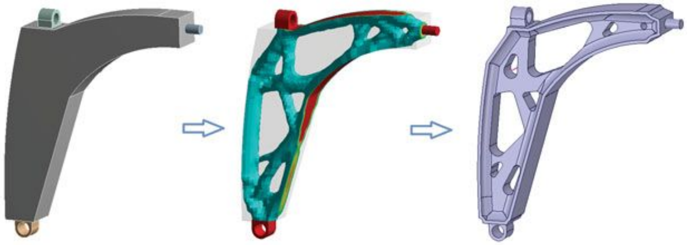
香港中文大學
The Chinese University of Hong Kong

MAEG5160: Design for Additive Manufacturing

Lecture 18: Topology design of truss structures_2



Prof SONG Xu
Department of Mechanical and Automation Engineering,
The Chinese University of Hong Kong.

Lecture 18: Topology design of truss structures_2

2.4 Linear programming problems for single load problems

In the preceding section the minimum compliance truss topology problem was reformulated as a non-smooth, convex problem in the displacements only. This can now be used as a basis for generating a range of other equivalent problem statements.
The starting point is actually the bound formulation (4.25), which for the simpler case of no self-weight becomes, up to a scaling,

$$\begin{aligned} &\min_{\mathbf{u}} \left\{-\mathbf{f}^T\mathbf{u}\right\} \\ &\text{s.t.} : \frac{V}{2}\mathbf{u}^T\hat{\mathbf{K}}_i\mathbf{u} \leq 1, \quad i = 1, \ldots, m \ , \end{aligned}$$

i.e. a maximization of compliance, with constraints on the specific strain energies.
For the *single load* truss problem the element stiffness matrices are dyadic products and we get for the specific energies

$$\mathbf{u}^T\mathbf{K}_i\mathbf{u} = \left(\frac{\sqrt{E_i}}{l_i}\mathbf{b}_i{}^T\mathbf{u}\right)^2$$

Lecture 18: Topology design of truss structures_2

This special form implies that above can be written in LP-form as

$$\min_{\mathbf{u}} \{-\mathbf{f}^T\mathbf{u}\}$$
$$\text{s.t.}: -1 \leq \sqrt{\frac{VE_i}{2}}\frac{\mathbf{b}_i{}^T\mathbf{u}}{l_i} \leq 1, \quad i = 1,\dots,m\ , \tag{4.30}$$

which is a problem of maximizing the compliance with constraints on the strains $\varepsilon_i = l_i{}^{-1}\mathbf{b}_i{}^T\mathbf{u}$ in all bars. For suitable stress constraint values $\bar{\sigma}_i$ it turns out that problem (4.30) is the dual of the force formulation (4.9)

$$\min_{\substack{q_i^+ \geq 0,\, q_i^- \geq 0,\\ i=1,\dots,m\\ \mathbf{B}(\mathbf{q}^+ - \mathbf{q}^-) = \mathbf{f}}} \sum_{i=1}^{m} \frac{l_i}{\bar{\sigma}_i}(q_i^+ + q_i^-)\ . \tag{4.31}$$

Here the tension/compression forces q_i^+, q_i^- are the multipliers for the strain inequality constraints of (4.30). As seen in section 4.1.2, problem (4.31) is, after a change of variables, precisely the traditional minimum mass plastic design formulation (4.8). The developments described above show that the minimum compliance design problem for a single load case is equivalent to a minimum mass plastic design formulation, in the sense that for a solution $\mathbf{t}$, $\mathbf{q}$ to the minimum mass plastic design problem with data V, $\bar{\sigma}_i$, there corresponds a solution $\mathbf{t}_C$, $\mathbf{u}_C$ to the minimum compliance problem with data V_C, E_i. The precise relations are

$$\bar{\sigma}_i = \sqrt{E_i}, \quad t_C = \tfrac{V_C}{V}t, \quad \mathbf{u}_C = \tfrac{V_C}{V}\tilde{\mathbf{u}}\ ,$$

where $\tilde{\mathbf{u}}$ is the dual variable of the minimum mass plastic design problem corresponding to the static equilibrium constraint $\mathbf{Bq} = \mathbf{f}$.

Lecture 18: Topology design of truss structures_2

The member force formulations (4.8) and (4.9) are, as described earlier, the traditional formulations for single load truss topology optimization. These are, of course, very efficient formulations and could be solved using sparse, primal-dual interior point LP-methods or the simplex algorithm. The force methods are at first glance problems in plastic design, as kinematic compatibility is ignored, and their use in elastic design is justified by the possibility of finding statically determinate solutions. The equivalence between the force methods and the minimum compliance problem for the single load case shows that any solution to the force LP-formulation leads to a minimum compliance topology design, within the framework of elastic designs. Such designs are uniformly stressed designs, as well as having a constant specific energy in all active bars. The existence of basic solutions to the linear programming problem (4.9) implies that there exist minimum mass truss topologies with a number of bars not exceeding the degrees of freedom. If there exists such a basic solution with only non-zero forces (areas), this is a statically determinate truss. Otherwise, the truss will have a unique force field for the given load but will be kinematically indeterminate. In other words the truss may have rigid body (mechanism) response to certain loads other than the load for which it is designed; this may be the case even after nodes with no connected bars are removed.

Lecture 18: Topology design of truss structures_2

For the sake of completeness of presentation, note that in the reinforcement case without self-weight, the single load case problem can be reduced to a quadratic optimization problem with linear constraints

$$\min_{\mathbf{u},\,\tau} \left\{ \frac{1}{2}\mathbf{u}^T \left(\sum_{i \in S} s_i \mathbf{K}_i \right) \mathbf{u} - \mathbf{f}^T\mathbf{u} + \tau^2 \right\}$$

$$\text{s.t.}: \; -\tau \le \sqrt{\frac{VE_i}{2}}\frac{{\mathbf{b}_i}^T\mathbf{u}}{l_i} \le \tau, \quad i \in R\,.$$

Notice here that the matrix $\sum_{i \in s} s_i \mathbf{K}_i$ is positive semi-definite, but usually not positive definite. The problem statement (4.32) also represents a simplification of the minimum compliance problem for a single load case with *lower bounds* on the variables; the vector $\mathbf{s}$ represents the vector of lower bounds on the design variables.

Lecture 18: Topology design of truss structures_2

2.5 Reduction to problem statements in stresses only

In the following we will base our developments on the worst-case multiple load design formulation (4.28). In order to simplify notation we will refrain from covering the problem of reinforcement and the self-weight problem will also play a minor role in the following. However, we begin with a general treatment that covers truss, variable thickness sheet and sandwich plate design.

