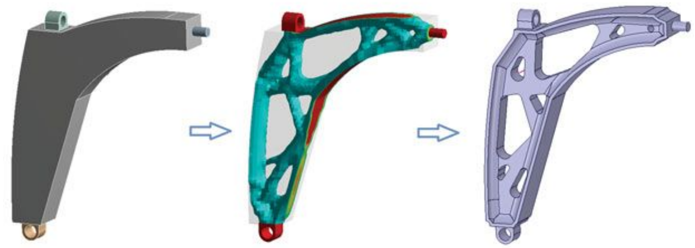
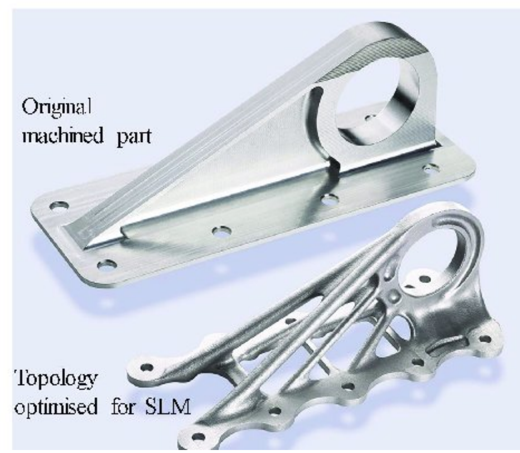
香港中文大學
The Chinese University of Hong Kong

MAEG5160: Design for Additive Manufacturing

Lecture 5: Finite Element for Topological Optimization





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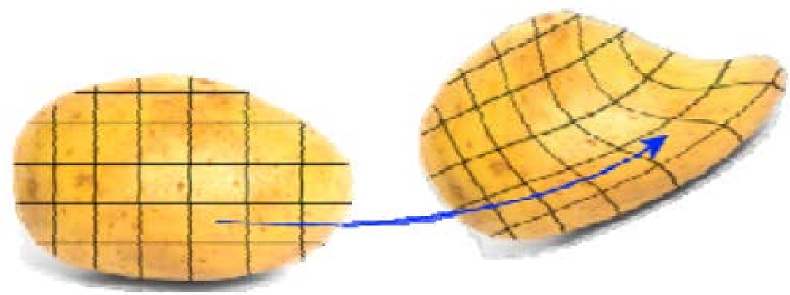
Lecture 5: Finite Element for Topological Optimization (without intensive mathematics)

1. A very brief guide into Continuum Mechanics

It starts with observations…



- **Deformations** (displacement)
  - Vector function that maps a material point into its new coordinate, i.e.

$$\boldsymbol{u} = [u(x,y,z), v(x,y,z), w(x,y,z)]^T$$

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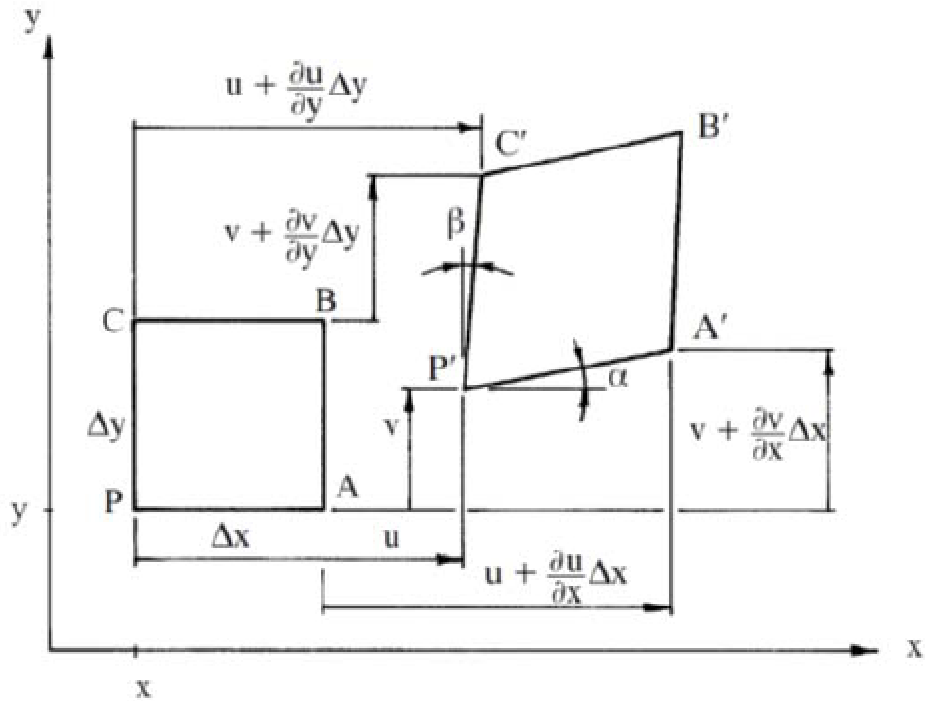
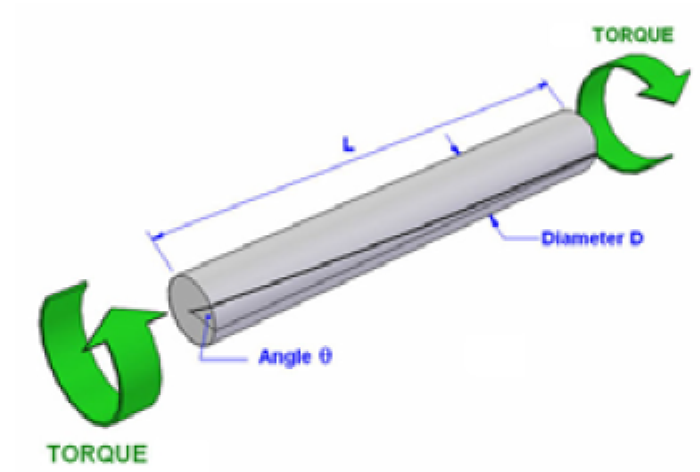
- **Strains** (measurable) - relative deformation





$$\epsilon_x = \frac{\partial u}{\partial x}, \qquad \epsilon_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

$$\epsilon_y = \frac{\partial v}{\partial y}, \qquad \epsilon_{xz} = \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)$$

$$\epsilon_z = \frac{\partial w}{\partial z}, \qquad \epsilon_{yz} = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)$$

