香港中文大學
The Chinese University of Hong Kong

MAEG5160: Design for Additive Manufacturing

Lecture 9: Extensions and Applications

Prof SONG Xu
Department of Mechanical and Automation Engineering,
The Chinese University of Hong Kong.

Extensions and Applications

Foreword

Previously, we discussed the basics of the topology optimization method applied to compliance minimization. Due to the simple form of the compliance minimization problem, this problem was used as the fundamental test case in the initial developments of the topology optimization method. Despite its simplicity, the compliance minimization problem gives rise to non-trivial theoretical and numerical problems such as checkerboards, mesh-dependency and existence issues, and convergence to local minima. These problems have to be dealt with before one can proceed to more advanced applications and objective functions.

Here we describes a range of advanced applications, emphasizing problem formulations and solution procedures. New numerical and theoretical problems like for example localized modes in low-density regions, one-node connected hinges, instability of low density elements for geometrical nonlinear modelling, etc., appear for the more advanced applications, and we also discuss methods to avoid them.

Extensions and Applications

Here we mainly use the SIMP approach to interpolate between solid and void material since this approach has proven to generalize easily to alternative applications. Unless otherwise stated we use filtering of sensitivities to obtain checkerboard-free and mesh-independent designs. Also, the solution procedure follows the methodology described before. This means that the optimization is based on the use of density as the primary variable, with state equations and associated sensitivity analysis being treated as a function call. For this reason we write all optimization problems with only the density as a free variable. Nonetheless, for easy identification of problem structure we still include the state equations in the formulation. Finally note that we throughout this chapter base all formulations on a discretized FE format.

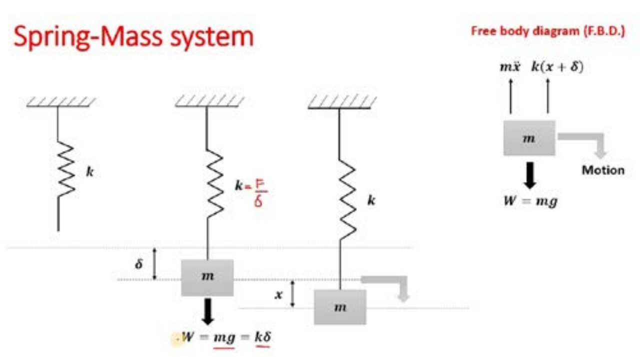
$$
\begin{aligned}
&\min_{\mathbf{u},\rho_e} \mathbf{f}^T\mathbf{u} \\
&\text{s.t.}: \left(\sum_{e=1}^{N} \rho_e^p \mathbf{K}_e\right)\mathbf{u} = \mathbf{f}\ , \\
&\qquad \sum_{e=1}^{N} v_e\rho_e \le V, \quad 0 < \rho_{\min} \le \rho_e \le 1, \quad e = 1,\dots,N\ .
\end{aligned}
$$

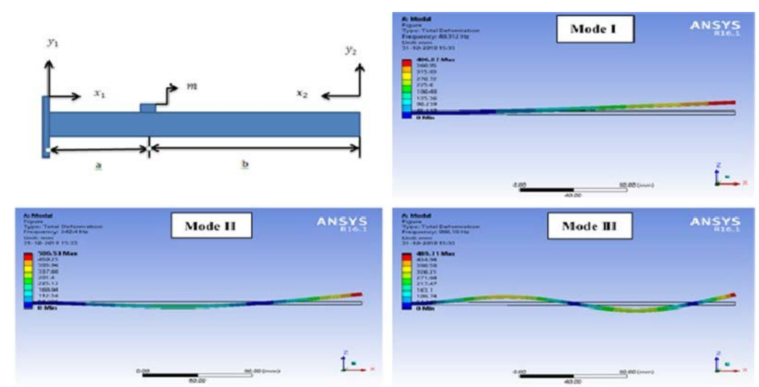
Extensions and Applications

1. Problems in dynamics

One of the first applications of the topology optimization method outside of compliance minimization was in eigenvalue optimization for free vibrations. This problem is relevant for the design of machines and structures subjected to dynamic loads. For example, one may wish to keep the eigenfrequencies of a structure away from the driving frequency of an attached engine or one may wish to keep the fundamental eigenfrequencies well above possible disturbance frequencies. Also, structures with high fundamental eigenfrequency tend to be reasonable stiff for all conceivable loads and therefore maximization of the fundamental frequency results in designs that are also good for static loads.

In specialized cases, one may wish to maximize the dynamic response of a structure. This may be the case in sensors where the output signal is dependent on the vibration amplitude, in actuators where resonance phenomena may increase performance or in musical instruments and loudspeakers where the radiated sound power (over a wide spectrum of frequencies) should be maximized.





