be reduced to

$$\min_{\mathbf{u}} \max_{i=1,\ldots,m} \left\{ \frac{V}{2} \left[\mathbf{u}^T \hat{\mathbf{K}}_i \mathbf{u} - 2\mathbf{g}_i^T \mathbf{u} \right] - \mathbf{f}^T \mathbf{u} \right\} . \qquad (4.24)$$

This is an *unconstrained*, *convex* and *non-smooth* problem, in the displacement variable $\mathbf{u}$ only, with optimal value minus one half of the optimal value for the problem (4.12). It can equivalently be written as the *smooth*, constrained and convex problem

$$\min_{\mathbf{u},\, \tau} \left\{ \tau - \mathbf{f}^T \mathbf{u} \right\}$$

$$\text{s.t.} : \frac{V}{2} \left[\mathbf{u}^T \hat{\mathbf{K}}_i \mathbf{u} - 2\mathbf{g}_i{}^T \mathbf{u} \right] \leq \tau, \quad i = 1, \ldots, m .$$

Lecture 17: Topology design of truss structures

This problem has a large number of constraints, but these can be efficiently handled via interior point methods. For completeness let us state the equivalent problems for the weighted average, multiple load truss case

$$\min_{\mathbf{u}^k}\left[\max_{i=1,\dots,m}\left\{\sum_{k=1}^{M} w^k\left(\frac{V}{2}{\mathbf{u}^k}^T\mathbf{K}_i\mathbf{u}^k-{\mathbf{f}^k}^T\mathbf{u}^k\right)\right\}\right]$$

One can think of the resulting displacements only problems shown above as equilibrium problems for a structure with a non-smooth, convex strain energy. Philosophically speaking, this strain energy is the strain energy for a "self-optimized" structure which automatically adjusts its topology and sizing so as to minimize compliance for the applied load(s).

It is possible to show existence of solutions to the problems and to prove the equivalence between problem statements. The solutions are not unique and it is quite well-known that there are normally "many" solutions (actually subspaces of solutions). The equivalence of the problems is understood in the sense that for a solution u to for example problem (4.24) and the corresponding set *J(u)* of active bars, there exists a corresponding set of bar volumes t satisfying the optimality condition

$$\sum_{i\in J(\mathbf{u})} t_i\hat{\mathbf{K}}_i\mathbf{u}=\mathbf{f}+\sum_{i\in J(\mathbf{u})} t_i\mathbf{g}_i\ ,\quad \sum_{i\in J(\mathbf{u})} t_i=V\ ,$$

$$t_i=0,\ i\notin J(\mathbf{u});\quad t_i\geq 0,\ i=1,\dots,m\ ,$$

and these optimality conditions are precisely the optimality conditions for the min-max problem (4.24).

Lecture 17: Topology design of truss structures

For the worst case multiple load problem it is possible to generate a displacements only formulation in the form

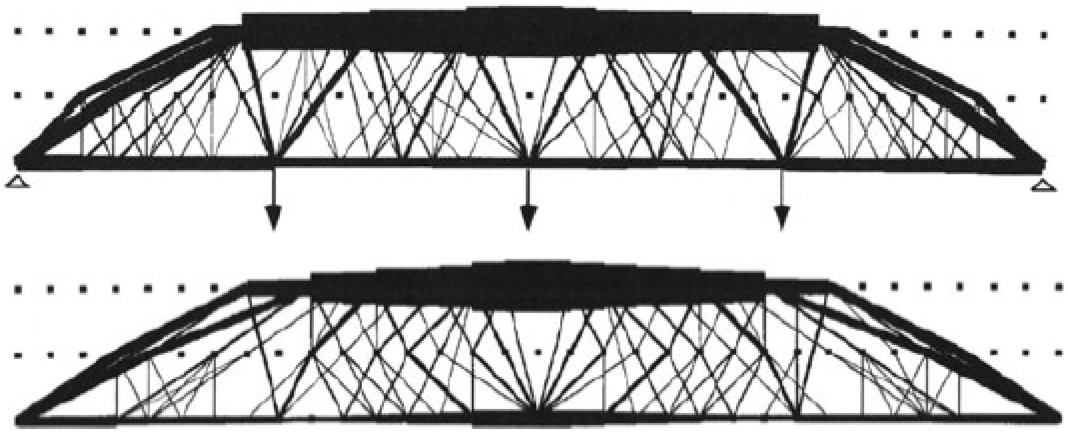
$$\min_{\substack{\mathbf{u}^k,\, \lambda^k \geq 0 \\ \sum_{k=1}^{M} \lambda^k = 1}} \left[\max_{i=1,\dots,m} \left\{ \sum_{k=1}^{M} \lambda^k \left(\frac{V}{2} \mathbf{u}^{k^T} \mathbf{K}_i \mathbf{u}^k - \mathbf{f}^{k^T} \mathbf{u}^k \right) \right\} \right]$$

where we have used weighting parameters $\lambda^k \geq 0$, $k = 1, \dots, M$. Solutions to this problem can likewise be proved to exist and the optimal value of problem (4.27) equals minus one half the extremal value of problem (4.4) The direct equivalence between the two problems (in the sense discussed for the single load problem) may fail if a multiplier λ^k equals zero in the optimal solution to problem (4.27). In this case we cannot guarantee equilibrium for this load condition, as the equilibrium will not necessarily be enforced by the necessary conditions of optimality. However, a set of bar areas can be identified by considering the loads with non-zero multipliers, and a minimum compliance truss will be generated for these loads. This makes it natural to consider a slightly perturbed version of (4.4) and (4.27), where the multipliers are constrained as $\lambda^k \geq \varepsilon > 0$, $k = 1, \dots, M$. For the resulting perturbed version of problem (4.27) we can write

$$\min_{\substack{\lambda^k \geq \varepsilon \\ \sum_{k=1}^{M} \lambda^k = 1}} \left(\min_{\mathbf{u}^k} \left[\max_{i=1,\dots,m} \left\{ \sum_{k=1}^{M} \lambda^k \left(\frac{V}{2} \mathbf{u}^{k^T} \mathbf{K}_i \mathbf{u}^k - \mathbf{f}^{k^T} \mathbf{u}^k \right) \right\} \right] \right)$$

indicating that the inner problem in the displacements could be solved using the methods that can be used for the single load case, with the outer problem solved using algorithms for convex non-differentiable optimization problems.

Lecture 17: Topology design of truss structures



The difference (and similarity) between multiple load case treated in the weighted average formulation (equal weights) (top) and treated in the worst case min-max formulation (below). Optimal truss topologies for transmitting three vertical forces to two fixed supports for a long slender rectangular ground structure of aspect ratio 16 (like a long span bridge), with 33 by 3 equidistant nodes and all 2818 possible non-overlapping connections. In this figure, the vertical scale has been distorted in order to being able to show the results

Thank you for your attention