- Def.: $\epsilon := \dfrac{\Delta L}{L}$ (Linear!) - general:

(elongations - rotations)

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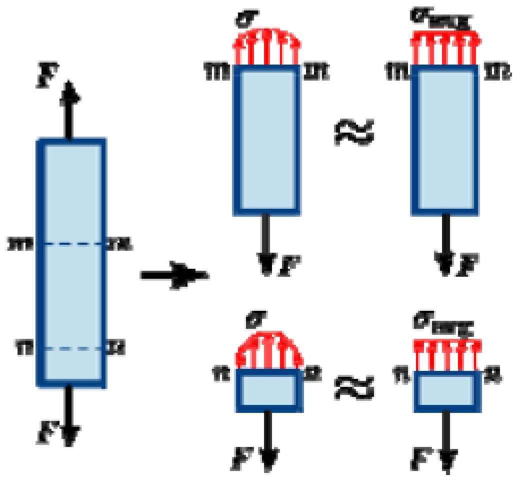
- **Stresses** (NOT measurable):



Important – the stress depends on the point (position) AND the orientation of cut-surface.

- Def.: $\sigma_{avg} := \dfrac{F}{A}$ or $\sigma = \lim_{A \to 0} \dfrac{F}{A}$

- General stress state: (similar to strains)

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_x & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_y & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_z \end{bmatrix}$$

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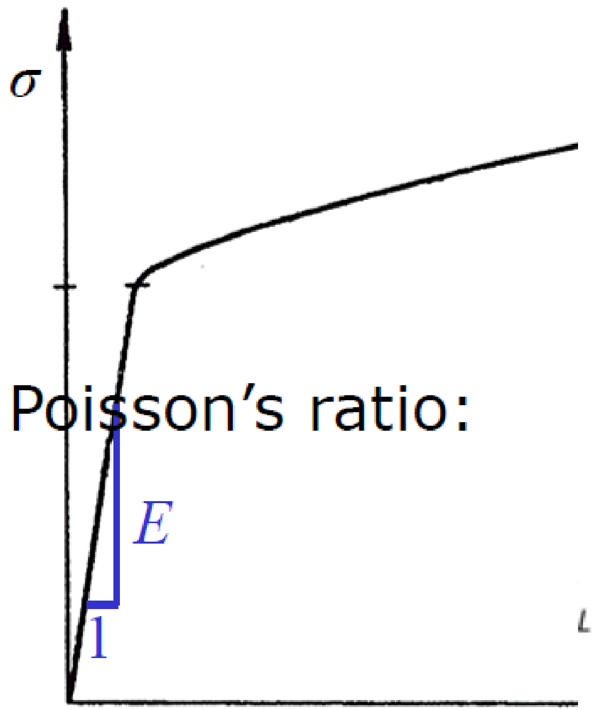
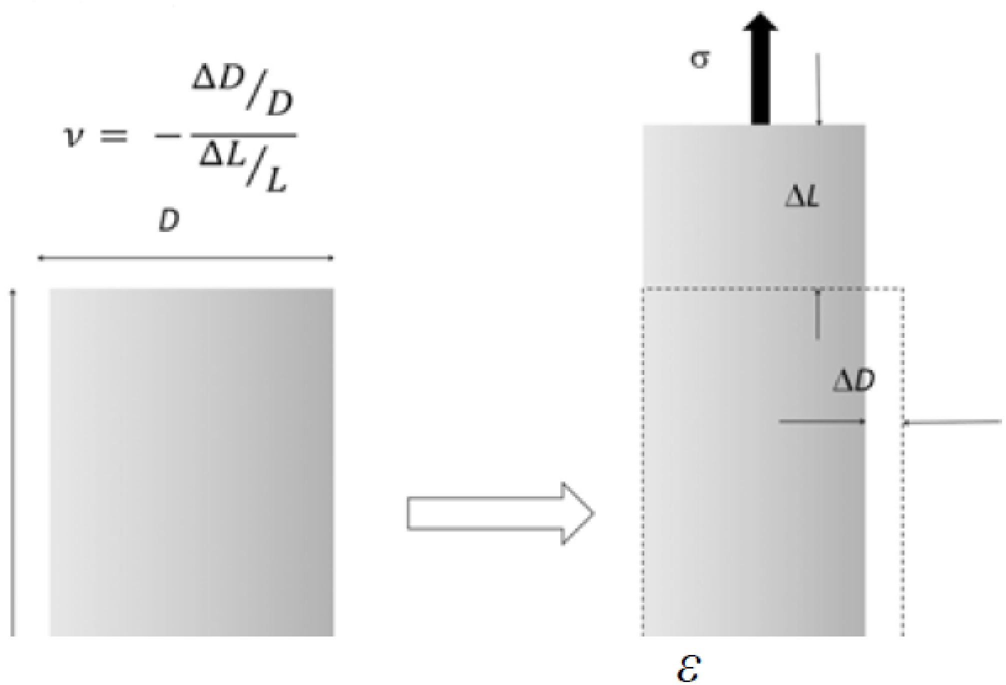
- Hooke's law – linear, isotropic materials:
  Just two independent material parameters