Extensions and Applications

1.1 Free vibrations and eigenvalue problem

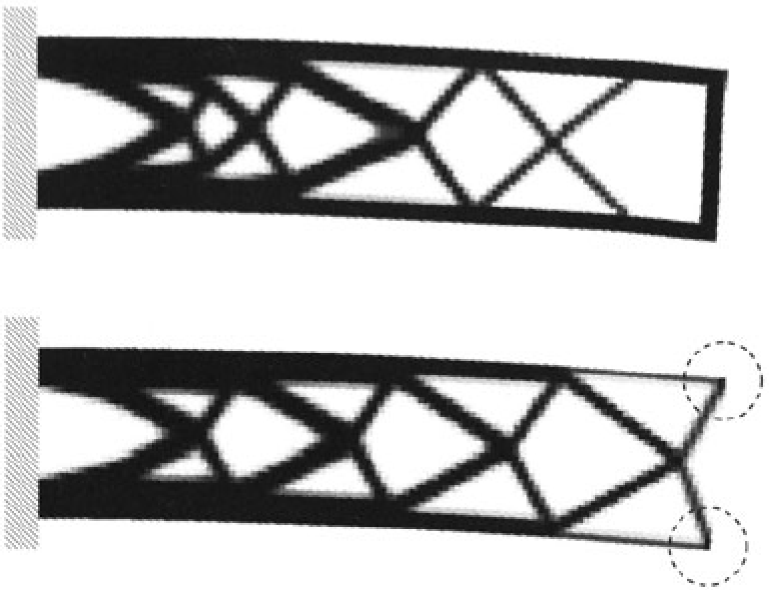
A commonly used design goal for dynamically loaded structures is the maximization of the fundamental eigenvalue λ_{min}

$$\max_{\boldsymbol{\rho}} \left\{ \lambda_{min} = \min_{i=1,\dots,N_{dof}} \lambda_i \right\}$$
$$\text{s.t.} : (\mathbf{K} - \lambda_i \mathbf{M})\boldsymbol{\Phi}_i = \mathbf{0}, \quad i = 1,\dots,N_{dof} \ ,$$
$$\sum_{e=1}^{N} v_e \rho_e \leq V, \qquad 0 < \rho_{min} \leq \rho_e \leq 1, \quad e = 1,\dots,N \ ,$$

where $\mathbf{K}$ and $\mathbf{M}$ are the system stiffness and mass matrices, respectively and $\boldsymbol{\Phi}_i$ is the eigenvector associated with the i'th eigenvalue. In practice one does not solve for all N_{dof} modes of the eigenvalue problem. Only the first up to 10 modes will usually play a role in determining the dynamical response of a structure.

Note that the problem *as stated* has a trivial solution: one can in principle obtain an infinite eigenvalue by removing the entire structure. Therefore, the eigenvalue problem is often used in "reinforcement" problems where parts of the structure are fixed to be solid or there is a finite minimum thickness of the structure like a fixed shell thickness in the reinforcement optimization of an engine hood. Alternatively, non-structural masses may be added to parts of the design domain.

Extensions and Applications



Top: A reinforcement problem. Maximization of the fundamental eigenvalue of a 5-bay tower structure where the outer frame structure is fixed to be solid.
Below: Maximization of the fundamental eigenvalue of a structure with nonstructural masses (each with a mass of 10% of the distributable mass) attached on the rightmost corners. The structures are shown in their fundamental mode of vibrations.

Extensions and Applications

An alternative to the formulation is to apply the so-called bound formulation

$$\max_{\boldsymbol{\rho}} \quad \beta$$

$$\text{s.t.}: \quad \lambda_i \geq \beta, \quad i = 1, \ldots, N_{dof} \ ,$$

$$(\mathbf{K} - \lambda_i \mathbf{M})\mathbf{\Phi}_i = \mathbf{0}, \quad i = 1, \ldots, N_{dof} \ ,$$

$$\sum_{e=1}^{N} v_e \rho_e \leq V, \qquad 0 < \rho_{min} \leq \rho_e \leq 1, \quad e = 1, \ldots, N \ .$$

The sensitivities of a *single* modal eigenvalue are simply found as

$$\frac{\partial \lambda_i}{\partial \rho_e} = \mathbf{\Phi}_i^T \left[\frac{\partial \mathbf{K}}{\partial \rho_e} - \lambda_i \frac{\partial \mathbf{M}}{\partial \rho_e} \right] \mathbf{\Phi}_i \ ,$$

where it is assumed that the eigenvector has been normalized with respect to the kinetic energy, i.e. $\mathbf{\Phi}_i^T \mathbf{M} \mathbf{\Phi}_i = 1$.