Now returning to problem (4.28) we note that by a change of variables of $\mathbf{u}^k$ to $\frac{1}{\lambda^k}\mathbf{u}^k$, this problem can be stated as

$$\inf_{\substack{\lambda^k > 0 \\ \sum_{k=1}^{M} \lambda^k = 1}} \left(\min_{\mathbf{u}^k} \left[\max_{i=1,\dots,m} \left\{ \sum_{k=1}^{M} \left(\frac{V}{2} \frac{1}{\lambda^k} \mathbf{u}^{k^T} \mathbf{K}_i \mathbf{u}^k - \mathbf{f}^{k^T} \mathbf{u}^k \right) \right\} \right] \right) , \qquad (4.33)$$

which is now *jointly* convex on the feasible set in both the multipliers λ^k and the displacements. Here we have used the inf-operator to indicate the use of a decreasing sequence of lower bounds on the multipliers λ^k. The presence of the infimum over the multipliers indicates that it is a natural choice to use interior penalty methods for a computational procedure for solving of this problem, as will be described later.

Lecture 18: Topology design of truss structures_2

We shall now show that by deriving the dual formulations of (4.33) one can for the *truss* case generate what amounts to stress based min-max minimum compliance formulations. The basis for this derivation is again, as in the earlier development, the dyadic structure of the individual member stiffness matrices. Expressing the maximization over the bar numbers (the inner problem) with a bounding variable and using auxiliary variables $c^k_i = b_i^T u^k$ (the member elongations), the equivalent convex dual problem can be derived to have the form

$$\begin{aligned} &\inf_{\mathbf{t}} \min_{\mathbf{q}^k} \left[\max_{k=1,\dots,m} \left\{ \frac{1}{2} \sum_{i=1}^{m} \frac{l_i^2}{E_i} \frac{(q_i^k)^2}{t_i} \right\} \right] \\ &\text{s.t.} : \mathbf{B}\mathbf{q}^k = \mathbf{f}^k, \quad k = 1,\dots,M \ , \\ &\qquad \sum_{i=1}^{m} t_i = V, \quad t_i > 0, \quad i = 1,\dots,m \ . \end{aligned} \tag{4.34}$$

With $\bar{\lambda}^k$, $\bar{\mathbf{u}}^k$ denoting the Lagrange multipliers for a bound constraint formulation of the maximization over k and the equilibrium constraint, respectively, we can for an optimum $\mathbf{q}^k$, $\mathbf{t}$ of (4.34) with $\bar{\lambda}^k > 0$, $k = 1,\dots,M$, identify $\mathbf{u}_k = \bar{\mathbf{u}}^k / \bar{\lambda}^k$, $\mathbf{t}$ as a solution to our original problem statement (4.4) in displacements and bar areas. Also, we can show, from the Karush-Kuhn-Tucker optimality conditions that $q_i^k = \frac{E_i}{l_i^2} t_i \mathbf{b}_i^T \mathbf{u}^k$, i.e. compatibility of stresses and displacements is automatically assured.

Lecture 18: Topology design of truss structures_2

Finally, we will consider the *elimination of the bar volumes* from the problem (4.34), by directly solving for these variables. This corresponds to the elimination of bar volumes in the displacements (strain) based formulation as carried out. Expressing the maximization over load cases by a maximization over a convex combination of weighting factors

$$\min_{\substack{\mathbf{q}^k \\ \mathbf{B}\mathbf{q}^k = \mathbf{f}^k}} \quad \inf_{\substack{\mathbf{t} > 0 \\ \sum_{1=1}^{m} t_i = V}} \quad \max_{\substack{\lambda^k \geq 0 \\ \sum_{k=1}^{M} \lambda = 1}} \left\{ \sum_{k=1}^{M} \lambda^k \left[\frac{1}{2} \sum_{i=1}^{m} \quad \frac{l_i^2}{E_i} \frac{(q_i^k)^2}{t_i} \right] \right\}$$

we can derive the optimal values of the bar volumes as

$$t_i = V \sqrt{\frac{l_i^2}{E_i} \sum_{k=1}^{M} \lambda^k (q_i^k)^2} \left[\sum_{i=1}^{m} \sqrt{\frac{l_i^2}{E_i} \sum_{k=1}^{M} \lambda^k (q_i^k)^2} \right]^{-1}$$

Inserting it in the previous equation we obtain the following problem in the member forces only

$$\min_{\mathbf{q}^k} \quad \max_{\substack{\lambda^k \geq 0 \\ \sum_{k=1}^{M} \lambda^k = 1}} \left\{ \frac{1}{2V} \left[\sum_{i=1}^{m} \left(\frac{l_i}{\sqrt{E_i}} \sqrt{\sum_{k=1}^{M} \lambda^k (q_i^k)^2} \right) \right]^2 \right\}$$

$$\text{s.t.} : \mathbf{B}\mathbf{q}^k = \mathbf{f}^k, \quad k = 1, \ldots, M \ .$$

Lecture 18: Topology design of truss structures_2

For the single load case we recover the traditional linear programming formulation (4.9) in the disguised form

$$\min_{\substack{\mathbf{q} \\ \mathbf{Bq}=\mathbf{f}}} \left\{ \frac{1}{2V} \left[\sum_{i=1}^{m} \left(\frac{l_i}{\sqrt{E_i}} |q_i| \right) \right]^2 \right\}$$

Rescaling the objective function and taking the square root of the objective function results in (4.9). Note that we have again seen that the stress constraint values for the plastic topology problem should be chosen as $\sqrt{E_i}$
Also, as (4.38) was obtained by direct duality without rescaling, one can see that the optimal value II of the optimal compliance will relate to the optimal value ψ, of the minimum mass plastic design problem as

$$\Psi^2 = \Pi V$$

Note that (4.38) is the natural formulation for the stress *only* reformulation of the minimum compliance problem stated as a corresponding equilibrium problem for a structure with a non-smooth, convex complementary energy.

Lecture 18: Topology design of truss structures_2

2.6 Extension to contact problems

The discussion above can be extended to problems involving unilateral contact, as we shall briefly outline in the following. The most natural setting for unilateral contact problems is a displacement based formulation. For an unilateral contact condition of the form $\mathbf{Cu}<0$, the minimum compliance problem for contact problems becomes

$$\max_{\substack{\mathbf{t}\geq 0 \\ \sum_{i=1}^{m} t_i = V}} \min_{\substack{\mathbf{u} \\ \mathbf{Cu}\leq 0}} \left\{ \frac{1}{2}\sum_{i=1}^{m} t_i \mathbf{u}^T \mathbf{K}_i \mathbf{u} - \mathbf{f}^T \mathbf{u} \right\}$$

where only the inner equilibrium problem is altered. The problem of finding the stiffest structure among all structures with constant contact pressure is also considered and in this case the unilateral constraint should be of the scalar form $\mathbf{l}_c\mathbf{Cu}<0$, corresponding to a total gap constraint $\mathbf{l}_c\mathbf{d} = 0$, where $\mathbf{d}$ is an initial gap which is designed to achieve constant pressure. This case is also covered by the statement (4.39), by proper choice of $\mathbf{C}$.