- Stiffness: $\sigma = E\epsilon$ (E in [Pa])



- Poisson's ratio:



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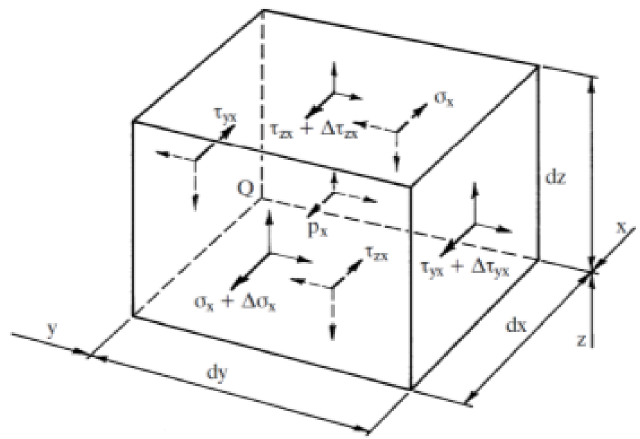
Governing equations (using Newton's 2nd law)



The linear system of partial differential equations:

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + p_x = 0$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + p_y = 0$$

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + p_z = 0$$

or

$$(\lambda + \mu)\nabla(\nabla \cdot \boldsymbol{u}) + \mu^2 \nabla^2 \boldsymbol{u} + \boldsymbol{p} = \boldsymbol{0}$$

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}$$

$$\mu = \frac{E}{2(1+\nu)}$$

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- Essential since it allows us to interpolate, e.g. stiffness, density, conductivity, …

$$E(\rho) = E_{\min} + \rho^p (E_{\max} - E_{\min})$$

Different problems need different interpolations

- Principle of virtual work

$$\int_\Omega \delta \boldsymbol{\epsilon}^T \boldsymbol{E}(\rho) \boldsymbol{\epsilon} \mathrm{d}\Omega - \int_\Omega \delta \boldsymbol{u}^T \boldsymbol{P} \mathrm{d}\Omega + \int_{\Gamma_T} \delta \boldsymbol{u}^T \boldsymbol{t} \mathrm{d}\Gamma_T = 0$$

- The finite element method (FEM)

$$\boldsymbol{K}(\boldsymbol{\rho})\boldsymbol{U} = \boldsymbol{F}$$

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- The von Mises stress (or equivalent tensile stress):

$$\sigma_{vM} = \sqrt{3J_2} \quad \text{or}$$

$$\sigma_{vM}^2 = \frac{1}{2}\left[(\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2 + 6(\sigma_{xy}^2 + \sigma_{xz}^2 + \sigma_{yz}^2)\right]$$

- The strain energy and compliance:

$$U = \frac{1}{2}\int_\Omega \boldsymbol{\sigma}^T \epsilon \mathrm{d}\Omega \quad \text{and} \quad C = \boldsymbol{u}^T \boldsymbol{F} = \boldsymbol{u}^T \boldsymbol{K} \boldsymbol{u}$$

- Stiffness vs compliance: $E = \frac{\partial \sigma}{\partial \epsilon}$ vs $C = \frac{\partial \epsilon}{\partial \sigma}$

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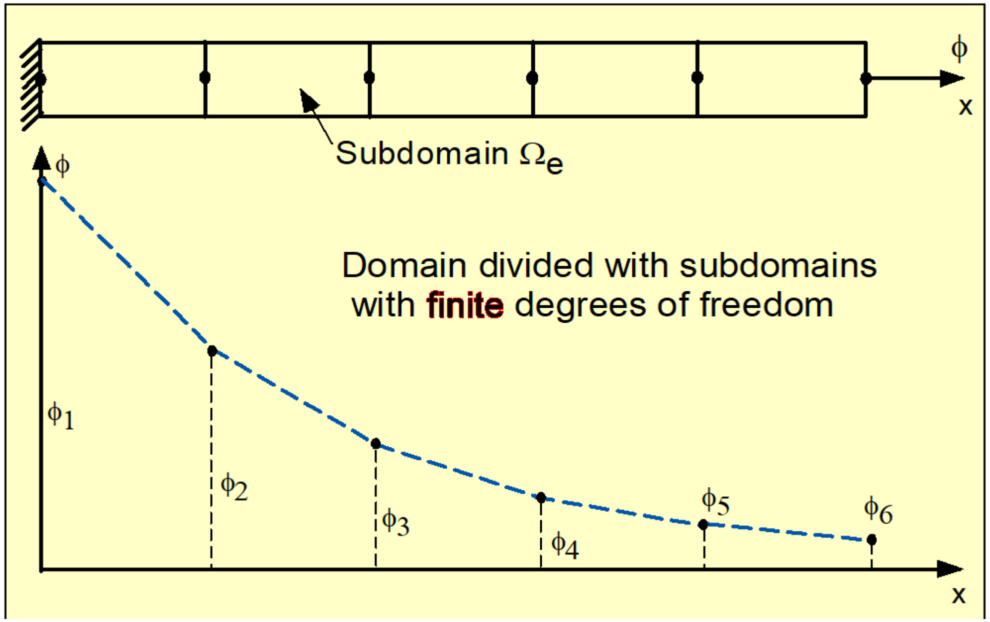
2. Another brief guide into Finite Element Method

- A continuous function of a continuum (given domain Ω) having infinite degrees of freedom is replaced by a discrete model, approximated by a set of piecewise continuous functions having a finite degree of freedom.

