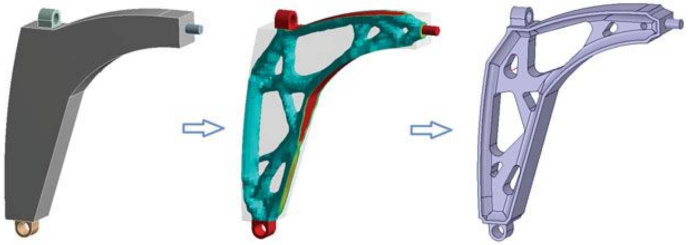
香港中文大學
The Chinese University of Hong Kong

MAEG5160: Design for Additive Manufacturing

Lecture 7: Topology Optimization (TO) by distribution of isotropic material (continue)



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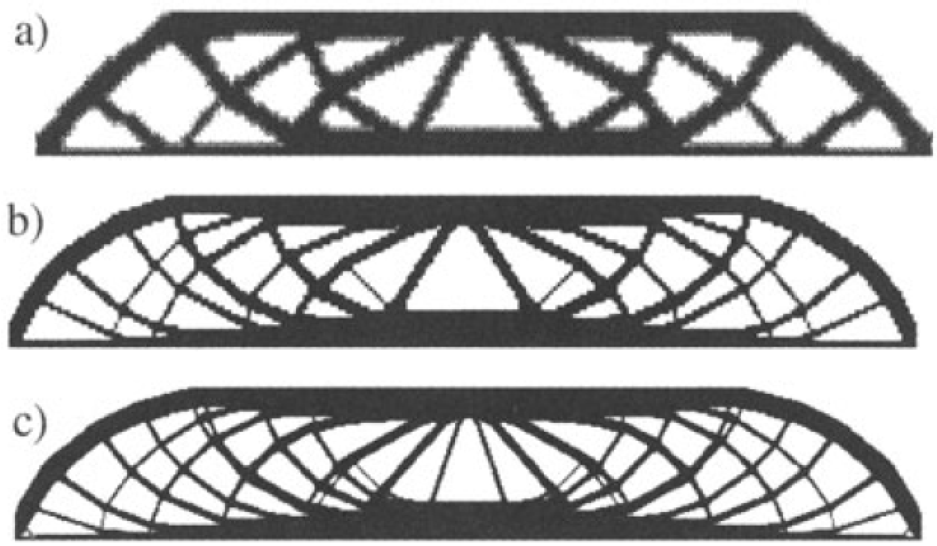
Topology Optimization (TO) by distribution of isotropic material

In the following we will discuss two important issues that significantly influences the computational results that can be obtained with the material distribution based topology design procedure. These are (1) *the appearance of checkerboards* and (2) *the mesh-dependency of results*. The former refers to the formation of regions of alternating solid and void elements ordered in a checkerboard like fashion and is related to the discretization of the original continuous problem. Mesh-dependence concerns the effect that qualitatively different optimal solutions are reached for different mesh-sizes or discretization and this problem is rooted in the issue of existence of solutions to the continuous problem.

1. Mesh-refinement and existence of solutions



Dependence of the optimal topology on mesh refinement for the MBB beam example. Solution for a discretization with a) 2700, c) 4800 and d) 17200 elements.

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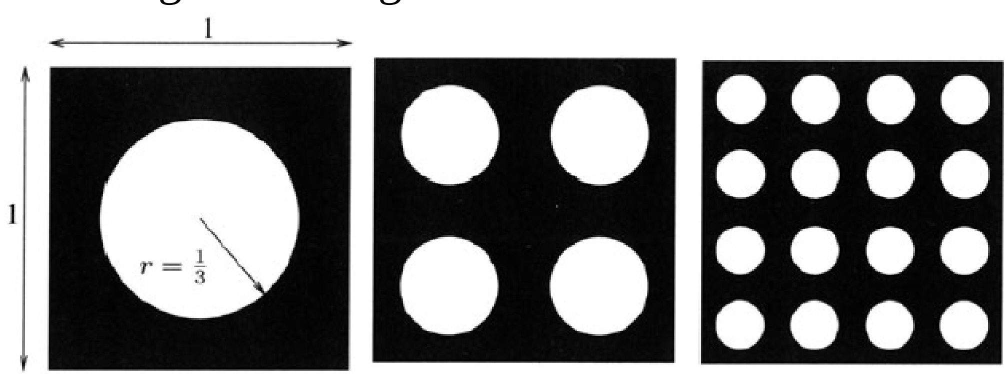
The approach to generate macroscopic and mesh-independent 0-1 solutions is to reduce the space of admissible designs by some sort of global or local restriction on the variation of density, thus effectively ruling out the possibility for fine scale structures to form. The techniques that have been suggested for enforcing such a restriction fall into three generic classes of methods. These consists of either *adding constraints to the optimization problem*, *reducing directly the parameter space for the designs*, or *applying filters in the optimization implementation*. For most of these methods, existence of solutions and also convergence of the FE approximations have been proved, providing a solid foundation. Note that the alternative to a restriction of the design space is to extend the space by allowing composites as admissible designs. For minimum compliance this lives up to our requirement of independence of mesh refinement, but also gives designs with large areas of "grey". This is thus not an option if 0-1 designs are the goal.



$V = 1 - 1 \cdot \pi \frac{1}{3^2} = 0.65$ $\quad V = 1 - 4 \cdot \pi \frac{1}{6^2} = 0.65$ $\quad V = 1 - 16 \cdot \pi \frac{1}{12^2} = 0.65$

$P = 1 \cdot 2\pi \frac{1}{3} = 2.09$ $\quad P = 4 \cdot 2\pi \frac{1}{6} = 4.19$ $\quad P = 16 \cdot 2\pi \frac{1}{12} = 8.38$

An example of how smaller holes increase the perimeter, for a fixed volume. V is the volume and P is the perimeter of internal holes

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Perimeter control: The perimeter of a mechanical element Ω_{mat} is, vaguely speaking, the sum of the lengths/areas of all inner and outer boundaries. Constraining the perimeter clearly limits the number of holes that can appear in the domain, and existence of solutions to the perimeter controlled topology optimization is actually assured for both the discrete 0-1 setting and the interpolated version using SIMP. Also, it has been implemented for both situations and for 2-D as well as 3-D problems. For the SIMP method, one can impose a constraint that mimics such a perimeter bound in the form of an upper bound on the *total variation*, *TV(p)* , of the density ρ. In case the function ρ is smooth, the total variation constraint is a L^1 bound on its gradient:

$$TV(\rho) = \int_{\mathbf{R}^n} \|\nabla\rho\| \, \mathrm{d}x \leq P^*$$

For a 0-1 design, the total variation of ρ coincides with the perimeter of Ω^{mat} when ρ is 1 in Ω^{mat} and 0 elsewhere (in $\mathbf{R}^n, n = 2(3)$). In this case the constraint is expressed as

$$TV(\rho) = \sup\left\{\int_{\mathbf{R}^n} \rho \operatorname{div}\varphi \mathrm{d}x \,\middle|\, \varphi \in C_c^1(\mathbf{R}^n, \mathbf{R}^n),\ \|\varphi\| \leq 1\right\} \leq P^*$$

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where $C_c^1(\mathbf{R}^n, \mathbf{R}^n)$ denotes compactly supported vector valued C^1 functions.