Lecture 18: Topology design of truss structures_2

The introduction of *design independent* constraints in the inner problem of (4.39) does not change the saddle point property of the problem. It does not make sense for contact problems to assume that the stiffness matrix is positive definite for at least one design. Instead, one has to assume that the applied force does not give raise to rigid body motions and that the applied force is not entirely applied at the potential contact nodes. With this assumption we have existence of solution and an equivalent displacements only problem in the form

$$\min_{\substack{\mathbf{u} \\ \mathbf{C}\mathbf{u} \leq \mathbf{0}}} \max_{i=1,\ldots,m} \left\{ \frac{V}{2} \mathbf{u}^T \mathbf{K}_i \mathbf{u} - \mathbf{f}^T \mathbf{u} \right\}$$

This problem can be solved by equivalent means as (4.25): the extra linear constraints **Cu** ≤0 does not influence the efficiency. Consider now the worst-case multiple load problem in the formulation which includes contact

$$\inf_{\substack{\lambda^k > 0 \\ \sum_1^M \lambda^k = 1}} \min_{\substack{\mathbf{u}^k \\ \mathbf{C}^k \mathbf{u}^k \leq \mathbf{0}}} \max_{i=1,\ldots,m} \left\{ \sum_{k=1}^{M} \left(\frac{V}{2} \frac{1}{\lambda^k} \mathbf{u}^{k\,T} \mathbf{K}_i \mathbf{u}^k - \mathbf{f}^{k\,T} \mathbf{u}^k \right) \right\}$$

Here we have related each load case to a potentially different contact condition. Computing the dual of the equilibrium problem, we obtain the complementary energy formulation in the form

Lecture 18: Topology design of truss structures_2

$$\inf_{\mathbf{t}} \min_{\mathbf{q}^k, \mathbf{p}^k} \left[\max_{k=1,\dots,M} \left\{ \frac{1}{2} \sum_{i=1}^{m} \frac{l_i^2}{E_i} \frac{(q_i^k)^2}{t_i} \right\} \right]$$

$$\text{s.t.} : \mathbf{B}\mathbf{q}^k = \mathbf{f}^k - {\mathbf{C}^k}^T \mathbf{p}^k, \quad \mathbf{p}^k \geq \mathbf{0}, \quad k = 1, \dots, M \ ,$$

$$\sum_{i=1}^{m} t_i = V, \quad t_i > 0, \quad i = 1, \dots, m \ ,$$

where the contact forces p^k also enter as variables. As for the non-contact case we can compute the optimal bar volumes and the resulting force-only formulation only change by the addition of the contact forces in the equilibrium constraint. For the single load case we get the disguised linear programming problem

$$\min_{\substack{\mathbf{q}, \mathbf{p} \geq \mathbf{0} \\ \mathbf{B}\mathbf{q} = \mathbf{f} - \mathbf{C}^T \mathbf{p}}} \left\{ \frac{1}{2V} \left[\sum_{i=1}^{m} \left(\frac{l_i}{\sqrt{E_i}} |q_i| \right) \right]^2 \right\}$$

For the displacement formulation one has, likewise, the LP formulation

$$\min_{\mathbf{u}} \left\{ -\mathbf{f}^T \mathbf{u} \right\}$$

$$\text{s.t.} : \mathbf{C}\mathbf{u} \leq \mathbf{0}; \qquad -1 \leq \sqrt{\frac{V E_i}{2}} \frac{\mathbf{b}_i^T \mathbf{u}}{l_i} \leq 1, \quad i = 1, \dots, m \ ,$$

taking the development "full circle" .

Lecture 18: Topology design of truss structures_2

We close this section by remarking that the minimum compliance problem with unilateral contact formulated as a

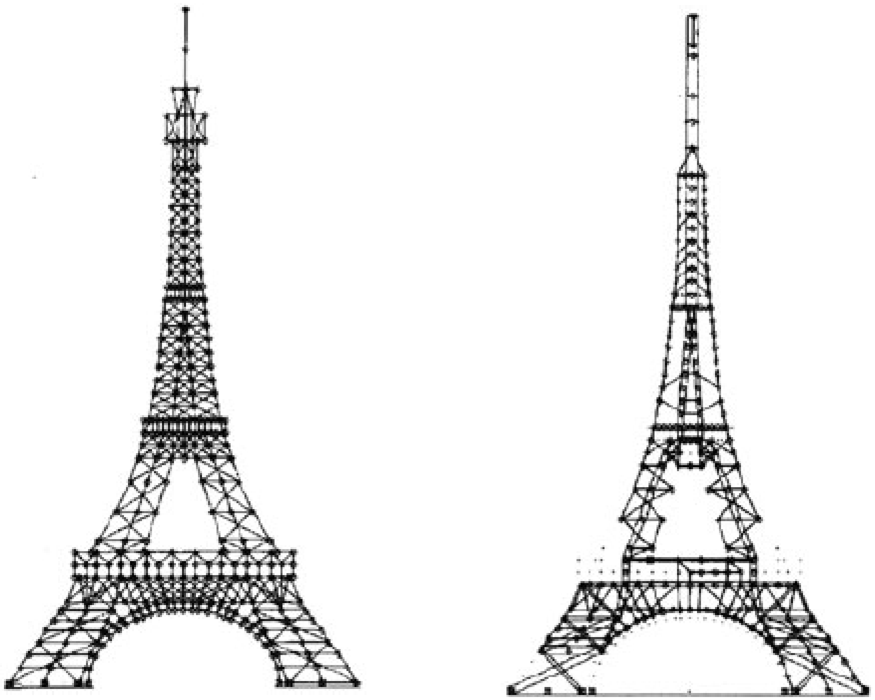
$$\sup_{\substack{\mathbf{t} > 0 \\ \sum_{i=1}^{m} t_i = V}} \left[\Phi(\mathbf{t}) = \min_{\substack{\mathbf{u} \\ \mathbf{C}\mathbf{u} \le \mathbf{d}}} \left\{ \frac{1}{2} \sum_{i=1}^{m} t_i \mathbf{u}^T \mathbf{K}_i \mathbf{u} - \mathbf{f}^T \mathbf{u} \right\} \right]$$

for positive definite stiffness matrices is actually a C^1-smooth problem in the bar volumes. Here we consider that an initial non-zero gap **d** is given. The derivatives of the functional Φ(t) (it is minus one half of the compliance) are given as

$$\frac{\partial}{\partial t_i} \Phi(\mathbf{t}) = \frac{1}{2} \mathbf{u}_*^T \hat{\mathbf{K}}_i \mathbf{u}_*, \text{ with } \mathbf{u}_* = \arg \min_{\substack{\mathbf{u} \\ \mathbf{C}\mathbf{u} \le \mathbf{d}}} \left\{ \frac{1}{2} \sum_{i=1}^{m} t_i \mathbf{u}^T \hat{\mathbf{K}}_i \mathbf{u} - \mathbf{f}^T \mathbf{u} \right\}$$

Note, however, that the displacements are not differentiable as functions of bar volumes, as the displacements are non-smooth at designs were there are active contact nodes with zero contact forces. This feature means that most other design problems which involves contact conditions are non-smooth problems. Nonetheless, directional derivatives can be computed.