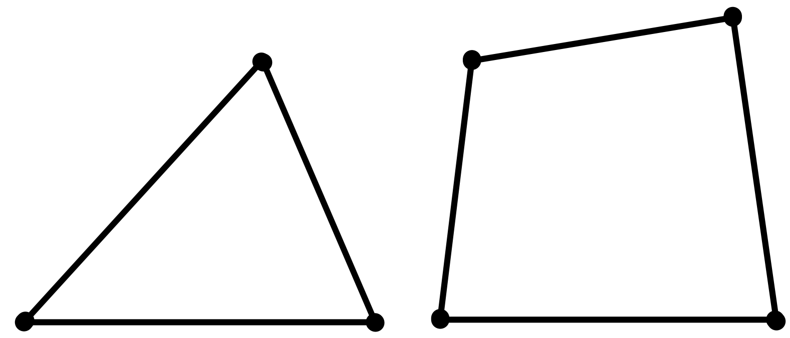
Example:
A bar subjected to some excitations like applied force at one end. Let the field quantity flow through the body, which has been obtained by solving governing DE/PDE, In FEM the domain Ω is subdivided into sub domain and in each sub domain a piecewise continuous function is assumed.



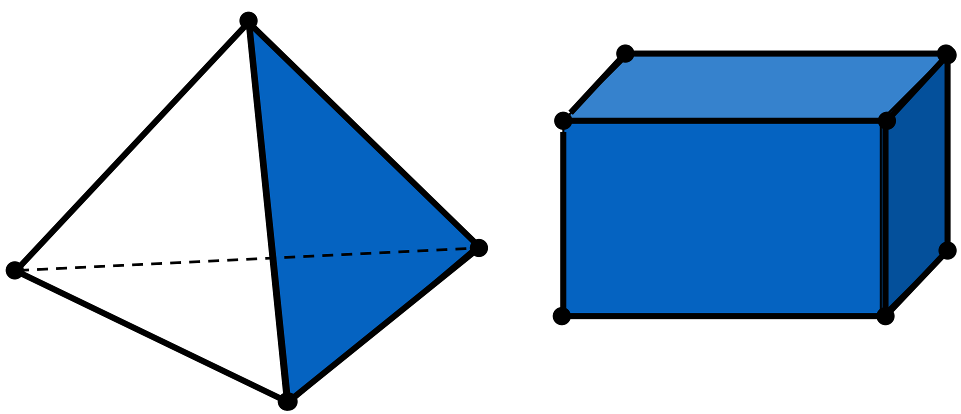
Common Types of Elements

<u>One-Dimensional Elements</u>
Line
Rods, Beams, Trusses, Frames

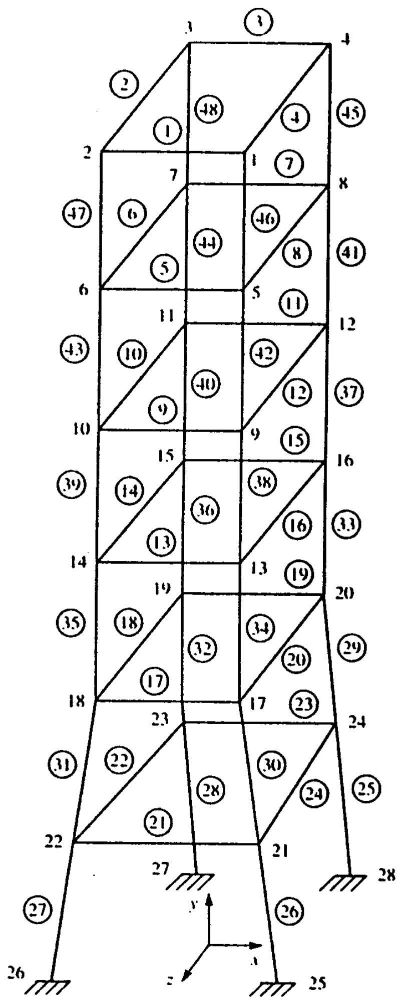
<u>Two-Dimensional Elements</u>
Triangular, Quadrilateral
Plates, Shells, 2-D Continua



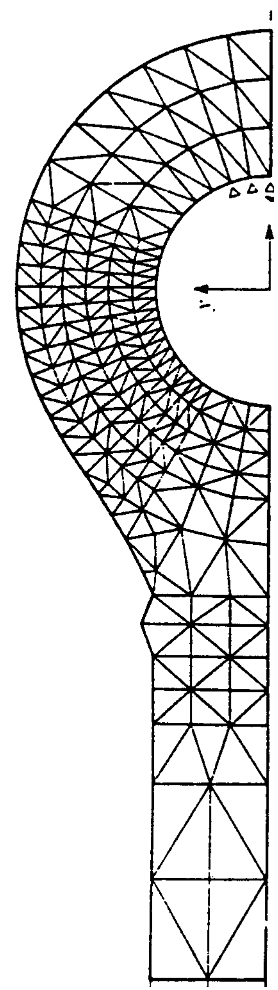
<u>Three-Dimensional Elements</u>
Tetrahedral, Rectangular Prism (Brick)
3-D Continua