For an element wise constant finite element discretization of the density the total variation can in 2-D be calculated as

$$P = \sum_{k=1}^{K} l_k \left(\sqrt{\langle \rho \rangle_k^2 + \epsilon^2} - \epsilon \right)$$

where $\langle \rho \rangle_k$ is the jump of material density through element interface k of length l_k and K is the number of element interfaces (here one should also count interfaces at the boundary of the domain Ω – else there will be bias towards having material at the borders of Ω). The parameter ϵ is a small positive number which is used to convert the non-differentiable absolute value into a differentiable term. This expression is exactly the total variation of the element-wise constant density when $\epsilon = 0$.

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It should be mentioned that there is an inherent problem of assuring isotropy in an implementation of a discretized perimeter measure. Thus, in a regular 2-D mesh of squares, a bound on the discretized expression will tend to favour structural edges parallel to those of the finite element mesh. This is caused by the effect that a straight edge of length 1 that is angled 45 degrees to the directions of the finite element mesh will be approximated by a jagged edge that has the perimeter $\sqrt{2}$. In contrast, the same edge has perimeter 1 when it is parallel to the mesh directions. In the limit of mesh refinement for a FE-mesh directed along the X_i-axes, the discretized perimeter measure is thus rather the proper discretization of what is referred to a "taxi-cab" perimeter measure:

$$TV_{\mathrm{taxicab}}(\rho) = \int_{\mathbf{R}^2} \left(\left|\frac{\partial \rho}{\partial x_1}\right| + \left|\frac{\partial \rho}{\partial x_2}\right| \right) \, \mathrm{d}x$$

This means that the numerical results will approach solutions of a continuum topology optimization problem statement that includes a "perimeter bound“ that actually measures the "length" of the boundary of the structure by projecting this onto the coordinate axes. This in turn implies that even though the perimeter constraint assures convergence with respect to mesh refinement, a dependence on the choice of mesh will nonetheless be seen. This effect, however, does not change with mesh refinement. This directional bias of the results can be reduced considerably by considering more involved discrete versions of the perimeter measure. The perimeter bound adds one extra constraint to the topology optimization problem and thus does not create any substantial problems for the use of an algorithm like MMA. However, it has been reported that the perimeter constraint can be quite difficult to approximate resulting in fluctuations in the design variables (this relates to the choice of the asymptotes of MMA). However, this can be solved by an internal loop procedure for the perimeter approximation which is computationally inexpensive compared to the equilibrium analysis. Finally, one should note that choosing the bounding value of the perimeter constraint can be rather tricky.

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Other methods of restricting gradients

One can also consider other types of gradient constraints for the SIMP method that ensure existence of solutions and convergence with mesh-refinement. These presuppose that ρ is sufficiently smooth for the bound to make sense and do not seem to have any equivalent for the discrete-valued 0-1 setting, in contrast to the perimeter measure discussed above. One possibility is to constrain the local density variation by imposing pointwise bounds on the derivatives of the function ρ *:*

$$\left|\frac{\partial \rho}{\partial x_i}\right| \leq G, \ \ (i = 1, 2, (3)) \ .$$

This scheme, which in essence constrains the L_{∞} norm of the gradient of ρ, does assure existence of solutions and convergence of the finite element scheme. The advantage of this gradient constraint is that it gives a well-defined local length scale. The constraint implies that a transition from void to void through full material has to take place over a distance that is longer than *2/G*, which is thus the width of the thinnest features of a feasible design. Unfortunately, an implementation results in a huge number of extra constraints in the optimization problem and the method must therefore be considered to be too slow for practical design problems, if implemented directly as constraints. However, if one approximates the L_{∞} constraint by a L^q constraint for suitable large q one can alternatively operate with just one global constraint (but choosing the constraint value can be tricky and requires experimentation for each design case).

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The basic concept of a slope constraint can also be enforced by an adaptive constraint strategy in the optimization algorithm that is similar to adding move limits. This means that one only works with the values of the box-constraints on the density ρ, which at the (K+1)th iteration step are modified to restrict the variations in the design

$$\rho_i^{K+1} \geq \max\{\rho_{\min}, \rho_{j(i)}^{K} - D_{i,\ j(i)} G\}$$

Here j(i) is the element number of the element with the highest density value among all elements adjacent to the element i at the prior iteration step, and $D_{i,\ j(i)}$ is the distance between the centers of the elements i and $j(i)$. This strategy does not add to the computational complexity of the optimization procedure. However, it does require that the applied optimization algorithm can handle (temporary) violations of the box constraints. Furthermore, it is unclear whether "playing" with the move-limits will jeopardize convergence of the algorithm.

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Another option is to implement a "global gradient constraint" by which we mean the norm of the function ρ in the Sobolov space $H^1(\Omega)$:

$$||\rho||_{H^1} = \left(\int_\Omega \left(\rho^2 + ||\nabla \rho||^2 \right) \, \mathrm{d}\Omega \right)^{\frac{1}{2}} \leq M \, .$$

Proof of existence when including this bound in the minimum compliance problem can be found (for three dimensional problems the proof requires that the power in SIMP satisfies $\rho < 3$). Note that we for any finite element discretization of the ground structure can choose a large enough bound *M* on the norm of ρ so that the norm constraint remains inactive, thus seemingly returning to the original formulation for this discretization. Thus implementation of it also requires utmost care and should involve experimenting with a range of values of the bound *M.* A global gradient constraint can also be formulated with the term ρ^2 removed from it, so that the constraint becomes a L^2 constraint on the gradient of ρ. Numerical experiments with global gradients in the setting of topology optimization can be found in Borrvall (2001), where also U constraints in general are considered.