Extensions and Applications

For a solution where the optimum eigenvalue is *single* modal, the implementation is straight forward although one should note that the sensitivities of eigenvalues, as opposed to the sensitivities of the compliance objective, may take negative as well as positive values. This is not a problem when using mathematical programming methods for the optimization but it requires a small modification for an application of the optimality criteria algorithm. The density update used in compliance minimization

$$\rho_{K+1} = \rho_K [B_K]^\eta = \rho_K \left[\frac{-\frac{\partial c}{\partial \rho_e}}{\Lambda \, v_e} \right]^\eta ,$$

must for eigenvalue maximization be changed to

$$\rho_{K+1} = \rho_K [B_K]^\eta = \rho_K \left[\frac{\max(0, \; -\frac{\partial \lambda_{min}}{\partial \rho_e})}{\Lambda \, v_e} \right]^\eta .$$

The bound-formulation problem may be solved using mathematical programming solvers like for example MMA.

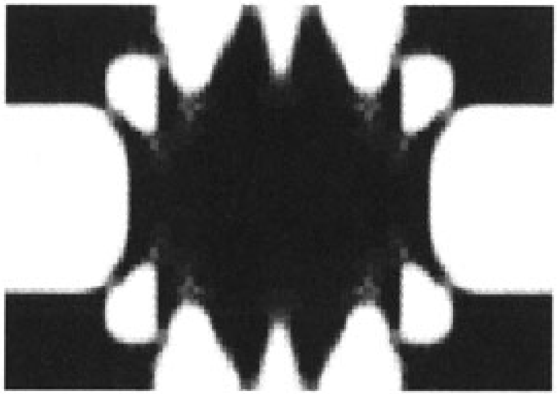
Extensions and Applications

For the formulations above, the optimized structures will often have a multi-modal eigenvalue and this may be critical for stability. In order to prevent multiple eigenmodes, one may require that the second eigenvalue is some percent bigger than the first, the third is some percent bigger than the second and so on. These constraints may easily be applied by rewriting the bound formulation to the format:

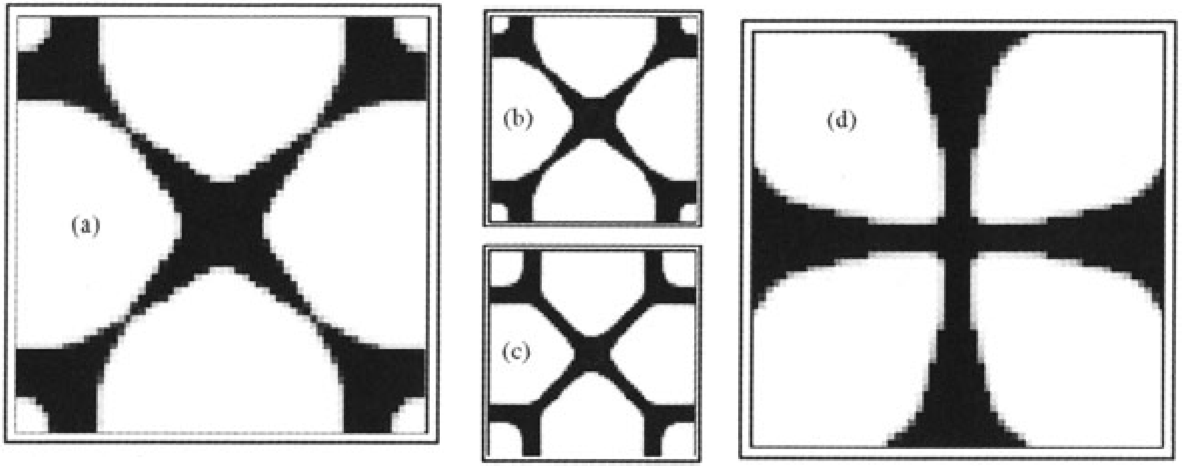
$$
\begin{aligned}
\max_{\boldsymbol{\rho}} \quad & \beta \\
\text{s.t.}: \quad & [\alpha]^i \, \lambda_i \geq \beta, \quad i = 1, \ldots, N_{dof} \, , \\
& (\mathbf{K} - \lambda_i \mathbf{M}) \boldsymbol{\Phi}_i = \mathbf{0}, \quad i = 1, \ldots, N_{dof} \, , \\
& \sum_{e=1}^{N} v_e \rho_e \leq V, \qquad 0 < \rho_{min} \leq \rho_e \leq 1, \quad e = 1, \ldots, N \, ,
\end{aligned}
$$

where e.g. $\alpha = 0.95$ (in (2.5) each eigenvalue is multiplied with α in the power i). In this way one also eliminates the problem of non-differentiability at the optimum for a multiple eigenvalue solution. However, one should be careful when using this method since the constraints may prevent the eigenmodes in switching during the optimization. Therefore convergence to better solutions may be jeopardized.