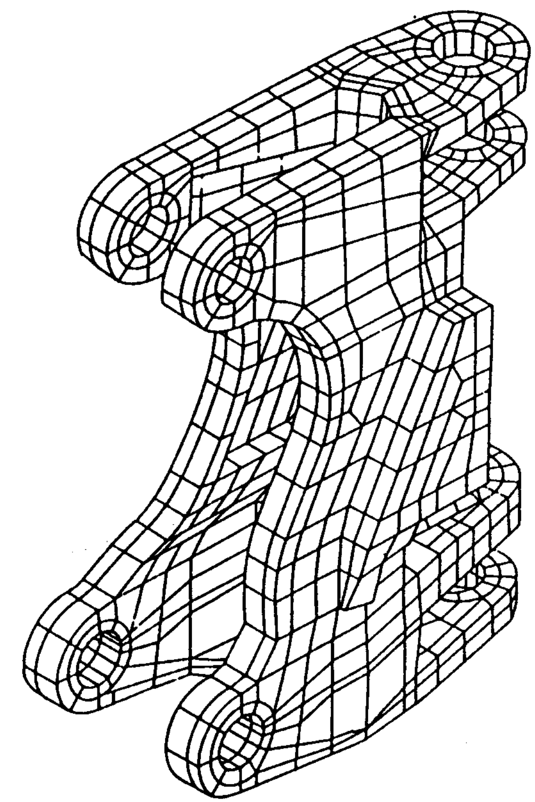
Discretization Examples



One-Dimensional Frame Elements



Two-Dimensional Triangular Elements



Three-Dimensional Brick Elements

Basic Steps in the Finite Element Method

- **Domain Discretization**
- **Select Element Type (Shape and Approximation)**
- **Derive Element Equations (Variational and Energy Methods)**
- **Assemble Element Equations to Form Global System**

$$[K]\{U\} = \{F\}$$

[K] = Stiffness or Property Matrix
{U} = Nodal Displacement Vector
{F} = Nodal Force Vector

- **Incorporate Boundary and Initial Conditions**
- **Solve Assembled System of Equations for Unknown Nodal Displacements and Secondary Unknowns of Stress and Strain Values**

Development of Finite Element Equation

- **The Finite Element Equation Must Incorporate the Appropriate Physics of the Problem**
- **For Problems in Structural Solid Mechanics, the Appropriate Physics Comes from Either Strength of Materials or Theory of Elasticity**
- **FEM Equations are Commonly Developed Using *Direct*, *Variational-Virtual Work* or *Weighted Residual* Methods**

Direct Method

Based on physical reasoning and limited to simple cases, this method is worth studying because it enhances physical understanding of the process

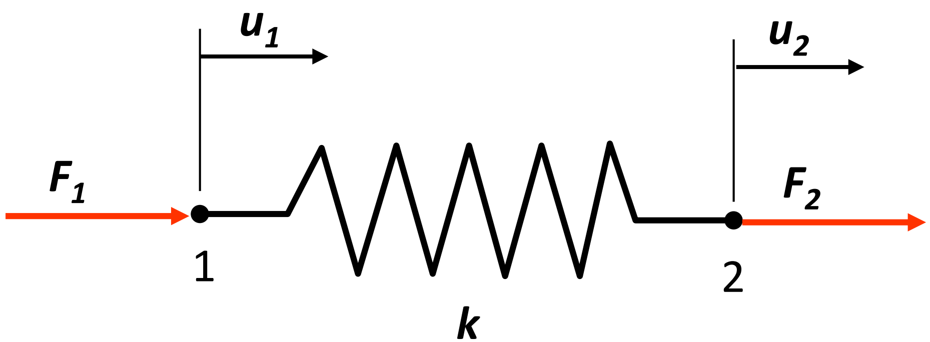
Variational-Virtual Work Method

Based on the concept of virtual displacements, leads to relations between internal and external virtual work and to minimization of system potential energy for equilibrium

Weighted Residual Method

Starting with the governing differential equation, special mathematical operations develop the "weak form" that can be incorporated into a FEM equation. This method is particularly suited for problems that have no variational statement. For stress analysis problems, a Ritz-Galerkin WRM will yield a result identical to that found by variational methods.

Simple Element Equation Example
Direct Stiffness Derivation



$$\text{Equilibrium at Node 1} \Rightarrow \; F_1 = ku_1 - ku_2$$
$$\text{Equilibrium at Node 2} \Rightarrow \; F_2 = -ku_1 + ku_2$$

or in Matrix Form

$$\begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

Stiffness Matrix

Nodal Force Vector

$$[K]\{u\} = \{F\}$$

One-Dimensional Bar Element

$$\text{Approximation}: u = \sum_k \psi_k(x) u_k = [\boldsymbol{N}]\{\boldsymbol{d}\}$$

$$\text{Strain}: \ e = \frac{du}{dx} = \sum_k \frac{d}{dx}\psi_k(x) u_k = \frac{d[\boldsymbol{N}]}{dx}\{\boldsymbol{d}\} = [\boldsymbol{B}]\{\boldsymbol{d}\}$$

$$\text{Stress-Strain Law}: \ \sigma = Ee = E[\boldsymbol{B}]\{\boldsymbol{d}\}$$

$$\int_\Omega \sigma \delta e dV = P_i u_i + P_j u_j + \int_\Omega f \delta u dV \ \Rightarrow$$

$$\{\boldsymbol{\delta d}\}^T \int_0^L A[\boldsymbol{B}]^T E[\boldsymbol{B}] dx \{\boldsymbol{d}\} = \{\boldsymbol{\delta d}\}^T \begin{Bmatrix} P_i \\ P_j \end{Bmatrix} + \{\boldsymbol{\delta d}\}^T \int_0^L A[\boldsymbol{N}]^T f dx \ \Rightarrow$$

$$\int_0^L A[\boldsymbol{B}]^T E[\boldsymbol{B}] dx \{\boldsymbol{d}\} = \{\boldsymbol{P}\} + \int_0^L A[\boldsymbol{N}]^T f dx$$