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Filtering the density

The techniques above impose explicit limitations on the allowable density distributions that can appear in the optimal design, and as such have to be catered for as constraints in the optimization formulation. An alternative to this is to directly limit the variations of the densities that appear in the set of admissible stiffness tensor E_{ad} by only admitting *filtered* densities in the stiffness. Thus the SIMP method is modified to the following reduced design space:

$$E_{ijkl}(x) = ((\rho * K)(x))^p E^0_{ijkl}, \ \rho \in L^\infty(\Omega) ,$$

$$(\rho * K)(x) = \frac{1}{< K >} \int_\Omega \rho(y) K(x-y) \, \mathrm{d}y , \quad < K > = \int_{R^n} K(y) \, \mathrm{d}y$$

$$\int_\Omega \rho(x) \mathrm{d}\Omega \leq V; \quad 0 \leq \rho(x) \leq 1, \quad x \in \Omega ,$$

where K is a convolution kernel, for example

$$K_r(x) = \begin{cases} 1 - \dfrac{||x||}{r} & \text{if} \quad ||x|| \leq r , \\ 0 & \text{otherwise} . \end{cases}$$

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The filter radius r is fixed in the formulation and implies the enforcement of a fixed length-scale in the stiffness distribution. The filtering means that the stiffness in a point x depends on the density $\rho(x)$ in all points of a neighborhood of x. This implies a smoothing of the stiffness fields in a fashion similar to a filtering of an image. The smoothing and the fixed scale means that this method gives existence of solutions and convergence with refinement of the FE mesh. Loosely speaking, the reason that the filter removes any fine scale behaviour of the density p is that such variations in the mechanical analysis (via the filtering $(\rho * K))$ appears like a grey which is penalized by SIMP. Generally this method results in density fields p that are bi-valued, but the stiffness distribution $(\rho * K)^P$ is more "blurred" with grey boundaries. In a sense this is an ambiguity, as the mechanical analysis is done on the filtered density. For implementation, the differences compared to the standard procedure are that the element stiffness matrices in the finite element analysis are defined by weighted averages of the stiffnesses of neighbouring elements, and the sensitivity information should be modified to cater for the redefined stiffness tensor (this means that the sensitivity of the compliance with respect to $\rho(x)$ will involve the mutual energy of a neighbourhood of x).

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Filtering the sensitivities

Computational experience has shown that filtering of the sensitivity information of the optimization problem is a highly efficient way to ensure mesh-independency. This means modifying the design sensitivity of a specific element, based on a weighted average of the element sensitivities in a *fixed* neighborhood. Such a filter is purely heuristic but it produces results very similar to for example those obtained by a local gradient constraint, it requires little extra CPU-time and it is very simple to implement as it does not add to the complexity of the optimization problem (no extra constraints need to be considered). Similar ideas of weighted averages have been used to ensure mesh-independence for simulations of bone-re-modelling and for analysis with plastic-softening materials.

The scheme works by modifying the element sensitivities of the compliance as follows:

$$\widehat{\frac{\partial f}{\partial \rho_k}} = \frac{1}{\rho_k \sum_{i=1}^{N} \hat{H}_i} \sum_{i=1}^{N} \hat{H}_i \; \rho_i \, \frac{\partial f}{\partial \rho_i}$$

where *N* is the total number of elements in the mesh and where the *mesh independent* convolution operator (weight factor) $\hat{H}_i$ is written as

$$\hat{H}_i = r_{\min} - \operatorname{dist}(k,i), \quad \{i \in N \mid \operatorname{dist}(k,i) \leq r_{\min}\}, \; k = 1, \ldots, N \, .$$

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In this expression, the operator dist(k, i) is defined as the distance between the center of the element *k* and the center of an element i. The convolution operator $\hat{H}_i$ is zero outside the filter area. The convolution operator for element i is seen to decay linearly with the distance from element *k*. It is worthwhile noting that the sensitivity converges to the original sensitivity when the filter radius r_{min} approaches zero and that all sensitivities will be equal (resulting in an even distribution of material) when r_{min} approaches infinity.
This filter is implemented in the Matlab code.

Unfortunately, the theoretical basis for the method is not yet understood. Also, it is unclear exactly what problem we are solving. However, numerous applications, many of which are shown in this monograph are based on this filtering method. It has been applied to 2 and 3 dimensional problems, to problems with up to 20 structural or other constraints, to problems involving multiple areas of physics and it has been an invaluable tool in designing extremal material structures. Furthermore, it gives results that are stable under mesh-refinement and maintain a minimum length-scale that is controlled by the filter radius r_{min}. Also, experience shows that the filter somehow improves the computational behaviour of the topology design procedures as it delays the tendency of the SIMP scheme to get "stuck" in 0-1 designs. In the following we will refer to this filtering as the *mesh-independence filter.*

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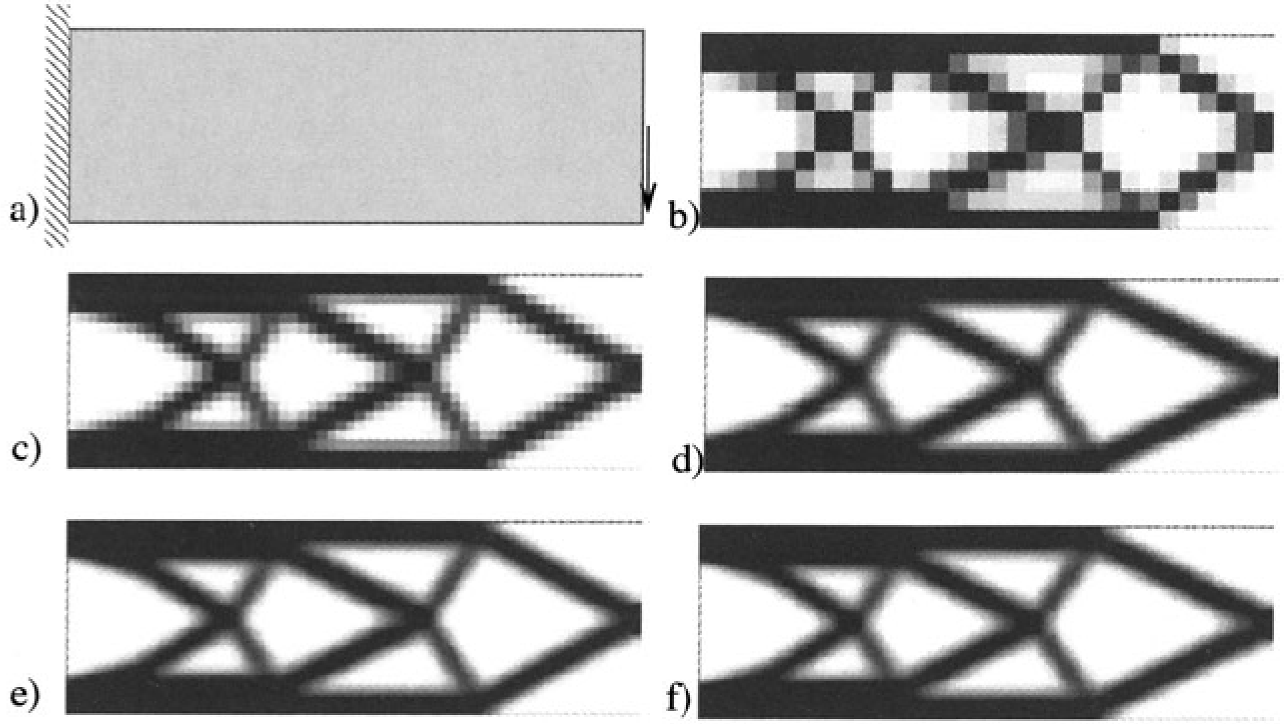