Extensions and Applications



Optimized topology for maximization of the fundamental eigenfrequency. The design domain is a simply supported Mindlin plate with a 10% non-structural mass at the center and volume constraint of 60% of the total volume



Maximization of the fundamental eigenfrequency for pre-stressed Mindlin plates. The plates are clamped at all edges, there is a 10% non-structural mass attached to the center and the distributable amount of material is 25% of the total volume. The pre-stress levels are a) $\sigma_{11} = \sigma_{22} = 0$, b) $\sigma_{11} = \sigma_{22} = 10$, c) $\sigma_{11} = \sigma_{22} = 25$ and d) $\sigma_{11} = \sigma_{22} = 100$

Extensions and Applications

1.2 Forced vibrations

In some situations one may want to minimize or maximize the dynamical response of a structure for a given driving frequency or frequency range. An example of the former could be for an airplane where the vibrations in the structure should be minimized at the frequency of the propeller. For the latter, examples are a sensor which should give a large output for a certain driving frequency or a clock frequency generator that should vibrate at a certain frequency for least possible input.

For solving this type of design problem we define the dynamic compliance as driving force times magnitude of the displacement and express the goal for the dynamical response in terms of this compliance. An optimization problem solving the problem of minimizing the dynamic compliance of a structure subject to periodic forces, $\mathbf{f}(\Omega)$, with frequency Ω, can then be written as

$$
\begin{aligned}
\min_{\boldsymbol{\rho}} \quad & \left\{c = (\mathbf{f}^T \mathbf{u})^2\right\} \\
\text{s.t. :} \quad & (\mathbf{K} - \Omega^2 \mathbf{M})\, \mathbf{u} = \mathbf{f}\ , \\
& \sum_{e=1}^{N} v_e \rho_e \leq V, \qquad 0 < \rho_{min} \leq \rho_e \leq 1, \quad e = 1, \ldots, N\ .
\end{aligned}
$$

Extensions and Applications

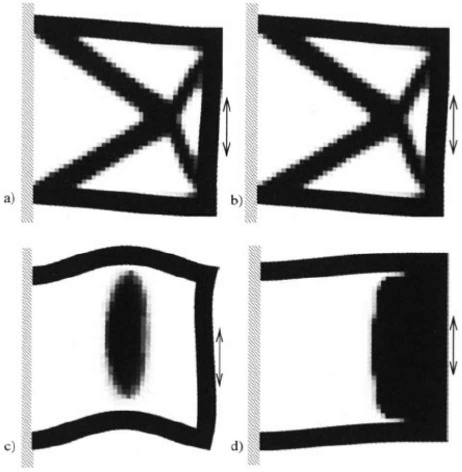
The sensitivities of the objective function may by use of the adjoint method be found as

$$\frac{\partial c}{\partial \rho_e} = \boldsymbol{\lambda}^T \left[\frac{\partial \mathbf{K}}{\partial \rho_e} - \Omega^2 \frac{\partial \mathbf{M}}{\partial \rho_e} \right] \mathbf{u} \ ,$$

where $\boldsymbol{\lambda}$ is the solution to the adjoint problem

$$(\mathbf{K} - \Omega^2 \mathbf{M}) \ \boldsymbol{\lambda} = -2(\mathbf{f}^T \mathbf{u}) \ \mathbf{f} \ .$$

We readily see from above that for low driving frequencies **Ω**, the results obtained should correspond roughly to the results of solving static problems (the term **Ω**^{2}M is a small perturbation to the stiffness matrix). However, for higher driving frequencies we should expect different resulting topologies. It can be shown that this formulation corresponds to forcing the closest eigenfrequency away form the driving frequency.



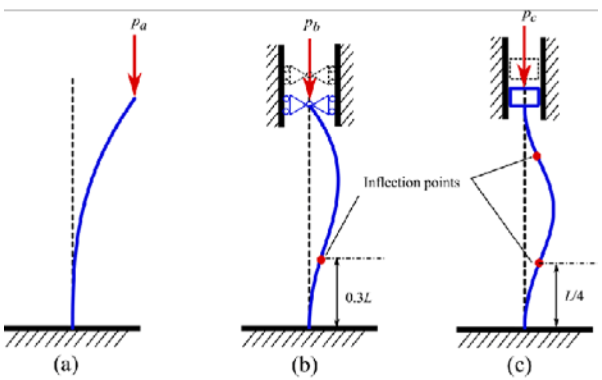
Optimized topologies for different driving frequencies . a) A zero driving frequency gives a statically stiff structure. b) while a small driving frequency forces the first eigenfrequency upwards resulting in a statically stiff structure. c) A larger driving frequency results in a tuned mass damper, d) and an even larger driving frequency forces the first eigenfrequency downwards and away from the driving frequency. All four examples where solved as reinforcement problems for a given outer frame and a stiffness ratio between black and white areas of 100:l. The structures are shown in their deformed states corresponding to the forced vibration mode.

