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Theory for Large Deformation

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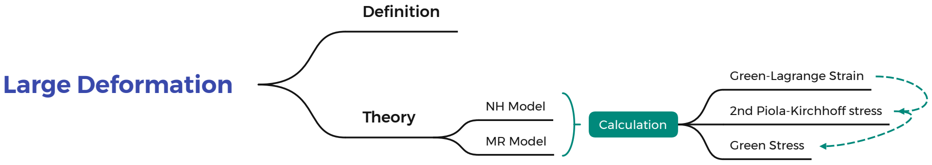

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Introduction

Large deformation

Definition

Large deformation happens when strains and/or rotations are large enough to invalidate assumptions inherent in infinitesimal strain theory.

Assumptions inherent in infinitesimal strain theory: the displacements of the material particles are assumed to be much smaller (indeed, infinitesimally smaller) than any relevant dimension of the body; so that its geometry and the constitutive properties of the material (such as density and stiffness) at each point of space can be assumed to be unchanged by the deformation.

Strain and deformation tensors

Displacement field & displacement gradient tensor

Definition

Displacement field:

$$\mathbf{u}\left(x, y\right) = \begin{Bmatrix} u \\ v \\ w \end{Bmatrix} \tag{1}$$

Definition

Displacement gradient tensor:

$$\mathbf{H}\left(x, y\right) = \begin{bmatrix} \frac{\partial u}{\partial X} & \frac{\partial u}{\partial Y} & \frac{\partial u}{\partial Z} \\ \frac{\partial v}{\partial X} & \frac{\partial v}{\partial Y} & \frac{\partial v}{\partial Z} \\ \frac{\partial w}{\partial X} & \frac{\partial w}{\partial Y} & \frac{\partial w}{\partial Z} \end{bmatrix} \tag{2}$$

Deformation gradient

Definition

The deformation gradient $\mathbf{F}$ is the derivative of each component of the deformed $\mathbf{x}$ vector with respect to each component of the reference $\mathbf{X}$ vector. For $\mathbf{x} = \mathbf{x}(\mathbf{X})$, then

$$F_{ij} = x_{i,j} = \frac{\partial x_i}{\partial X_j} = \begin{bmatrix} \frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} & \frac{\partial x_1}{\partial X_3} \\ \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} & \frac{\partial x_2}{\partial X_3} \\ \frac{\partial x_3}{\partial X_1} & \frac{\partial x_3}{\partial X_2} & \frac{\partial x_3}{\partial X_3} \end{bmatrix} \quad (3)$$

A slightly altered calculation is possible by noting that the displacement $\mathbf{u}$ of any point can be defined as

$$\mathbf{u} = \mathbf{x} - \mathbf{X} \quad (4)$$

and this leads to $\mathbf{x} = \mathbf{X} + \mathbf{u}$, and

$$\mathbf{F} = \frac{\partial}{\partial \mathbf{X}}(\mathbf{X} + \mathbf{u}) = \frac{\partial \mathbf{X}}{\partial \mathbf{X}} + \frac{\partial \mathbf{u}}{\partial \mathbf{X}} = \mathbf{I} + \mathbf{H} \quad (5)$$

Green-Lagrange strains

The length of a vector after the deformation L_f can be expressed with the length before the deformation L_o.

$$L_f^2 = \mathbf{x}^T\mathbf{x} = (\mathbf{FX})^T(\mathbf{FX}) = \mathbf{X}^T\mathbf{F}^T\mathbf{FX} = L_o^2\mathbf{M}^T\mathbf{F}^T\mathbf{FM} \tag{6}$$

where $\mathbf{M}$ is the vector of unit length of the original length direction.
The Green-Lagrange description of strain is the quadratic extension, which is defined as follows. Equation (6) is again substituted into the definition.

$$\epsilon = \frac{L_f^2 - L_o^2}{2L_o^2} = \frac{L_o^2\mathbf{M}^T\mathbf{F}^T\mathbf{FM} - L_o^2}{2L_o^2} = \mathbf{M}^T\underbrace{\frac{1}{2}\left(\mathbf{F}^T\mathbf{F} - \mathbf{I}\right)}_{\mathbf{E}}\mathbf{M} \tag{7}$$

Green-Lagrange strains

Definition

The Green-Lagrange strains **E** is defined as:

$$\mathbf{E} = \frac{1}{2}\left(\mathbf{F}^T \cdot \mathbf{F} - \mathbf{I}\right) \tag{8}$$

The Green-Lagrange strain tensor written out is

$$\mathbf{E} = \begin{bmatrix} E_{xx} & E_{xy} & E_{xz} \\ E_{xy} & E_{yy} & E_{yz} \\ E_{xz} & E_{yz} & E_{zz} \end{bmatrix} \tag{9}$$