Mesh-independent solutions of the cantilever problem using filtering of sensitivities. a) Design domain and load, b) 300, c) 600, d) 4800, e) 10.800 and f) 19.200 element discretization. Filter radius is 8.2% of the height of the design domain.

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Comparison of methods The perimeter, local gradient and filter methods produce very similar designs, but there are some differences. The perimeter control and the global gradient control schemes are global constraints and will allow the formation of locally very thin bars (albeit in limited numbers). The local gradient and filtering schemes are local constraints and will generally remove thin bars.

Predicting the value of the perimeter constraint for a new design problem must be determined by experiments, since there is no direct relation between local scale in the structure and the perimeter bound. If the perimeter bound is too tight, there may be no solution to the optimization problem. This problem is particularly difficult for three-dimensional problem. In contrast, the gradient and filtering schemes define a local length scale under which structural variation is filtered out. This local length scale corresponds to a lower limit on bar/beam widths. Such a possibility of imposing a minimum length scale is not only of importance for obtaining methods that are stable under mesh-refinement. Almost of greater importance is the possibility this gives for taking manufacturing considerations (machining constraints) into account. This can be in the form of minimum member size requirements for the material phase. This is important for the fabrication of MEMS (Micro Electro-Mechanical Systems), where mechanisms are etched or deposited by chemical processes. Also, minimum size of a void inclusion is crucial if a structure is machined out by milling processes.

Finally, we remark that the use of a fixed, finite dimensional set of designs is a direct way of assuring existence of solutions as well as stability with respect to mesh-refinement - the latter here then only means improving the analysis grid. The geometric resolution cannot be improved beyond what is contained in the initial design description.

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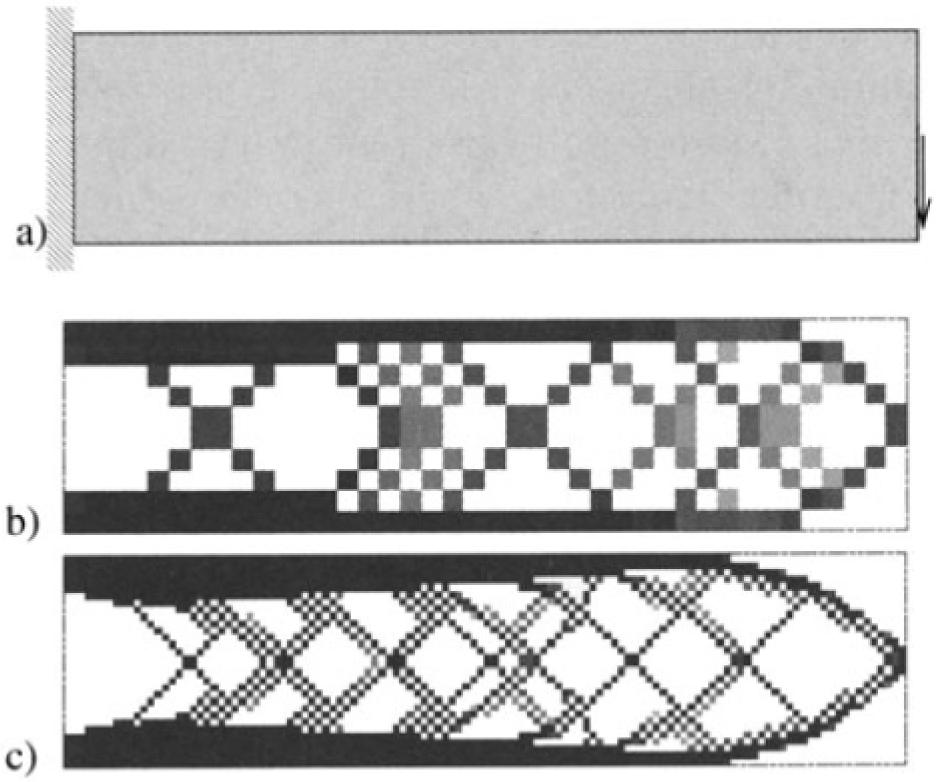
2. The checkboard problem



The checkerboard problem demonstrated on a long cantilever beam. a) Design problem, b) solution for 400 element discretization and c) solution for 6400 element discretization.

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Patches of checkerboard patterns appear often in solutions obtained by a direct implementation of the material distribution method that use the displacement based finite element method. Within a checkerboard patch of the structure the density of the material assigned to contiguous finite elements varies in a periodic fashion similar to a checkerboard consisting of alternating solid and void elements. Such patterns are also observed in the spatial distribution of the pressure in some finite element analyses of Stokes flows. It is now well understood that also for topology design the origin of the checkerboard patterns is related to features of the finite element approximation, and more specifically is due to bad numerical modelling that overestimates the stiffness of checkerboards.

The restriction methods already described also has the effect that checkerboarding is reduced or removed. The reason for this is that when one enforces a constraint on geometry (generally speaking in terms of the length of the boundary or in terms of gradient variation) that assure that solutions exist, one also obtains FE-convergence and checkerboards cannot be present for a fine enough mesh (more precisely, they can be made arbitrarily weak).