$$[\boldsymbol{K}]\{\boldsymbol{d}\} = \{\boldsymbol{F}\}$$

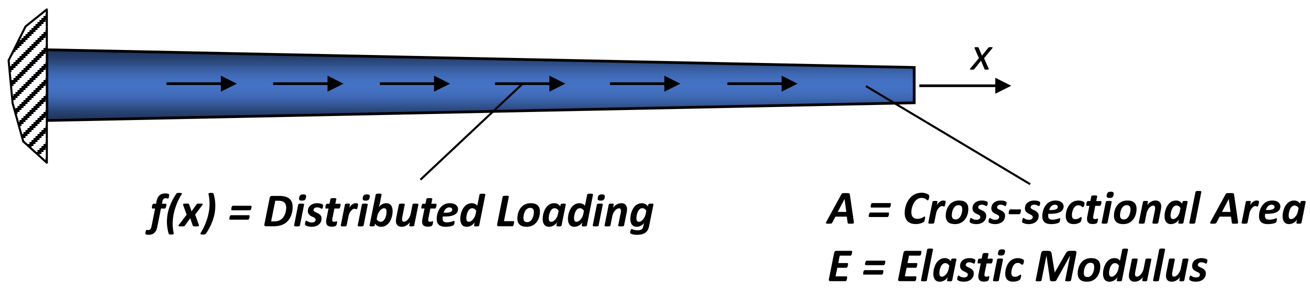
$$[K] = \int_0^L A[\boldsymbol{B}]^T E[\boldsymbol{B}] dx = \text{Stiffness Matrix}$$

$$\{\boldsymbol{F}\} = \begin{Bmatrix} P_i \\ P_j \end{Bmatrix} + \int_0^L A[\boldsymbol{N}]^T f dx = \text{Loading Vector}$$

$$\{\boldsymbol{d}\} = \begin{Bmatrix} u_i \\ u_j \end{Bmatrix} = \text{Nodal Displacement Vector}$$

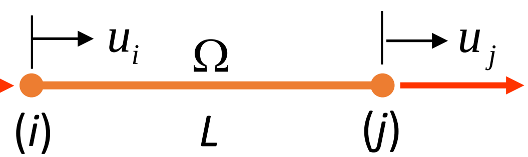
One-Dimensional Bar Element

Axial Deformation of an Elastic Bar



Typical Bar Element

$$P_i = -AE\frac{du_i}{dx} \qquad P_j = -AE\frac{du_j}{dx}$$



(Two Degrees of Freedom)

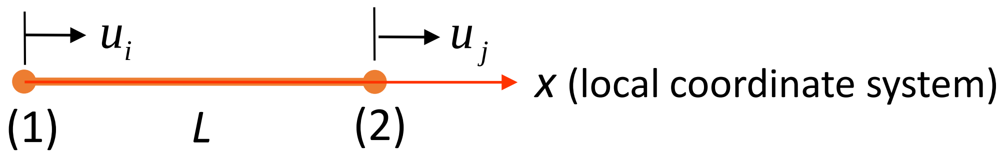
Virtual Strain Energy = Virtual Work Done by Surface and Body Forces

$$\int_V \sigma_{ij}\delta e_{ij}dV = \int_{S_t} T_i^n \delta u_i dS + \int_V F_i \delta u_i dV$$

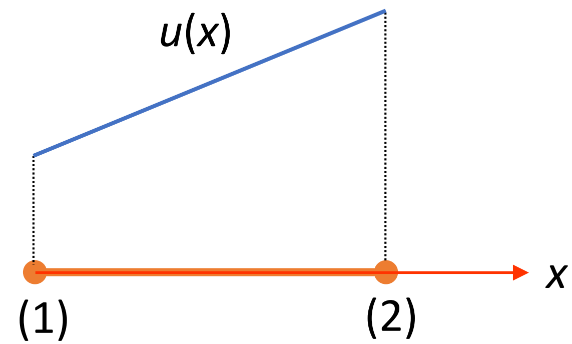
For One-Dimensional Case

$$\int_\Omega \sigma\delta e dV = P_i u_i + P_j u_j + \int_\Omega f\delta u dV$$

Linear Approximation Scheme



Approximate Elastic Displacement



$$u = a_1 + a_2 x \quad \Rightarrow \quad \begin{array}{l} u_1 = a_1 \\ u_2 = a_1 + a_2 L \end{array}$$

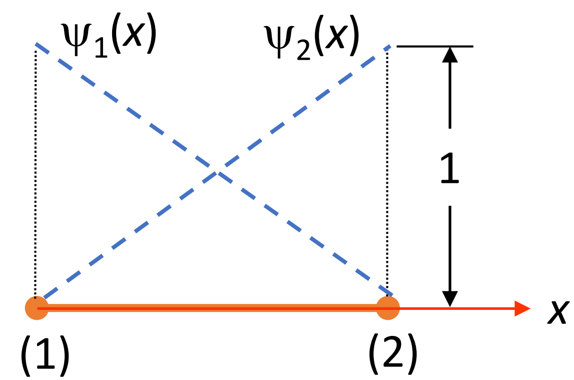
$$\Rightarrow u = u_1 + \frac{u_2 - u_1}{L} x = \left(1 - \frac{x}{L}\right) u_1 + \left(\frac{x}{L}\right) u_2$$

$$= \psi_1(x) u_1 + \psi_2(x) u_2$$

$$u = \begin{bmatrix} \psi_1 & \psi_2 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{bmatrix} 1 - \frac{x}{L} & \frac{x}{L} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = [\boldsymbol{N}]\{\boldsymbol{d}\}$$

$[\boldsymbol{N}]$ = Approximation Function Matrix

$\{\boldsymbol{d}\}$ = Nodal Displacement Vector



$\psi_k(x)$ – Lagrange Interpolation Functions

Element Equation

Linear Approximation Scheme, Constant Properties

$$[K]=\int_0^L A[\boldsymbol{B}]^T E[\boldsymbol{B}]dx = AE[\boldsymbol{B}]^T[\boldsymbol{B}]\int_0^L dx = AE\begin{Bmatrix} -\frac{1}{L} \\ \frac{1}{L} \end{Bmatrix}\begin{Bmatrix} -\frac{1}{L} & \frac{1}{L} \end{Bmatrix}L = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