Lecture 18: Topology design of truss structures_2



The flexibility in choice of ground structure. Optimal design of a well-known structure. Left hand picture shows the ground structure and the right hand picture the optimal topology for a single downward load at the top of the structure. The example shows that it is crucial to consider multiple load cases for realistic structures.

Lecture 18: Topology design of truss structures_2

3. Computational procedures and examples

The availability of efficient methods to solve large (sparse) LP problems makes it natural to solve the single load truss topology design problem using the LP formulations. For problems with multiple loads and/or bounded bar areas, for the reinforcement problem as well as for the FEM case, we cannot obtain a linear programming formulation of the minimum compliance problem and we are forced to solve such problems by other means. Problems of the form (4.1)-(4.5) and (4.11)-(4.13) generalize most easily to more general design situations involving stress and displacement constraints but they are large scale and non-convex. The optimality criterion method is a good and easily programmed option for solving this problem in the minimum compliance setting, if suitable lower bounds on the bar volumes are imposed. Problems (4.24)-(4.28) and (4.25), (4.29) are convex and have the size of the degrees of freedom of the ground structure; the former are non-differentiable and unconstrained and the bound formulations are differentiable, but at the cost of a high number of constraints. Below we shall present a specialized and physically intuitive algorithm for solving problem (4.24); it can be easily implemented to take advantage of the sparsity of the Matrices K_i, but is actually not efficient compared to other methods based on the smooth formulations. Problem (4.24) has for some time been used as a "difficult" test case for general purpose algorithms for min-max optimization or non-differentiable optimization. The most efficient approach (when LP-codes cannot be applied) is to use modern penalty methods for problems (4.25), (4.29) which can be solved by such general purpose algorithms. Alternatively, the SDP format (4.20)-(4.22) can be used. In both cases sparsity and the fact that the number of variables is much lower than the number of constraints should be utilized. It should be emphasized that the truss topology design problem is a very challenging mathematical programming problem with structure and properties which are a test for even the best of algorithms.

Lecture 18: Topology design of truss structures_2

3.1 An optimality criteria method

For the continuum problems treated earlier the optimality criteria method is an effective and general mean for solving minimum compliance problems. Also for truss topology design this is a simple computational procedure, but it is not as effective as other approaches based on interior point or SDP techniques. However, it is physically intuitive as the method assigns material to members proportionally to the specific energy of each member in order to reach the situation of constant specific energy in the active bars. Thus each iteration step consists of the following

$$\text{For } t_i^{K-1} \text{ given, compute } \mathbf{u}_{K-1} \text{ from the equilibrium eqs.}$$

$$\text{Find } \Lambda^K \text{ so } \sum_{i=1}^{m} \max\left\{ t_i^{K-1} \frac{\mathbf{u}_{K-1}{}^T \hat{\mathbf{K}}_i \mathbf{u}_{K-1}}{\Lambda^K},\ t_{\min} \right\} = V\ ,$$

$$\text{Update } t_i^K = \max\left\{ t_i^{K-1} \frac{\mathbf{u}_{K-1}{}^T \hat{\mathbf{K}}_i \mathbf{u}_{K-1}}{\Lambda^K},\ t_{\min} \right\} .$$

The linearity of stiffness and volume in the bar areas implies that the optimality criteria algorithm for the single load case can be viewed as a fully stressed design algorithm, and it is as such a fix point algorithm. Also, the method can be viewed as an implementation of a sequential quadratic programming technique. Also, the similarity to convex approximation techniques as MMA has been outlined.

Lecture 18: Topology design of truss structures_2

The optimality criteria method involves assembly of the global stiffness matrix as well as solving the equilibrium problem at each iteration step, and this part of the algorithm is the most time consuming. Note, that for $t_{min} \sim 0.0$ the algorithm can utilize that the volume is linear in the design variables, so that satisfying the volume constraint is just a rescaling of variables. However, the algorithm does not take advantage of the fact that also the stiffness matrix is linear in the design variables. Also for the single load case truss topology problem (4.1) we have that the matrices K_i are dyadic products and this is not used either.

3.2 A non-smooth descent method

One can also devise physically intuitive algorithms that work with the displacements as the primary variables. The basis is then the equivalent problems (4.24)-(4.28). We will here describe an "f-steepest descent" method for these non-smooth problems. The algorithm is actually inefficient, but, as mentioned, it is physically intuitive, and it is closely related to the optimality criteria algorithm. Even though the algorithm solves a problem in the displacement variables u it also generates the bar volumes t from an inner problem. This contrasts to the standard procedure in optimal structural design where one solves for the design variables, with the displacements removed via the state equation and adjoint equation. We describe the algorithm for the topology design problem with external loads as well as loads due to self-weight. Thus the algorithm for problem (4.24)

Lecture 18: Topology design of truss structures_2

$$\min_{\mathbf{u}} \left[F(\mathbf{u}) = \frac{V}{2} \max_{i=1,\ldots,m} \left\{ \mathbf{u}^T \hat{\mathbf{K}}_i \mathbf{u} - 2\mathbf{g}_i^T \mathbf{u} \right\} - \mathbf{f}^T \mathbf{u} \right]$$

consists of the following very intuitive steps:

0. Compute an initial guess of displacement field $\mathbf{u}$, for example by solving the equilibrium equations for a feasible set of bar volumes $\mathbf{t}$.
1. For present $\mathbf{u}$, compute $\Lambda^*(\mathbf{u}) = \max_{i \in R} \left\{ \mathbf{u}^T \hat{\mathbf{K}}_i \mathbf{u} - 2\mathbf{g}_i^T \mathbf{u} \right\}$, and indices

$$J(\mathbf{u}) = \left\{ i \in R | \mathbf{u}^T \hat{\mathbf{K}}_i \mathbf{u} - 2\mathbf{g}_i^T \geq \Lambda^*(\mathbf{u}) - \epsilon \right\}$$