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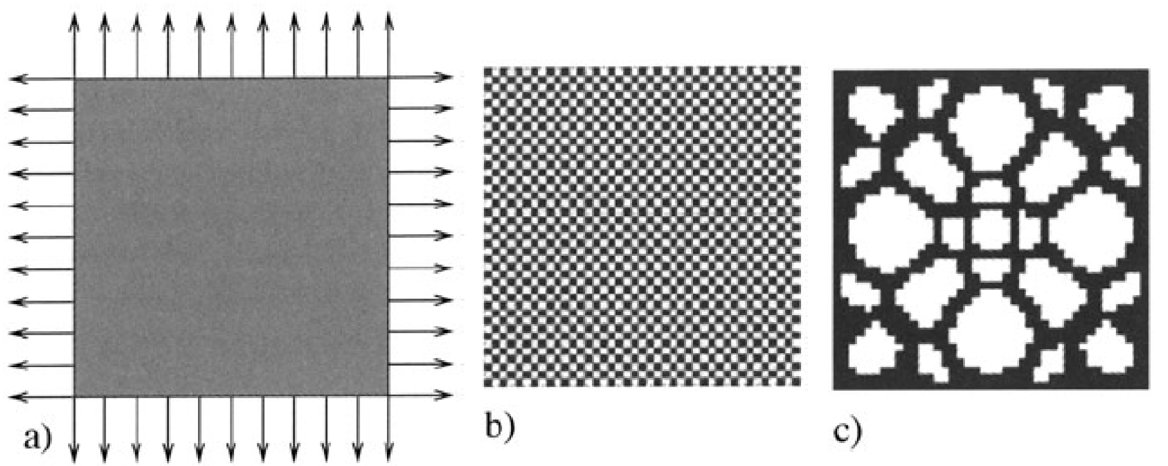
The stiffness of checkerboards The most direct explanation as to why checkerboards appear in topology design is that such lay-outs of material have an artificially high stiffness when analyzed in certain discretized formulations. Thus it turns out that a checkerboard of material in a uniform grid of square Q4 elements has a stiffness which is comparable to the stiffness of a $p = 1/2$ *variable thickness* sheet, for any applied loads (or prescribed strains). For the minimum compliance problem of an infinite medium, this means that for a Q4 discretization of displacements and any discrete as well as the continuum description of p, the corresponding optimization problem has the checkerboard version (matched to the Q4 mesh) as an optimal design. Thus it is not surprising that one in general sees that optimization generates these non-physical checkerboards when Q4-displacement elements are used.



The checkerboard problem demonstrated on a square structure subject to biaxial stress and modelled by Q4 elements. a) Design problem, b) solution without checkerboard control and c) solution with sensitivity filtering. All volume fractions are 50% and the resulting compliances for a variable thickness plate ($p = 1$ in SIMP) (a) is 2.67; for the checkerboard structure (b) 2.81; and for the non-checkerboard structure (c) it is 6.16. Even in this finite lay-out the non-physical checkerboard - modelled by Q4 elements - is almost as stiff as the sheet.

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Checkerboards and choice of FE spaces The problem of finding the optimal topology by the material distribution method is a *two* field problem. It involves finding the optimal distribution of material described by the density ρ (or stiffness tensor E) as well as the displacement field u of this optimal design. It is in this connection useful to remember that the displacement based minimum compliance problem we consider can be cast in the form

$$\max_{E \in \mathsf{E}_{ad}} \min_{v \in U} \left\{ \frac{1}{2} \int_{\Omega} E_{ijkl} \varepsilon_{ij}(v) \varepsilon_{kl}(v) \mathrm{d}\Omega - l(v) \right\}$$

and a numerical implementation operates on a discretized version of this min-max type problem for a functional of two variables. It is well-known (cf., Stokes flow), that the finite element analysis of such problems can cause problems, being unstable and being prone to the development of checkerboard patterns for one of the fields. The so-called Babuska-Brezzi (B-B) condition has been developed as a criterion that will guarantee that a finite element discretization results in a stable numerical scheme. Unfortunately, the functional of the topology design problem is not quadratic in the two fields and it is also not concave-convex. Thus one cannot directly apply standard saddle point theory and the related application of the Babuska-Brezzi condition to the present situation. However, these problems aside, taking a direct analogy to the similar problem in Stokes flow indicates nonetheless that certain combinations of finite element discretizations will be unstable and some stable. This has been confirmed by numerical experiments for both the SIMP model, for cases with composites and for variable thickness sheets. The analogy suggests that the use of higher order finite elements for the displacement function is a viable method to avoid the checkerboard problem and checkerboards are typically prevented when using 8 or 9-node quadrilaterals for the displacements in combination with an element wise constant discretization of density. An analysis based on a patch test of the finite element models substantiates this finding. These patch tests are based on a B-B type analysis of a linearized, incremental form of the necessary conditions, corresponding to an incremental, quadratic approximation of the saddle point problem, and the tests give information on the performance of various combinations of finite element approximations of the two field problem at hand. We also note that it is possible to extend the full mathematical analyses of mixed FE developed for the Stokes' flow problem to the variable thickness sheet problem.

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The use of higher order finite elements in topology design results in a substantial increase in CPU-time, even though this is not today a serious problem for 2-D problems. But it is still productive to employ alternative and computationally more economical methods. Many such methods have been proposed and typically include some flavour of a mesh related filtering of the densities.

One is to change the discretization of the density field to be given by the nodal values of the squares that define the mesh for the displacements; the element density is then the *average* of the nodal values. A sensitivity of compliance with respect to one of these densities will then depend on the energies in the four neighboring elements, and the design description is in nature similar to filtering methods. It can be shown that for a finite element discretization based on square elements, this idea corresponds to imposing a local gradient constraint, where G is equal to two times the element size. This means that there always will be a rim of at least one grey element between solid and void elements. Obviously, this also means that this nodal based averaging technique does not imply mesh-independence. Note that a scheme that uses a density interpolation of nodal values does *not* have the desired effect.
Another idea is to use non-conforming elements for the displacement fields, effectively giving correct zero stiffness to an infinite checkerboard also in the discretized problem.
Finally, we remark that theoretical studies of the appearance of checkerboards in three-dimensional problems are yet to be carried out. However numerical experience shows that checkerboards also appear for this case.

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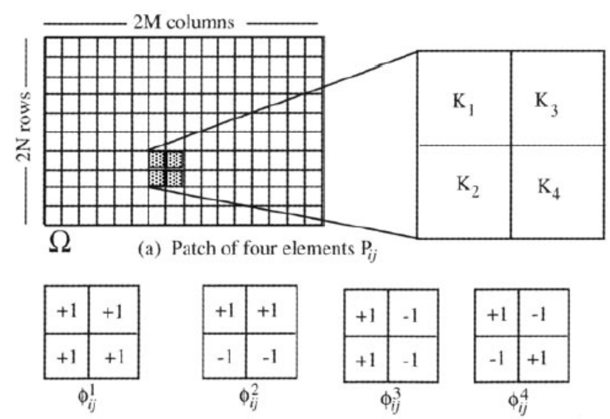
Removing checkerboards in a patch In order to save CPU-time, but still obtain checkerboard free designs, it has been suggested to employ a patch technique inspired by a method applied for the similar problems in Stokes Flow. This technique has in practical tests shown an ability to damp the appearance of checkerboards.