Extensions and Applications

2. Buckling problems

Another important problem in structural optimization is the maximization of the fundamental buckling load of a structure. The solution of the buckling problem and its associated numerical problems have many features in common with the dynamical problems discussed in the previous section.

Limiting ourselves to considering only linear modelling, i.e. small displacements, the standard objective is to maximize the minimum critical load P_{crit} (or equivalently to minimize $1/P_{crit}$). Typically the optimization problem is formulated as[1]



$$\begin{aligned}
\min_{\boldsymbol{\rho}} : \quad & \left\{ \frac{1}{P_{crit}} = \max_{i=1,\ldots,N_{dof}} \frac{1}{P_i} \right\} \\
\text{s.t.} : \quad & \left[\mathbf{G}(\mathbf{u}) - \frac{1}{P_i} \mathbf{K} \right] \boldsymbol{\Phi}_i = \mathbf{0}, \quad i = 1, \ldots, N_{dof} \ , \\
& \sum_{e=1}^{N} v_e \rho_e \leq V, \qquad 0 < \rho_{min} \leq \rho_e \leq 1, \quad e = 1, \ldots, N \ ,
\end{aligned}$$

where $\boldsymbol{\Phi}_i$ is the eigenvector associated with the i'th critical load and $\mathbf{G}(\mathbf{u})$ is the so-called geometric stiffness matrix which depends on the displacements obtained from the linear, static equilibrium problem $\mathbf{Ku} = \mathbf{f}$.

Extensions and Applications

In practice one does not solve for all *Ndof* modes of the eigenproblem. In the beginning of the design iterations there is usually only one or two critical eigenvalues whereas towards the end, up to 10 eigenvalues may cluster above the most critical eigenvalue. The number of eigenvalues close to the most critical eigenvalue should be monitored during the iterations.

Alternatively, one may reformulate it to a bound-formulation

$$
\begin{aligned}
&\min_{\boldsymbol{\rho}} \quad \beta \\
&\text{s.t.}: \quad \frac{1}{P_i} \leq \beta, \quad i = 1, \ldots, N_{dof} \ , \\
&\qquad \left[\mathbf{G}(\mathbf{u}) - \frac{1}{P_i}\mathbf{K} \right] \boldsymbol{\Phi}_i = \mathbf{0}, \quad i = 1, \ldots, N_{dof} \ , \\
&\qquad \sum_{e=1}^{N} v_e \rho_e \leq V, \qquad 0 < \rho_{min} \leq \rho_e \leq 1, \quad e = 1, \ldots, N \ .
\end{aligned}
$$

The sensitivities of a *single* modal eigenvalue are found as

$$
\frac{\partial \lambda_{min}}{\partial \rho_e} = \boldsymbol{\Phi}_1{}^T \left[\frac{\partial \mathbf{G}}{\partial \rho_e} - \frac{1}{P_1} \frac{\partial \mathbf{K}}{\partial \rho_e} \right] \boldsymbol{\Phi}_1 + \mathbf{v}^T \frac{\partial \mathbf{K}}{\partial \rho_e} \mathbf{u},
$$

where $\mathbf{v}$ is the solution to the adjoint load problem

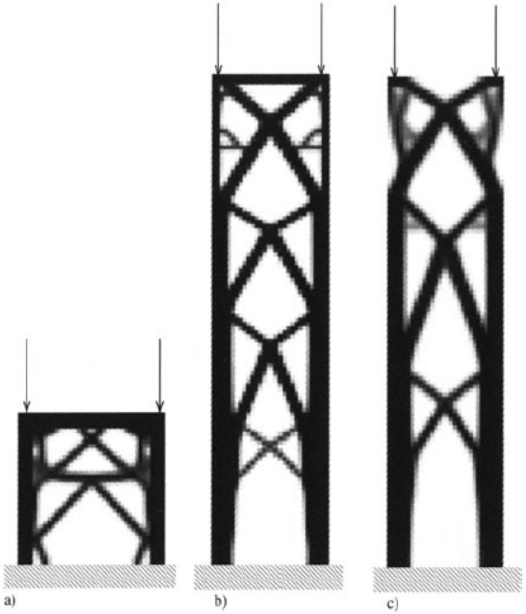
$$
\mathbf{K}\mathbf{v} = \boldsymbol{\Phi}_1^T \frac{\partial \mathbf{G}(\mathbf{u})}{\partial \mathbf{u}} \boldsymbol{\Phi}_1 .
$$