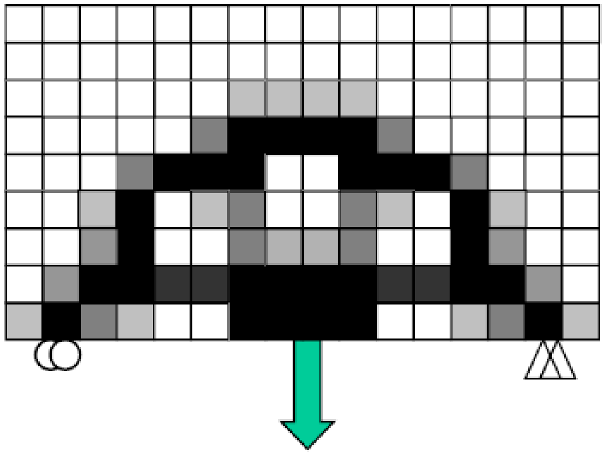
$$\{\boldsymbol{F}\}=\begin{Bmatrix} P_1 \\ P_2 \end{Bmatrix}+\int_0^L A[\boldsymbol{N}]^T f dx = \begin{Bmatrix} P_1 \\ P_2 \end{Bmatrix}+Af_o\int_0^L \begin{Bmatrix} -\frac{x}{L} \\ \frac{x}{L} \end{Bmatrix}dx = \begin{Bmatrix} P_1 \\ P_2 \end{Bmatrix}+\frac{Af_oL}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

$$\{\boldsymbol{d}\}=\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \text{Nodal Displacement Vector}$$

$$[\boldsymbol{K}]\{\boldsymbol{d}\}=\{\boldsymbol{F}\} \Rightarrow \frac{AE}{L}\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}=\begin{Bmatrix} P_1 \\ P_2 \end{Bmatrix}+\frac{Af_oL}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

Lecture 5: Finite Element for Topological Optimization (without intensive mathematics)

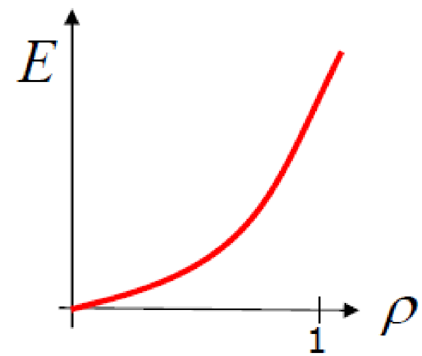
3. Finite Element for topological optimization: an example



*Discretized **SIMP** (Solid Isotropic Microstructure with Penalization for intermediate densities) **method***

$$
\begin{aligned}
\min_{\boldsymbol{\rho}} : \quad & \Phi(\boldsymbol{\rho}, \mathbf{U}(\boldsymbol{\rho})) \\
s.t. : \quad & \sum_{e=1}^{N} v_e \rho_e = \mathbf{v}^T \boldsymbol{\rho} \leq V^* \\
: \quad & g_i(\boldsymbol{\rho}, \mathbf{U}(\boldsymbol{\rho})) \leq g_i^*, \quad i = 1, ..., M \\
: \quad & \mathbf{0} \leq \boldsymbol{\rho} \leq \mathbf{1} \\
(: \quad & \mathbf{K}(\boldsymbol{\rho})\mathbf{U} = \mathbf{F}\ )
\end{aligned}
$$

Stiffness interpolation:



$$E(\rho_e) = E_1 + \rho_e^p (E_2 - E_1)$$

$$p > 1$$

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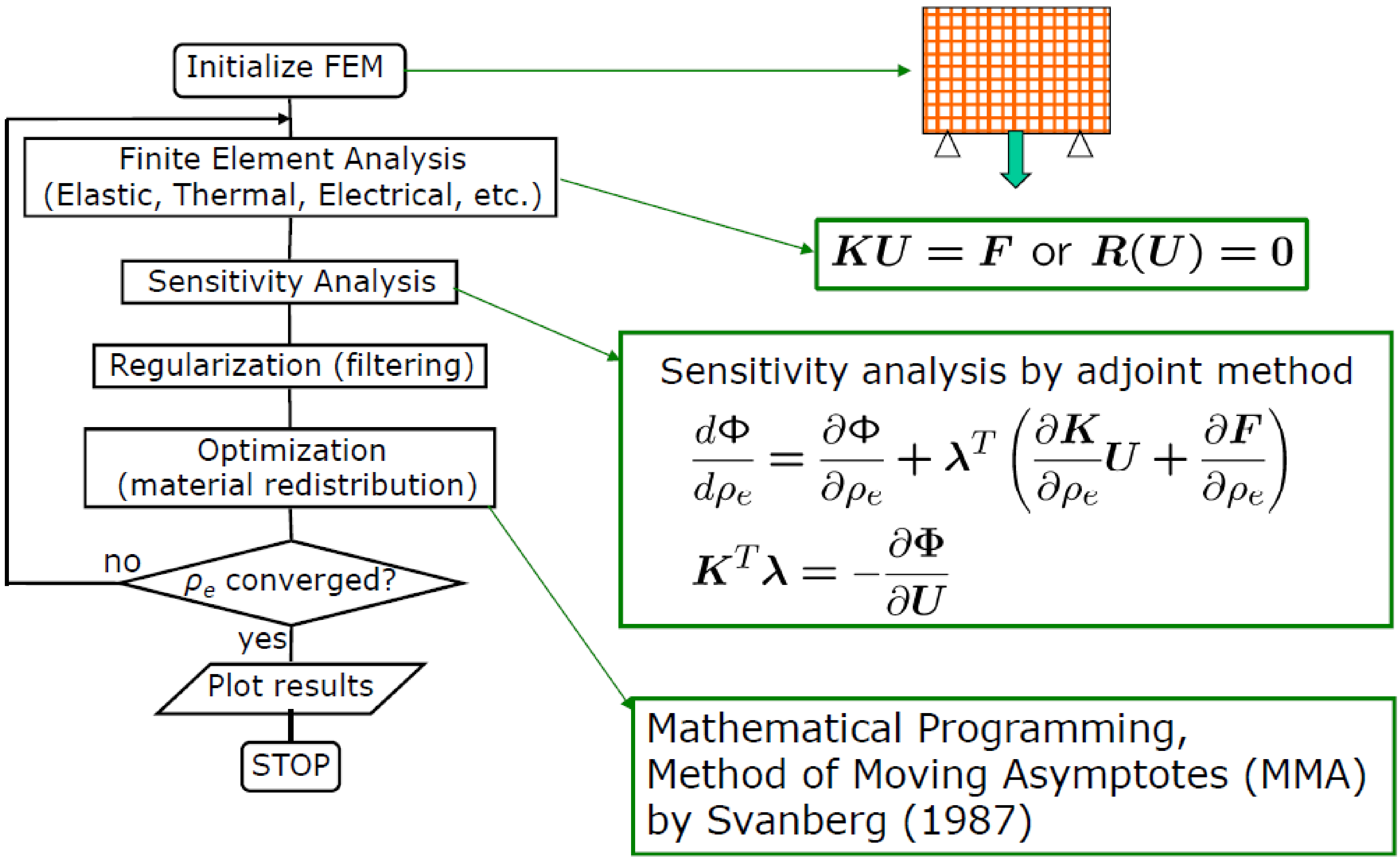