The strategy controls the formation of checkerboards in meshes of 4-node quadrilateral displacement elements coupled with constant material properties within each element. Thus one maintains the use of low order elements. However, the end result is the introduction of some type of element with a higher number of nodes, as the method in effect results in a "super-element" for the density and displacement functions in 4 neighbouring elements. In what follows we will assume that the design domain Ω is rectangular. It is discretized using a uniform mesh of square, 4-node isoparametric elements Kij , i = 1, ... ,2M, j = 1, ... ,2N where 2M and 2N are the (even) number of elements per side. Consider now, for odd i and j, a patch P ij of four contiguous elements Kl = Ki,j K2 = Ki+1,j K3 = Ki,j+l and K4 = Ki+1,j+l.

$$P_{ij} = K_1 \cup K_2 \cup K_3 \cup K_4$$



(a) Patch of four elements P_{ij}

(b) Basis functions associated with P_{ij}

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Associated with P_{ij} we introduce basis functions $\phi_{ij}^1, \phi_{ij}^2, \phi_{ij}^3$ *and* ϕ_{ij}^4 which take the values ± 1 in P_{ij} according to the pattern shown in Fig. 1.20 and are zero outside P_{ij}. Here we note that:

– The functions $\{\phi_{ij}^k\}$ constitute an orthogonal basis,
– A "pure" checkerboard pattern is of the form $u = \sum_{\mathrm{P}_{ij}} u_{ij}\phi_{ij}^4$.

This suggests that in order to avoid the formation of checkerboard patterns we need to restrict ρ to lie within the more restricted, checkerboard-free space

$$\bar{V} = \left\{ v \left| \begin{array}{l} v(x) = \sum_{\mathrm{P}_{ij}} (v_{ij}^1\phi_{ij}^1 + v_{ij}^2\phi_{ij}^2 + v_{ij}^3\phi_{ij}^3),\ (v_{ij}^1, v_{ij}^2, v_{ij}^3) \in \mathbf{R}^3, \\ i = 1, 3, \ldots 2N-1,\ j = 1, 3, \ldots 2M-1 \end{array} \right. \right\}$$

This restriction on ρ links the four elements in a patch, and the amount of material in $\mathrm{K}_1 \cup \mathrm{K}_4$ equals that of $\mathrm{K}_2 \cup \mathrm{K}_3$ and each is half of the total volume of the patch.

The coupling of the density distribution makes it difficult to apply the usual iterative optimality condition method. In MMA one can work directly with the design space $\bar{V}$ by using the $3MN$ parameters $(v_{ij}^1, v_{ij}^2, v_{ij}^3)$ as design variables. This, however, changes the simple bound constraints $0 \leq \rho \leq 1$ into a huge number of linear constraints on the parameters v, making this option impractical. Instead, the following simpler procedure, which has been applied in a variety of problems, can be employed for both algorithms:

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1. At each iteration of the optimization algorithm the cell size parameters within each element K_{ij} are updated using the usual update method (optimality criteria approach or an MMA step).
2. For each patch P_{ij} let $\{\rho_1, \rho_2, \rho_3, \rho_4\}$ be the be the updated densities in the four quadrants of the patch associated with the updated cell sizes (using the numbering of 1.20). We then seek, as the starting point for the next iteration, a new piece-wise constant and *checkerboard-free* density distribution within the patch, say $\bar{\rho}$, written as

$$\bar{\rho}(x) = \frac{1}{4}(\rho_1 + \rho_2 + \rho_3 + \rho_4)\phi^1 + \bar{v}_2\phi^2 + \bar{v}_3\phi^3, \; x \in \mathrm{P}_{ij} \; .$$

Here $\bar{\rho}$ is *checkerboard-free* (as $\bar{v}_4 = 0$) and it preserves material in the patch (as the coefficient of ϕ^1 is set as $\bar{v}_1 = \frac{1}{4}(\rho_1 + \rho_2 + \rho_3 + \rho_4)$). To determine the parameters $\bar{v}_2, \bar{v}_3$, we select $\bar{\rho}$ as the best L^2 approximation to ρ in P_{ij} under the constraints that $0 \leq \bar{\rho}_i \leq 1, i = 1,2,3,4$. This corresponds to a QP problem in two variables, with linear constraints. The solution can be found analytically, and is given as

$$\bar{\rho}_1 = \frac{1}{4}(3\rho_1 + \rho_2 + \rho_3 - \rho_4)\;, \qquad \bar{\rho}_3 = \frac{1}{4}(\rho_1 - \rho_2 + 3\rho_3 + \rho_4)\;,$$
$$\bar{\rho}_2 = \frac{1}{4}(\rho_1 + 3\rho_2 - \rho_3 + \rho_4)\;, \qquad \bar{\rho}_4 = \frac{1}{4}(-\rho_1 + \rho_2 + \rho_3 + 3\rho_4)\;,$$

if these values satisfies $0 \leq \bar{\rho}_i \leq 1$. If a $\bar{\rho}_i$ in these expressions is above 1, it is set to 1 and the corresponding diagonal density is adjusted to maintain the volume of the patch; negative values are handled likewise and are set to 0. The modification of the density outlined here has the flavour of a filtering in a post-processing step that is invoked at each step of the optimization procedure and should therefore be used with some caution. We note that it does not disturb areas of the domain where no checkerboard control is needed, and also remark again that the method corresponds to introducing a "super-element" of four Q4 elements with a total of 9 displacements nodal points and with **3** degrees of freedom for the density approximation.

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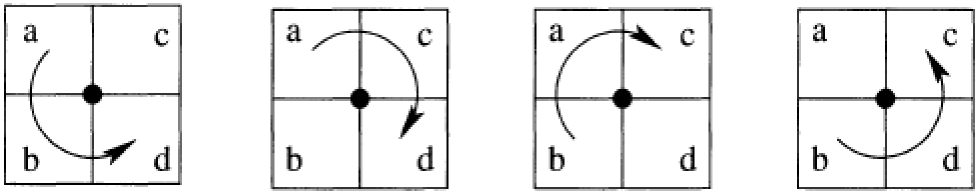
No Hinge: A checkerboard constraint

Previously, geometry control was achieved by defining one extra constraint for the optimization problem. This idea can also be implemented for checkerboard control, i.e., one defines a non-negative constraint function that should have value zero for the design to be free of checkerboards. Consider the patch of square elements below. Defining the function

$$m(x, y, z) = |y - x| + |z - y| - |z - x|$$

that is zero if the sequence of real numbers x, y, z is monotonic (increasing, decreasing or constant) and strictly positive otherwise, we can determine that the patch is free of checkerboard patterns, if just one of the numbers m(a, b, d), m(a, c, d), m(b, a, c) or m(b, d, c) is zero. This can be in turn be expressed as the condition that the number is zero.