2. Compute descent direction $\mathbf{d}$ as $\mathbf{d} = -\sum_{i \in J} t_i \left[\hat{\mathbf{K}}_i \mathbf{u} - \mathbf{g}_i \right] + \mathbf{f}$, where t_i, $i \in J$ are found from

$$\min_{\substack{t_i \geq 0,\, i \in J \\ \sum_{i \in J} t_i = V}} \left\{ \| \sum_{i \in J} t_i \left[\hat{\mathbf{K}}_i \mathbf{u} - \mathbf{g}_i \right] - \mathbf{f} \|^2 - \sum_{i \in J} t_i \mathbf{u}^T \left[\hat{\mathbf{K}}_i \mathbf{u} - 2\mathbf{g}_i \right] \right\}$$

3. If $\|\mathbf{d}\| \leq \delta$, stop. Else go to 4.
4. Compute a step size α^* for the update $\mathbf{u} := \mathbf{u} + \alpha \mathbf{d}$, by a line search (Golden Section method) with the function

$$\Psi(\alpha) = F(\mathbf{u} + \alpha \mathbf{d}) = \max_{i=1,\ldots,m} \left\{ \overline{a}_i \alpha^2 + \overline{b}_i \alpha + \overline{c}_i \right\} ,$$

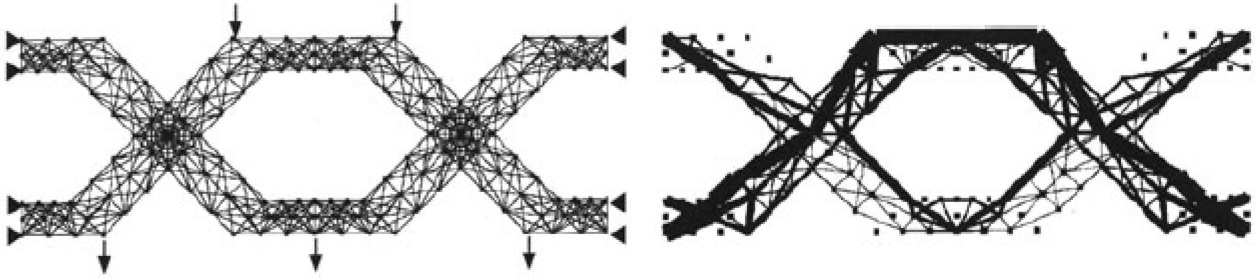
$$\overline{a}_i = \frac{V}{2} \mathbf{d}^T \hat{\mathbf{K}}_i \mathbf{d}, \quad \overline{b}_i = \left[V(\hat{\mathbf{K}}_i \mathbf{u} - \mathbf{g}_i) - \mathbf{f} \right]^T \mathbf{d} ,$$

$$\overline{c}_i = \left[\frac{V}{2} (\hat{\mathbf{K}}_i \mathbf{u} - 2\mathbf{g}_i) - \mathbf{f} \right]^T \mathbf{u} .$$

5. Update, $\mathbf{u} := \mathbf{u} + \alpha^* \mathbf{d}$, and go to step 1.

Lecture 18: Topology design of truss structures_2

Here, ε is a relaxation on the activity set ***J*** which is crucial to guarantee the convergence of the algorithm, and δ determines the accuracy of the solution (one works with decreasing sequences of these parameters) . Each iteration loop of the algorithm consists of first finding the set of almost active bars (Step 1) . The descent direction (Step 2) is then found by first finding the bar volumes of these bars which minimizes the error in equilibrium for the given estimate of displacement. The error is measured in a least squares sense and the descent direction is proportional to the residual of the equilibrium for this best fit of bar volumes. The algorithm can be implemented to take full advantage of sparsity, both in storage and in computations. For example one notes that the full stiffness matrix is not required. For a proof of the convergence of the algorithm.



An example of a complicated ground structure geometry, with 156 nodal points and 660 potential bars. The ground structure, supports and five loads are shown at the left. The resulting topology for a weighted average, multiple load problem formulation is shown right. The ground structure was generated by an interactive CAD-based programme

3.3 SDP (Semi-Definite Programming) and interior point methods

Methods working with bar volumes It is the provision of a lower bound on the bar volumes that allows for the use of the very effective optimality criterion method. A similar efficiency can be obtained by considering the problem of taking the infimum of the compliances for all truss structures with positive bar volumes

$$\inf_{\substack{t_i \geq 0 \\ \sum_{i=1}^m t_i = V}} \left\{ \Phi(\mathbf{t}) = \mathbf{f}^T \left[\sum_{i=1}^m t_i \hat{\mathbf{K}}_i \right]^{-1} \mathbf{f} \right\}$$

As shown earlier this problem is convex and this in combination with the inf-form makes it ideally suited for interior point barrier methods as this will imply that the positivity constraint on the bar volumes will be satisfied automatically. Problem (4.43) does not lend itself to the use of sparse techniques, as the Hessian of the objective function Φ(t) is full. However, the Hessian of the constraint $\sum_{i=1}^m t_i \hat{\mathbf{K}}_i u = \mathbf{f}$

is sparse. Sparsity can thus be utilized if the problem (4.5) in both the displacement and design variables is solved using an interior point method. Even though the latter problem is not convex, finding a stationary solution provides also a stationary point for problem (4.43), and thus a minimizer for this convex problem. This approach extends readily to all the problem types described above. The use of an interior barrier method for problem (4.43) involves the use of a suitable sequence of penalty parameters, which in effect corresponds to imposing a constraint of the type

Lecture 18: Topology design of truss structures_2

$$t_i \geq t_{\min} > 0, i = 1, \ldots, m$$

for a suitable small lower bound value t_{min}. This can make it troublesome to identify precisely which bars are active in the optimal topology. However, convergence of the designs (and relevant displacements) as we take the limit t_{min} ->0 is guaranteed.

For the worst-case multiple load problem (4.4), formulated as a smooth problem using a bound formulation with bounding variable α, a possible logarithmic barrier function is of the form

$$\min_{\mathbf{t},\, \alpha} \left\{ -\sum_{k=1}^{M} \ln(\alpha - \mathbf{f}^{k^T} \left[\sum_{i=1}^{m} t_i \hat{\mathbf{K}}_i \right]^{-1} \mathbf{f}^k) - \sum_{i=1}^{m} \ln(t_i) - \ln(\alpha_{\max} - \alpha) \right\}$$

where α_{max} is a suitable guaranteed upper bound on the optimal value of the problem. We also note that the SDP variations of the topology design problem can be solved by algorithms developed for such problems; such techniques have many common features with the logarithmic barrier approach just outlined.