Extensions and Applications

As was the case for dynamical problems, artificial modes may also here appear in low density regions where the (non-linear) geometrical stiffness is high compared to the linear stiffness. To avoid the problem of artificial local modes one can ignore the geometrical stiffness of low-density elements. This approach corresponds to ignoring the mass of low-density elements in the vibration problem. This cut-off method seems to stabilize the problem but may cause oscillations of the algorithm due to abrupt changes in the values of the objective function and sensitivities. However, a smooth version of this approach can be obtained by writing the interpolation schemes in a slightly different way for the two stiffness matrices:

$$\text{For matrix } \mathbf{K} \ : E_{\mathbf{K}} = [\rho_{\min} + (1 - \rho_{\min})\, \rho^p]\, E^0 \ ,$$
$$\text{For matrix } \mathbf{G} \ : E_{\mathbf{G}} = [\rho^p]\, E^0 \ ,$$

where $\rho_{\min}$ is the minimum density normally imposed in the topology design problems. This method seems to eliminate the problem for our test cases.



Optimal re-enforcement of portal frames for maximum fundamental buckling load. a) 40 by 40 elements discretization. b) and c) 120 by 30 element discretizations where b) is a re-enforcement problem where the outer frame is fixed to be solid and c) allows a free distribution of 50% material. The buckling load for the second tower c) is 1 % higher than for the first tower b) .

Extensions and Applications

3. Stress concentration

Imposing stress constraints on topology optimization problems is an extremely important topic. However, several challenges must be overcome in order to solve the problem efficiently. This section discusses some possible solution methods. However, the best way to solve stress constrained problems has probably yet to be suggested.

3.1 A stress criterion for the SIMP model

For the 0-1 formulation of the topology design problem a stress constraint is well-defined, but when a material of intermediate density is introduced, the form of the stress constraint is not a priori given.

A stress criterion for the SIMP model should be as simple as possible (like for the stiffness-density relation) , and the isotropy of the stiffness properties should be extended to the stress model. Moreover, for physical relevance, it is reasonable that the criterion should mimic consistent microstructural considerations. This leads one to apply a stress constraint for the SIMP model (with exponent p) that is expressed as a constraint of the von Mises equivalent stress

$$\sigma_{\mathrm{VM}} \leq \rho^p \sigma_l \quad \text{if } \rho > 0$$

Extensions and Applications

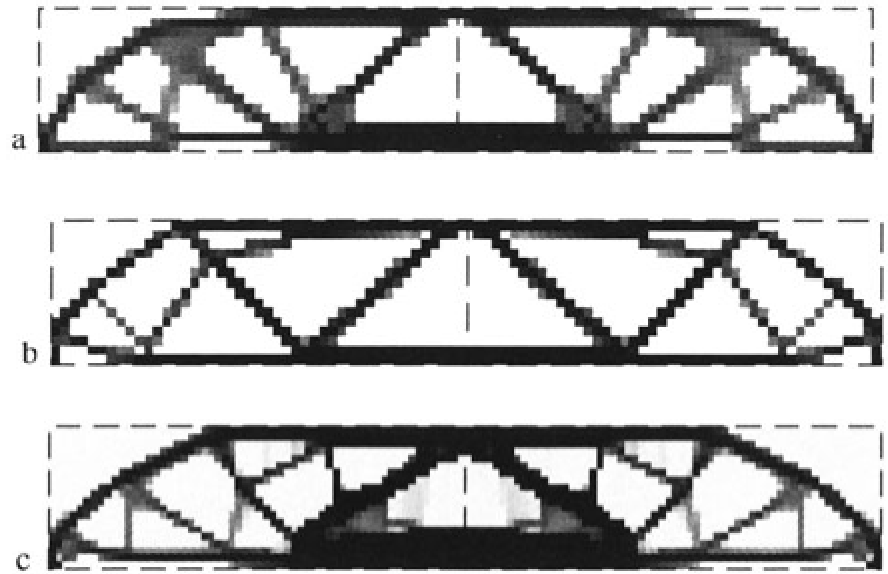
This constraint reflects the strength attenuation of a porous medium that arises when an average stress is distributed in the local microstructure, meaning that "local" stresses remain finite and non zero at zero density. This results in a reduction of strength domain by the factor ρ^p. We see that the same exponents are used for the stiffness interpolation and the stress constraint. Choosing another exponent is not consistent with physics and using an exponent that is less than *p* can for example lead to an artificial removal of material.