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A design defined by a density ρ that is element wise constant on a mesh of quadrilaterals with N interior nodes will thus be free of checkerboards if it satisfies the constraint

$$\sum_{k=1}^{N} h(\rho_{k,a}, \rho_{k,b}, \rho_{k,c}, \rho_{k,d}) = 0$$

where $\rho_{k,e}$, $e = a, b, c, d$ are the material densities in the elements connected to the node k. This constraint can thus be added to our optimization problem to assure checkerboard free solutions. It can also be used to remove "artificial" hinges in mechanism design, see section 2.6. As we have seen in other situations, an implementation using gradient based optimization techniques requires a replacement of the absolute value by a smooth substitute, for example $|x| \simeq \sqrt{x^2 + \epsilon^2} - \epsilon$ with $\epsilon = 0.1$. With this modification a sensitivity analysis of the constraint is straightforward, but rather tedious (Poulsen 2002a).

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Checkerboard control by filtering of sensitivities

The filtering technique for gradients can also be cast in a version that only constrains checkerboards, without imposing a mesh independent length scale. This just requires that one adjusts the filter to exactly making the design sensitivity of a specific element depend on a weighted average over the element itself and its *eight* direct neighbours. This is a is very efficient method for removing checkerboards.

$$\widehat{\frac{\partial f}{\partial \rho_k}} = \frac{1}{\rho_k \sum_{i=1}^{N} \hat{H}_i} \sum_{i=1}^{N} \hat{H}_i \; \rho_i \; \frac{\partial f}{\partial \rho_i}$$

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3. Non-uniqueness, local minima and dependence on data

It is important to observe that most problems in topology design (as in structural problems in the large) are not convex. Moreover, many problems have multiple optima, i.e. *non-unique solutions*. An example of the latter is the design of a structure in uni-axial tension. Here a structure consisting of one thick bar will be just as good as a structure made up of several thin bars with the same overall area. The non-convexity typically means that one can find several different local minima (which is what the gradient based algorithms locate) and one can obtain different solutions to the same discretized problem when choosing different starting solutions and different parameters of the algorithms. Most global optimization methods seem to be unable to handle problems of the size of a typical topology optimization problem. Based on experience, it seems that *continuation methods* must be applied to ensure some sort of stable convergence towards reliably good designs.

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The idea of *continuation methods* is to gradually change the optimization problem from an (artificial) convex (or quasi-convex) problem to the original (non-convex) design problem in a number of steps. In each step a gradient-based optimization algorithm is used until convergence. This is useful in many types of problems. Examples are the use of a continuation method were the structure first is optimized allowing regions consisting of composites, and after convergence, a penalization scheme is gradually introduced to obtain a 0-1 design. Likewise, for SIMP it is advisable to start out with $\rho = 1$ and then slowly raise the value of p through the computations until the final design is arrived at. For the perimeter constraint it is also beneficial to perform a gradual decrease of value of the constraint on the perimeter. For the mesh-independence filter, it is normally recommended to start with a large value of the filter size *rmin* (which gives designs with blurry edges) and to gradually decrease it, to end up with a well-defined 0-1 design. Finally, it is extremely important to observe that the results that one obtains with topology design of course depends on the data that one decides on using before applying the optimization procedure. Thus a change of the geometry of the design domain, the choice of load and boundary conditions can result in drastical changes in the "optimal design" that an algorithm may produce. Similar effects can be seen from variations of perimeter constraint values or filter parameters, etc. This is actually not that surprising as we are dealing with very "nasty" optimization problems, but in topology design this effect is just much more noticeable than in many other types of structural optimization problems.

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4. Combining topology and shape design

Traditionally, in shape design of mechanical bodies, a shape is defined by the oriented boundary curves or boundary surfaces of the body and in shape optimization the optimal form of these boundaries is computed. This approach is very well established and the literature is extensive. On the other hand, we have just seen how the material distribution formulation can give a good estimate of the boundary of a structure, but here a reasonable prediction of the finer details of the boundaries requires very large FEM models. Also, the inherent large scale nature of the topology optimization method is such that the objectives used for the optimization should be global criteria, e.g. compliance, volume, average stress, etc., so that the effectiveness of the dual optimizers can be maintained by treating problems with a moderate number of constraints. For example, the focal point in the presentation so far has been the minimization of the compliance of a structure subject to a constraint on the volume of the structure. On the other hand, the description of the body by boundary curves and surfaces allows the finer details of the body to be controlled by a moderate number of design variables (e.g., spline control points) so this setting is better suited for studying problems such as the minimization of the maximum value of the displacements or of the Von Mises equivalent stress in the body.