Lecture 18: Topology design of truss structures_2

Methods working with displacements Barrier function methods and especially the so-called" Penalty /Barrier /Multiplier (PBM) Method" can also to great advantage be used for the displacements only formulations of the form (4.24). In order to apply the PBM method to for example the min-max multiple load truss topology design problem, the formulation (4.33) is used in a form where the discrete maximization over bar numbers is removed by a bound formulation

$$\inf_{\substack{\lambda^k > 0,\, \mathbf{u}^k,\, \tau \\ \sum_{k=1}^{M} \lambda^k = 1}} \left\{ V\tau - \sum_{k=1}^{M} \mathbf{f}^{k\,T} \mathbf{u}^k \right\}$$

$$\text{s.t.}: \sum_{k=1}^{M} \frac{1}{2\lambda^k} \mathbf{u}^{k\,T} \mathbf{K}_i \mathbf{u}^k - \tau \le 0, \quad i = 1, \dots, m\,.$$

Note that (4.44) is a smooth convex optimization problem. It can be shown from the Karush-Kuhn-Thcker conditions of problem (4.44) that the Lagrange multipliers for the constraints on the specific energies are precisely the optimal volumes of the bars in the optimal topology. Hence the optimal bar volumes are approximated directly at each iteration step of the PBM method by the Lagrange multipliers for these constraints. Notice that a further reformulation is handy, namely the formulation

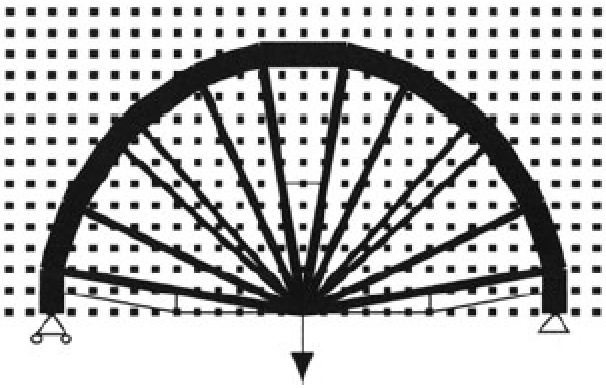
$$\inf_{\substack{s^k,\, \mathbf{x}^k,\, \tau \\ \sum_{k=1}^{M} (s^k)^2 = 1}} \left\{ V\tau - \sum_{k=1}^{M} s^k\, \mathbf{f}^{k\,T} \mathbf{x}^k \right\}$$

$$\text{s.t.}: \sum_{k=1}^{M} \mathbf{x}^{k\,T} \mathbf{K}_i\, \mathbf{x}^k - 2\tau \le 0, \quad i = 1, \dots, m\,,$$

Lecture 18: Topology design of truss structures_2

which is derived from previous equation by the transformation $s^k = \sqrt{\lambda^k}$, $\mathbf{x}^k = \mathbf{u}^k / \sqrt{\lambda^k}$

For a truss with *N* degrees of freedom, *m* potential bars and *M* load cases, the single load problem (4.29) has *N* variables and *m* constraints, while problem (4.45) has *NM*+*M*+1 variables and *m* non-linear constraints. The main computational effort in applying the PBM method is the minimization of the unconstrained penalty/barrier function. This is done using a Newton method, and it is interesting to note that the method does not require an increase in the number of Newton steps as the problem size increases. Note that each Newton step corresponds to solving a linear system of equations, which for the single load case is comparable in size to the linear system solved for one full equilibrium analysis step of the "Optimality Criteria Method" .



A detailed study of the design of a wheel with a 29 by 15 nodal lay-out with all 57770 possible non-overlapping connections.

Lecture 18: Topology design of truss structures_2

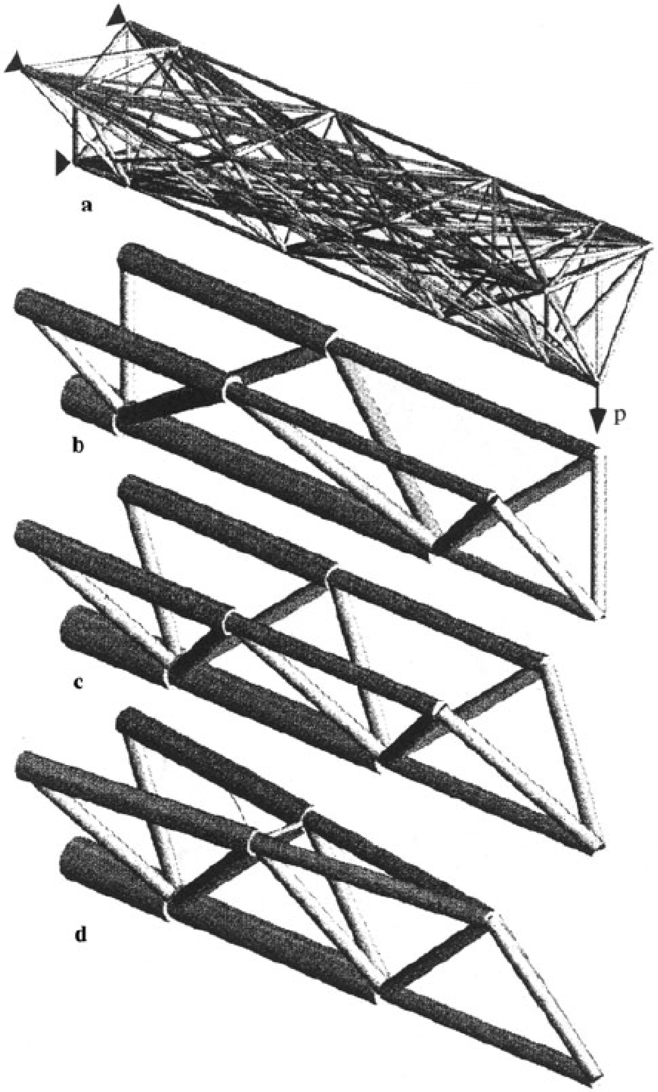
4. Extensions of truss topology design

4.1 Combined truss topology and geometry optimization

The topology design methods considered so far all employ the basic idea of a ground structure or reference design domain to obtain problem statements that are sizing problems for a fixed geometry. The choice of this reference geometry influences the result of the topology optimization making it important to consider sensitivity analysis of the optimal designs with respect to variation of the reference geometry, and even optimal design of this reference geometry may be fruitful in some situations.

In the ground structure approach to topology design of trusses the positions of nodal points are not used as design variables. This means that a high number of nodal points should be used in the ground structure to obtain efficient topologies. A drawback of the method is that the optimal topologies can be very sensitive to the layout of nodal points, at least if the number of nodal points is relatively low. This makes it natural to consider an extension of the ground structure approach and to include the optimization of the nodal point location for a given number and connectivity of nodal points. *With very efficient tools at hand for the topology design with fixed nodal positions it seems natural to treat the variation of nodal positions as an outer optimization in a two-level hierarchical formulation.* As the optimal value function of the topology compliance depends on the geometry variables in a non-smooth way, this outer minimization requires non-smooth optimization techniques.