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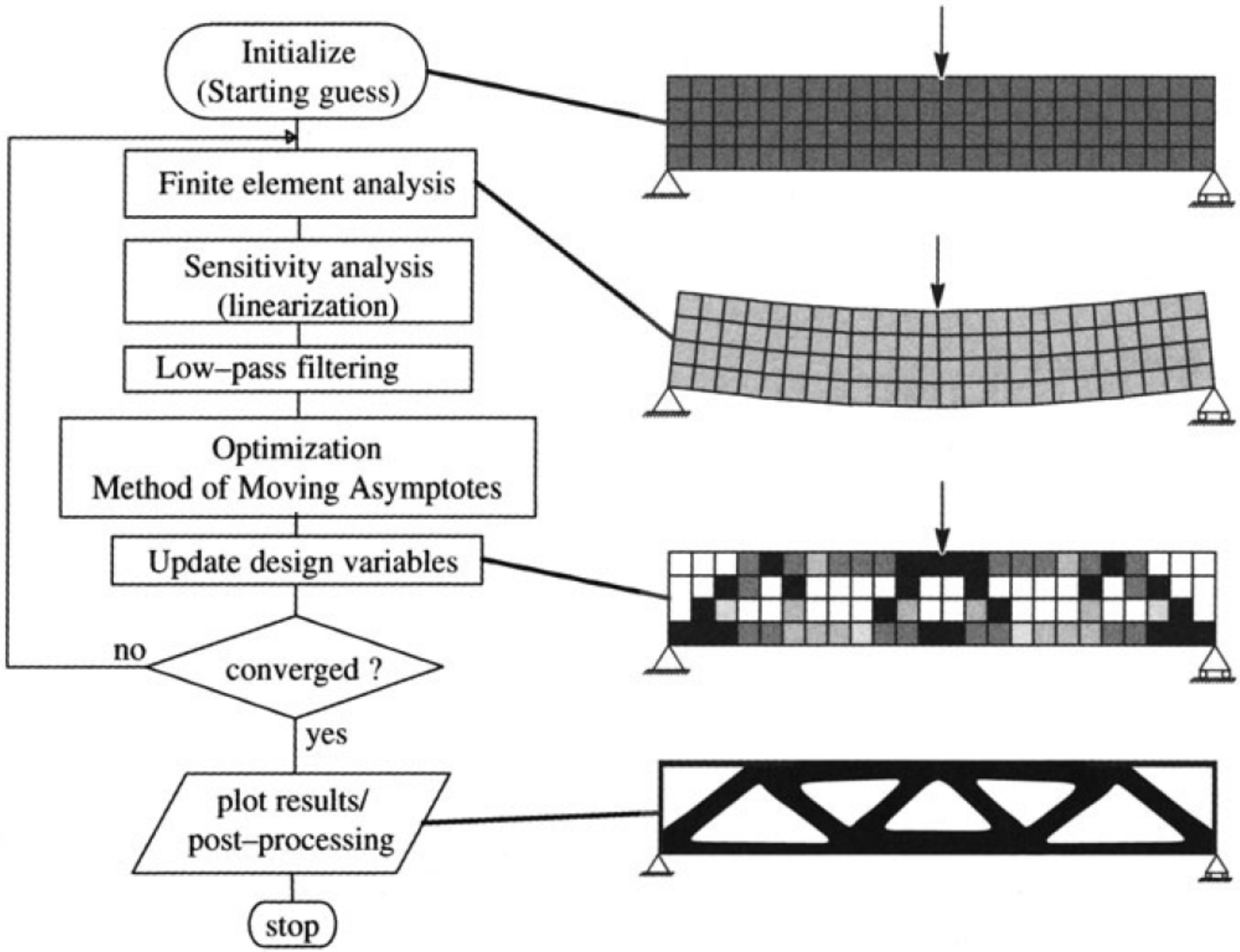

Thank you for your attention