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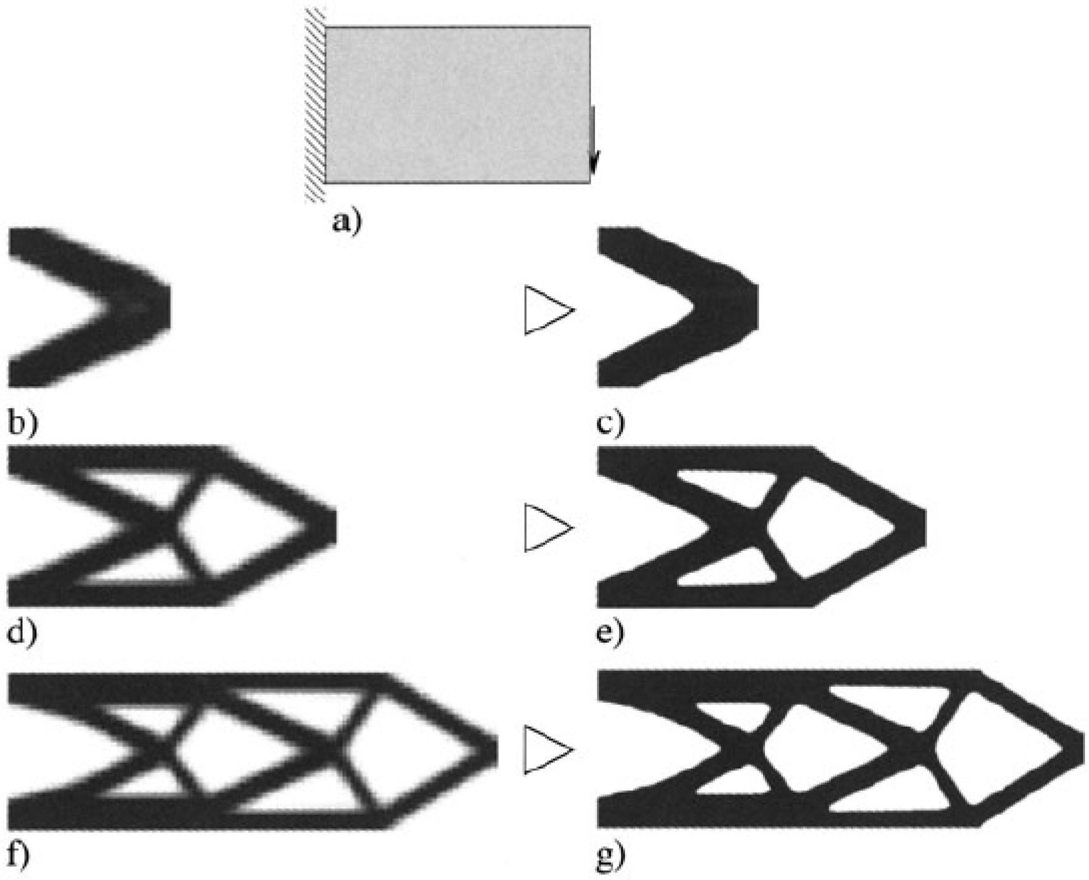

Postprocessing of grey-scale pictures by automatic (MATLAB) contour-plotting. Cantilever beam for different aspect ratios. b), d) and f): optimized topologies based on SIMP and filtering of sensitivities and c), e) and g): contour plots based on the grey-scale pictures

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It is thus for this type of situations natural to integrate the material distribution method and the boundary variations approach into one design tool, ***employing the topology optimization techniques as a pre-processor for boundary shape optimization***. The possibility of generating the optimal topology for a body can be used by the designer to select the shape of the initial proposed form of the body for the boundary variations technique. This is usually left entirely to the designer, but the material distribution method gives the designer a rational basis for his choice of initial form. As to be expected, the topology is of great importance for the performance of the structure, and it has turned out that - not unexpectedly - the compliance optimized topologies generated using topology design are very good starting points for optimization concerning several other criteria such as maximum stress, maximum deflection.

The direct integration of topology optimization and shape design methods is made difficult by the fact that the description of a structure by a density function is fundamentally different from a description by boundary curves or surfaces, as used in boundary variations shape optimization methods. In a CAD integrated shape optimization system, it is perhaps natural that the integration is based on the designer drawing the initial shape for the boundary variations technique directly on the top of a picture of the topology optimized structure, allowing for designer interaction. This also creates a design situation where the ingenuity of the designer is put to use for generating a "good" initial form from the topology optimization results. The term "good” in this context covers considerations such as ease of production, aesthetics, etc. that may not have a quantified form. However, automatic interfacing between the topology optimization method and other structural optimization methods is no doubt more productive. Here image processing and smooth surface generation are key technologies.

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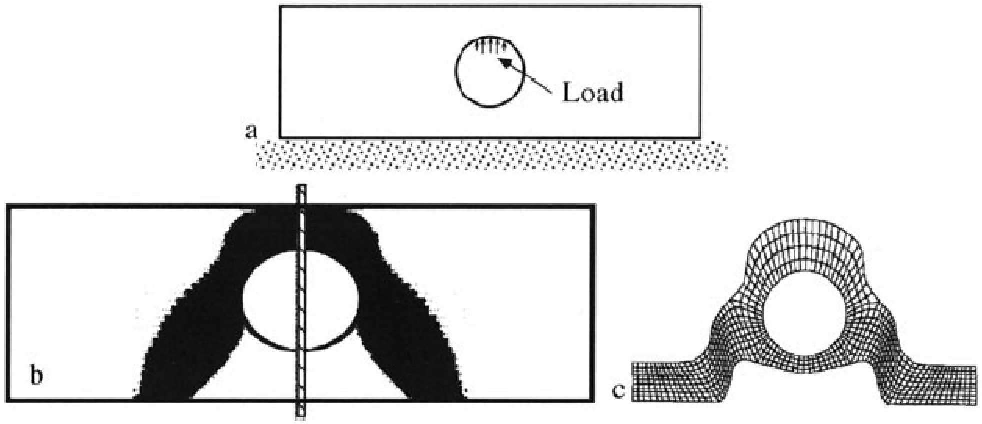
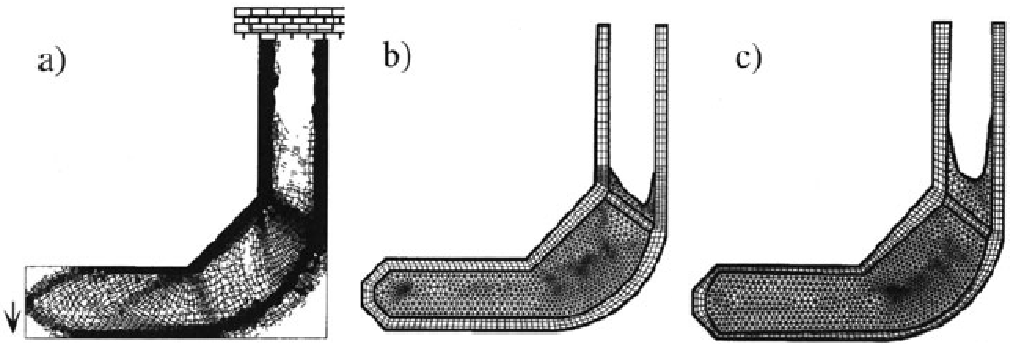


The optimal design of a bearing pedestal, using the homogenization approach integrated with the boundary shape design system CAOS. a) The reference domain, with loading. The rim of the inner hole was kept as a solid in the topology optimization. b) The result of the homogenization approach. c) The final design, after boundary shape design for minimum maximal Von Mises stress and after adding outer parts to the structure for fastening.



Optimized topology and shape design of a structure made of two materials, resulting in a sandwich structure. a) Optimized two-material topology computed using rank-3 layered materials. b) Initial design for a refinement using boundary shape optimization. All boundaries between skin and core are restricted to be piecewise straight lines. For the boundary design the weight is minimized without increasing the compliance relative to the optimal topology. c) Final shape optimized structure.

Thank you for your attention