Lecture 18: Topology design of truss structures_2



An example of a 3-D topology and geometry optimization for a beam carrying a single load. In a) we show the ground structure of nodal points and potential bars. Note that the ground structure has non-equidistant nodal point positions along the length axis of the "beam" . In a) we see the optimal topology for the fixed nodal lay-out of the ground structure, in c) a combined geometry and topology optimization with nodal positions restricted to move along the length axis of the 'beam". Finally, in d) the result of a combined geometry and topology optimization with totally free nodal positions is shown. The (non-dimensional) compliance values of the optimized designs are 1.00, 0.945 and 0.911, respectively.

Lecture 18: Topology design of truss structures_2

For the combined topology and geometry problem for trusses we have as the simplest formulation

$$\min_{\mathbf{u},\mathbf{a},\mathbf{x}} \mathbf{f}^T\mathbf{u}$$

$$\text{s.t. : } \sum_{i=1}^{m} a_i l_i(\mathbf{x})\mathbf{K}_i(\mathbf{x})\mathbf{u} = \mathbf{f} \ ,$$

$$\sum_{i=1}^{m} a_i l_i(\mathbf{x}) = V, \quad a_i \geq 0, \ i = 1, \ldots, m \ ,$$

$$\bar{x}_j^k \leq x_j^k \leq \hat{x}_j^k, \ j = 1, \ldots, n, \ k = 1, 2, (3) \ ,$$

which is just problem (4.5) rewritten as a problem depending also on the nodal positions X_j, j = 1, ... , n. The nodal positions are restricted to lie within certain bounds that should be chosen to make the resultant trusses realizable. As the member volumes are dependent on the nodal positions we have here reverted to the cross-sectional areas of the individual bars as design variables. Problem above can be solved as a unified problem considering the problem either as a unified analysis and design problem or as a standard structural optimization problem that can be solved through an adjoint method in the areas and nodal positions only (this requires the application of small lower bounds on the cross sectional areas). An alternative solution procedure is to apply a multilevel approach to the combined problem, treating the topology problem as the inner problem. Because of the size of the topology problem, earlier work has usually involved some form of heuristics to speed up the very significant amount of computations involved. Here we consider combining the effective truss topology design methods described earlier with appropriate tools from non-smooth analysis.

Lecture 18: Topology design of truss structures_2

For a fixed set of nodal positions we choose here the displacements only, form (4.24) of the topology design problem (without self-weight) and thus write previously as a two-level problem

$$\min_{\substack{\mathbf{x} \\ \bar{\mathbf{x}} \le \mathbf{x} \le \hat{\mathbf{x}}}} \left(-\min_{\mathbf{u}} \left[\max_{i=1,\ldots,m} \left\{ \frac{V}{2} \mathbf{u}^T \mathbf{K}_i(\mathbf{x}) \mathbf{u} - \mathbf{f}^T \mathbf{u} \right\} \right] \right)$$

The inner topology problem in the displacements u can effectively be solved (for fixed x) by one of the computational methods described previously. The main part remaining is then, of course, the minimization of the so-called master function

$$\mathcal{F}(\mathbf{x}) = -\min_{\mathbf{u}} \left[\max_{i=1,\ldots,m} \left\{ \frac{V}{2} \mathbf{u}^T \mathbf{K}_i(\mathbf{x}) \mathbf{u} - \mathbf{f}^T \mathbf{u} \right\} \right]$$

on the outer level. The number of variables (the nodal positions) in this outer problem will usually be moderate. However, there are two decisive drawbacks. There is no reason for F to be convex and F is not differentiable everywhere. Hence we cannot expect to find more than a local minimum of F and we have to work with codes from non-smooth optimization. These codes require that for each iterate x we can compute a so-called sub-gradient as a substitute for the gradient. Using tools from non-smooth calculus it is easily seen that this causes no difficulties for the above min-max function F.

Lecture 18: Topology design of truss structures_2

The two-level approach becomes especially attractive if we consider the single load truss topology problem for which the member stiffness matrices are dyadic products. Then *F(x)* reduces to a parametrized linear programming problem

$$\mathcal{F}(\mathbf{x}) = \max_{\mathbf{u}} \left\{ \mathbf{f}^T \mathbf{u} \,\middle|\, -1 \leq \sqrt{\frac{VE_i}{2}} \frac{\mathbf{b}_i(\mathbf{x})^T \mathbf{u}}{l_i(\mathbf{x})} \leq 1, \quad i = 1, \ldots, m \right\}$$

The sub-gradient in this case is basically the derivative with respect to x of the Lagrange function for this LP-problem. Hence we get a sub-gradient "for free" when solving (4.30) for a given set of nodal positions x.

4.2 Truss design with buckling constraints

Here, we formulate a problem of optimum truss topology design including a constraint on the *global stability* of the structure. We use here the so-called *linear buckling model* as the model of stability. This means that we can express the stability condition as the condition $\mathbf{K}(\mathbf{t}) + \mathbf{G}(\mathbf{u}, \mathbf{t}) \succeq \mathbf{0}$,

where **u** solves the small-deflection equilibrium equation $\mathbf{K}(\mathbf{t})\mathbf{u} = \mathbf{f}$

and where **G** is the standard *geometric matrix* of the truss. Note that *local* buckling constraints require a separate treatment.

Lecture 18: Topology design of truss structures_2

Now we can add this stability constraint to the compliance constrained minimum weight truss topology problem to obtain a problem of the form

$$\min_{\mathbf{t},\mathbf{u}} \sum_{i=1}^{m} t_i$$

$$\text{s.t.}: \mathbf{K}(\mathbf{t})\mathbf{u} = \mathbf{f}, \quad \mathbf{f}^T\mathbf{u} \le \tilde{\Phi}\,,$$

$$\mathbf{K}(\mathbf{t}) + \mathbf{G}(\mathbf{u},\mathbf{t}) \succeq \mathbf{0}, \quad t_i \ge 0,,\ i = 1,\dots,m\,.$$

Here Φ is the given maximal value of the compliance. Note that we can always find a feasible pair (t, u), as one can always make the truss stable and also stiff enough by adding enough material to each member (in this sense compliance and stability are not competing objectives). We have seen that one can use a linear matrix inequality to eliminate the displacement vector from an equilibrium equation that is combined with a compliance constraint. We also here rewrite above using this idea and arrive at new problem in the variable **t** only:

$$\inf_{\mathbf{t}>\mathbf{0}} \sum_{i=1}^{m} t_i$$

$$\text{s.t.}: \widetilde{\mathbf{A}}(\mathbf{t},\tilde{\Phi}) \succeq \mathbf{0}, \quad \mathbf{A}(\mathbf{t}) + \widetilde{\mathbf{G}}(\mathbf{t}) \succeq \mathbf{0}\,,$$