The classical stress-constrained optimization problem consists of finding the minimum weight structure that satisfies the stress constraint and which is in elastic equilibrium with the external forces, that is, we have a design problem in the form

$$
\begin{aligned}
&\min_{\boldsymbol{\rho}} \sum_{e=1}^{N} v_e \rho_e \\
&\text{s.t.}: \mathbf{K}\mathbf{u} = \mathbf{f}, \\
&\quad (\sigma_e)_{\text{VM}} \leq \rho_e^p \, \sigma_l \text{ if } \rho > 0\,, \qquad 0 < \rho_{min} \leq \rho_e \leq 1, \quad e = 1, \ldots, N
\end{aligned}
$$

where the stress for example is evaluated at the center-node of the individual FE elements.

Extensions and Applications



The MBB beam. a) stress design and b) compliance design with 45x15 finite elements, and c) stress design with 60x20 finite elements

Stress concentration For stress constraints one has to pay special attention to domains or problems that introduce a stress singularity (like in the inner corner of an L-shaped domain). The real difficulty for such situations is not so much in the optimization part but more the numerical problem of capturing the high stress at the corner. The optimization solution is of course strongly dependent on the quality of the analysis, and for most applications the stress constrained design optimization should be coupled with a much more refined analysis, using for example mesh adaptation.

Extensions and Applications

3.2 Solution aspects

Constraint relaxation The so-called "singularity" problem associated with stress constraints requires special care when dealing with topology design problems. It was first identified for truss problems and arises from a "degeneracy" or an "irregularity" of the design space. The key effect is that the feasible set in the design space contains degenerated appendices where constraint qualification does not hold. This means that classical optimization algorithms are unable to reach the optima that are located in these regions. In other words, a standard optimization algorithm is not able to completely remove some low density regions and to find the true optimal topologies.

One approach to circumvent this complication is to reformulate the problem as a sequence of problems that have nicer properties and which can give solutions that converge to the true design (like a continuation method). First, we note that in a topology design problem, the stress constraints should *only* be imposed if material is present. To eliminate the condition $\rho > 0$ from the constraint, one considers a modified formulation:

$$\rho \left(\frac{\sigma_{\mathrm{VM}}}{\rho^{p} \sigma_{l}} - 1 \right) \leq 0$$

Extensions and Applications

For bars in a truss, this is equivalent to considering forces instead of stresses. Unfortunately, this reformulation does not change the problems with constraint qualification, and additional measures are required. One method is to rewrite the stress constraints using the E-relaxation approach. This relaxation is a perturbation of the original problem where the original stress constraints are replaced by the following relaxed stress constraints and associated side constraints:

$$\rho\left(\frac{\sigma_{\mathrm{VM}}}{\rho^p\sigma_l}-1\right)\leq\epsilon(1-\rho)\ ,\quad \epsilon^2=\rho_{min}\leq\rho$$

Where ε is given. For any ε>0, the E-relaxed problem with the constraints is characterized by a design space that is not any longer degenerate, and the factor (1 - *p)* on ε assures that the real stress constraint is imposed for $p = 1$. It is thus possible to reach a *local* optimum with optimization algorithms. If we can find the global optimum, then for ε→0, the sequence of feasible domains and their optimal solutions converge continuously towards the original degenerate problem and its associated optimal solution.

The solution procedure thus now consists in solving a sequence of optimization problems, for decreasing ε, in a *continuation* approach similar to what is done with barrier and penalty functions. The implementation process is here driven by the minimum density $\rho_{min} = \varepsilon^2$ and choosing a quite large initial minimum density is necessary to obtain reasonable results. We remark here that the method may fail if the problem is such that there are many local minima for the relaxed problems. This may happen even for rather simple truss examples. An alternative approach is to not rely on gradient based techniques.