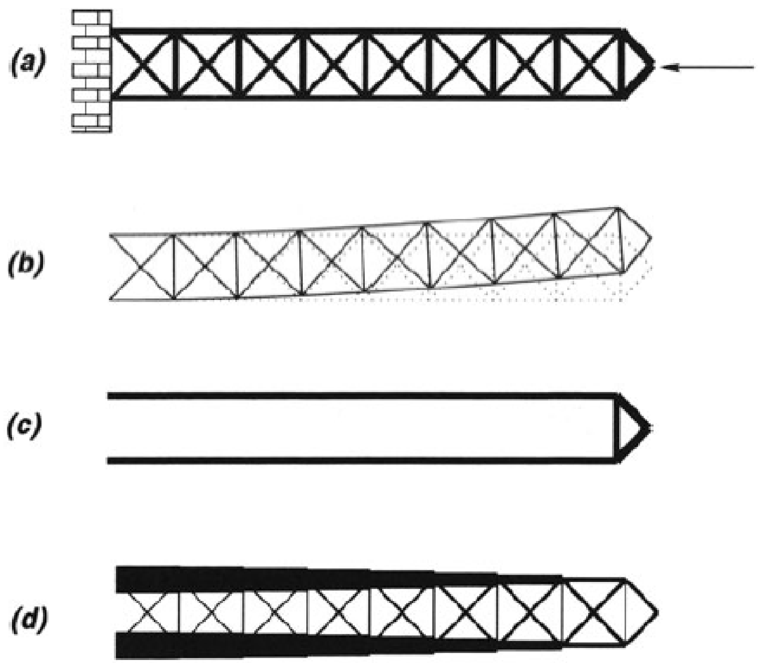
$$t_i \ge 0,\ i = 1,\dots,m\,.$$

where A is defined in (4.19) and where $\widetilde{\mathbf{G}}(\mathbf{t}) = \mathbf{G}(\mathbf{K}(\mathbf{t})^{-1}\mathbf{f},\mathbf{t})$

This problem is, due to the buckling constraint, not a standard convex SDP. However, local minima can still be found efficiently by a non-convex version of a PBM algorithm for SDP problems.

Lecture 18: Topology design of truss structures_2

An example of what can be achieved with this formulation is illustrated in Figure below. Here the initial design has $t_i = 1000/m$, $i = 1, \ldots, m$, with a corresponding compliance of 0.177 and a critical force (0.397) that is smaller than one, meaning that the truss is unstable. The standard truss optimization without stability constraint gives a design that is twice as light as the previous one but absolutely unstable. Finally, by truss optimization with stability constraint one can obtain a design that at a volume of 1179.6 is a bit heavier than the first one. However, it is stable under the given load. To see fully the effect of the stability constraint, we have for this example chosen the upper bound for the compliance so that the compliance constraint is not active. For truss (and frame) models *local* buckling of the individual members is also an important aspect to take into consideration.



The effect of a constraint on the global buckling load, a) shows the initial truss with its buckling mode in b). c) is the optimal truss without stability constraint and, d) is the optimal truss with a stability constraint.

Lecture 18: Topology design of truss structures_2

4.3 Control of free vibrations

The SDP approach from the previous section can easily be adapted to optimization of trusses with constraints on the free vibration frequencies. Here we formulate the problem of minimizing the weight (volume) of a truss subject to a compliance constraint and such that the lowest eigenfrequency is bigger than or equal to a prescribed value. This latter condition is written as

$$\lambda_i \geq \bar{\lambda}, \qquad i = 1, \ldots, n,$$

where λ_i are the eigenvalues of the problem $(\mathbf{K}(\mathbf{t}) - \lambda \mathbf{M}(\mathbf{t}))\mathbf{u} = 0$. Here $\mathbf{M}(\mathbf{t}) = \mathbf{M}_s(\mathbf{t}) + \mathbf{M}_0$ is the mass matrix of the truss; $\mathbf{M}_0$ is the part corresponding to non-structural mass and the (lumped) structural mass matrix is denoted by $\mathbf{M}_s(\mathbf{t})$.

The condition above can be written as a matrix inequality $\mathbf{K} - \bar{\lambda}\mathbf{M} \succeq 0$,

and, in parallel to previous problem, a minimum weight truss design problem with free vibration constraints can be formulated as follows

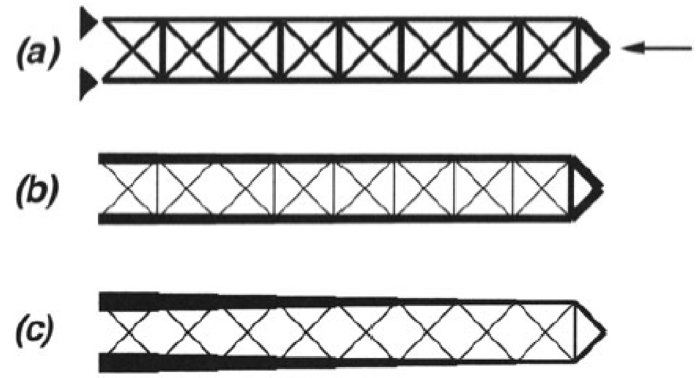
$$\min_{\mathbf{t} \geq \mathbf{0}} \sum_{i=1}^{m} t_i$$

$$\text{s.t. : } \mathbf{K}(\mathbf{t}) - \bar{\lambda}\mathbf{M}(\mathbf{t}) \succeq 0, \quad \widetilde{\mathbf{A}}(\mathbf{t}, \tilde{\Phi}) \succeq \mathbf{0} \ .$$

Lecture 18: Topology design of truss structures_2

This problem includes only linear matrix inequalities (the mass matrix does not depend on u) and is then much easier to solve than the stability problem. However, note that, in contrast to the stability problem, we may now have problems with feasibility. When **λ** is too big one may not be able to find any design **t** satisfying the vibration constraint; both of the matrices **A(t)** and **M(t)** are positive semidefinite and their eigenvalues grow with the same rate for increasing **t**. Note that the previous formulation avoids the technicalities of a formulation that directly and explicitly involves the eigenvalues. The non-differentiability of eigenvalues is circumvented by the application of interior point methods to the matrix-inequality constraint.

As an example we consider again a slender truss fixed on the left-hand side and subject to axial force on the right-hand side; there is a non-structural mass 1.0 at the column tip. The lowest eigenfrequency corresponding to the truss shown in right Figure (a) is $\lambda_{min} = 5.0\times10^{-5}$, while λ_{min} corresponding to the optimal truss without vibration constraint is zero (the truss is a mechanism, i.e. previous figure c). If we solve previous equation with $\lambda_{min} = 5.0\times10^{-5}$, the resulting truss weighs 0.5684 (see (b)). When we try to increase the minimum eigenvalue, setting $\lambda = 1.0\times10^{-4}$, we get the truss shown in (c) that weighs 0.8746.



Initial truss (a) and two optimal trusses with respect to free vibration constraint with $\lambda = 5.0\times10^{-5}$ (b) and $\lambda = 1.0\times10^{-4}$ (c)

Thank you for your attention