Extensions and Applications

Implementation aspects The local stress criterion adds a large number of constraints to what is already a large scale optimization problem. Thus it is important to apply an active set strategy which at each iteration step preselects the potentially dangerous stress constraints to be considered. At the beginning of the optimization process, the selection is large, but at the end of the optimization the set of active constraints is stable and it can be restricted to a fraction of the elements. For working with MMA (or CONLIN) it is also important to treat the stress constraint in a form which is suited for the approximation strategies of these. Thus the stress constraints should be written as (observe that the density variables are strictly positive for ε > 0):

$$\frac{\sigma_{\mathrm{VM}}}{\rho^p \sigma_l} - \frac{\epsilon}{\rho} + \epsilon \leq 1$$

A global stress constraint An alternative to working with the local constraints is to use global U constraints that for large *q* approximates the local constraints. This can be implemented in the form

$$\left[\sum_{e=1}^{N} \max\left(\left\{0\,,\ \frac{\sigma_{\mathrm{VM}}}{\rho^p \sigma_l} - \frac{\epsilon}{\rho} + \epsilon\right\}\right)^q\right]^{1/q} \leq 1$$

This is just one constraint, so the savings in computational effort is immense. The difficulty is the numerical problems associated with using large *q*. Computational experiments shows that *q=4* is a good choice; however, for problems with very localized high stresses (like an L-shape) one cannot assure that the stress is below the critical value in all areas. Nonetheless, the designs one can obtain are quite reasonable compared to using the very cumbersome local constraint.

Extensions and Applications

4. Pressure Loads

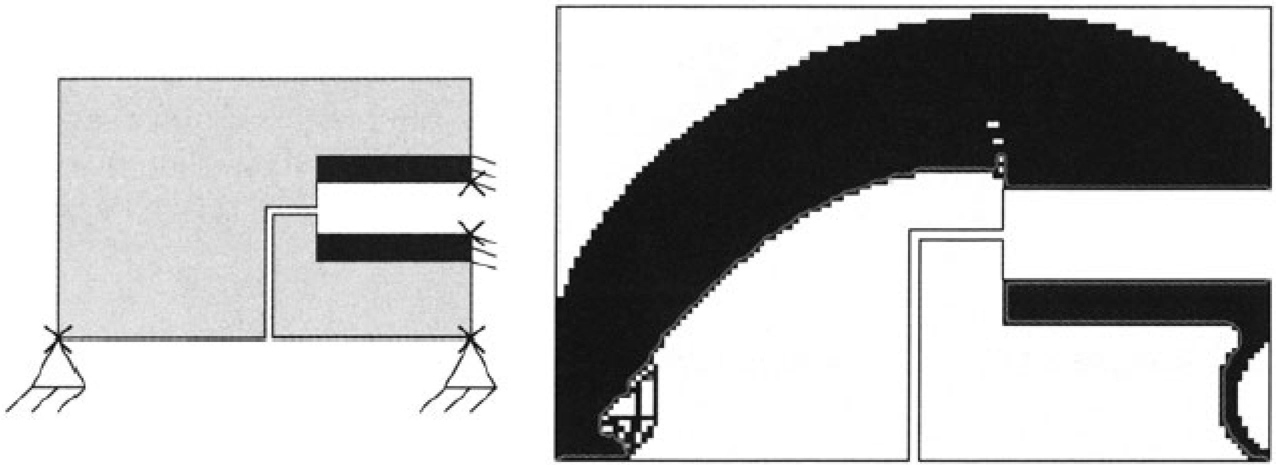
An example of design dependent loads is pressure loads. Since the direction as well as the position of attack of the pressure loads depend on the boundary between solid and void and because the boundaries are not well defined in topology optimization problems, topology design with pressure loads is a highly challenging problem. The optimization problem is the classical one of compliance minimization of a structure where the design parameters are the volumetric material densities throughout the design domain. The novel aspect here lies in the type of loading considered which occurs if free structural surface domains are subjected to forces where both the direction, the location, and the size can change with the material distribution. Examples are pressure and fluid flow loading with the direction and location of the load changing with and following the structural surface. The compliance of the structure is written as:

$$c(u) = \int_{\Omega} fu \mathrm{d}\Omega + \int_{\Gamma_t} tu \mathrm{d}\Gamma + \int_{\Gamma_p} pu \mathrm{d}\Gamma$$

where an extra term representing the design dependent load – here a pressure p – acting on parts of the boundary Γ_p of the *material domain*.

Extensions and Applications

The optimization process is performed by successive iterations making use of the finite element analysis model with fixed mesh on the one hand, and the design model with the parametrized iso-volumetric density surface for the pressure loading on the other. The load surfaces in the design model are controlled by the density distribution in the finite element model and in turn fully determine the global load vector on the finite element model. Thus the sensitivity analysis is based on both the analysis model and the design model. In the sensitivity analysis also the sensitivities of the load vector with respect to a design change must be evaluated, and this is done analytically. The problem is solved by an optimality criteria method.



Optimization of an inlet. Two separate parts of the structural surface are subjected to pressure loads. The design domain with the pressure initially distributed to narrow white internal channel is shown to the left. To the right is shown the optimized topology for a volume fraction of 40%. The pressurized surfaces are marked with grey lines

Thank you for your attention