Strain tensor

$$E_{xx} = \frac{\partial u}{\partial X} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial X}\right)^2 + \left(\frac{\partial v}{\partial X}\right)^2 + \left(\frac{\partial w}{\partial X}\right)^2\right] \tag{10}$$

$$E_{yy} = \frac{\partial v}{\partial Y} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial Y}\right)^2 + \left(\frac{\partial v}{\partial Y}\right)^2 + \left(\frac{\partial w}{\partial Y}\right)^2\right] \tag{11}$$

$$E_{zz} = \frac{\partial w}{\partial Z} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial Z}\right)^2 + \left(\frac{\partial v}{\partial Z}\right)^2 + \left(\frac{\partial w}{\partial Z}\right)^2\right] \tag{12}$$

$$E_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial Y} + \frac{\partial v}{\partial X}\right) + \frac{1}{2}\left[\frac{\partial u}{\partial X}\frac{\partial u}{\partial Y} + \frac{\partial v}{\partial X}\frac{\partial v}{\partial Y} + \frac{\partial w}{\partial X}\frac{\partial w}{\partial Y}\right] \tag{13}$$

$$E_{xz} = \frac{1}{2}\left(\frac{\partial u}{\partial Z} + \frac{\partial w}{\partial X}\right) + \frac{1}{2}\left[\frac{\partial u}{\partial X}\frac{\partial u}{\partial Z} + \frac{\partial v}{\partial X}\frac{\partial v}{\partial Z} + \frac{\partial w}{\partial X}\frac{\partial w}{\partial Z}\right] \tag{14}$$

$$E_{yz} = \frac{1}{2}\left(\frac{\partial v}{\partial Z} + \frac{\partial w}{\partial Y}\right) + \frac{1}{2}\left[\frac{\partial u}{\partial Y}\frac{\partial u}{\partial Z} + \frac{\partial v}{\partial Y}\frac{\partial v}{\partial Z} + \frac{\partial w}{\partial Y}\frac{\partial w}{\partial Z}\right] \tag{15}$$

Small deformation vs large deformation

$$
\begin{aligned}
\text{Green Strain} &= \text{Small Strain Terms} &+& \quad \text{Quadratic Terms} \\
E_{xx} &= \frac{\partial u}{\partial X} &+& \frac{1}{2}\left[\left(\frac{\partial u}{\partial X}\right)^2+\left(\frac{\partial v}{\partial X}\right)^2+\left(\frac{\partial w}{\partial X}\right)^2\right] \\
E_{yy} &= \frac{\partial v}{\partial Y} &+& \frac{1}{2}\left[\left(\frac{\partial u}{\partial Y}\right)^2+\left(\frac{\partial v}{\partial Y}\right)^2+\left(\frac{\partial w}{\partial Y}\right)^2\right] \\
E_{zz} &= \frac{\partial w}{\partial Z} &+& \frac{1}{2}\left[\left(\frac{\partial u}{\partial Z}\right)^2+\left(\frac{\partial v}{\partial Z}\right)^2+\left(\frac{\partial w}{\partial Z}\right)^2\right] \\
E_{xy} &= \frac{1}{2}\left(\frac{\partial u}{\partial Y}+\frac{\partial v}{\partial X}\right) &+& \frac{1}{2}\left[\frac{\partial u}{\partial X}\frac{\partial u}{\partial Y}+\frac{\partial v}{\partial X}\frac{\partial v}{\partial Y}+\frac{\partial w}{\partial X}\frac{\partial w}{\partial Y}\right] \\
E_{xz} &= \frac{1}{2}\left(\frac{\partial u}{\partial Z}+\frac{\partial w}{\partial X}\right) &+& \frac{1}{2}\left[\frac{\partial u}{\partial X}\frac{\partial u}{\partial Z}+\frac{\partial v}{\partial X}\frac{\partial v}{\partial Z}+\frac{\partial w}{\partial X}\frac{\partial w}{\partial Z}\right] \\
E_{yz} &= \frac{1}{2}\left(\frac{\partial v}{\partial Z}+\frac{\partial w}{\partial Y}\right) &+& \frac{1}{2}\left[\frac{\partial u}{\partial Y}\frac{\partial u}{\partial Z}+\frac{\partial v}{\partial Y}\frac{\partial v}{\partial Z}+\frac{\partial w}{\partial Y}\frac{\partial w}{\partial Z}\right]
\end{aligned}
$$

Neo-Hookean Models & Mooney-Rivlin Models

Alternative stress measures

- **Cauchy Stress Tensor** is known as true stress because it refers to the current deformed geometry as a reference force and area. Cauchy stress tensor has a dimension of 3×3.
- **First Piola-Kirchoff Stress** is defined differently, in this case, the stress vector is defined by the force, initial area, and unit normal vector $\mathbf{N}$ in the undeformed geometry (initial geometry). First Piola-Kirchoff Stress is of 3×3 dimension. However, it is different as Piola-Stress tensor is not symmetric contrary to Cauchy Stress Tensor. In short, First Piola-Kirchoff Stress vector refers to force in the current geometry (deformed) and area in the initial geometry (undeformed).
- **Second Piola-Kirchoff Stress** is derived using the product of transpose of the inverse of the deformation gradient with First-Piola Kirchoff Stress Vector. Second Piola-Kirchoff Stress Tensor is a symmetric tensor compared to First Piola-Kirchoff Stress Tensor, which is defined entirely in the reference configuration.

2nd Piola-Kirchhoff stress

The Second Piola-Kirchoff stress tensor is the derivative of the Helmholtz free energy with respect to the Green strain tensor multiplied by the density (spatial or material depending on the system orientation chosen) for a thermoelastic body without any internal constraints.

Definition

2nd Piola-Kirchhoff stress:

$$\boldsymbol{\sigma}^{\mathrm{PK2}} = \rho_o \frac{\partial \Psi}{\partial \mathbf{E}^{\mathrm{el}}} \tag{16}$$

The Helmholtz free energy contains thermal energy and mechanical strain energy. But in most every discussion of Mooney-Rivlin coefficients, the thermal part is neglected, leaving only the mechanical part, W. (Actually, W is declared to represent $\rho_o\Psi$, not just Ψ). Second, since all of the deformations of a hyperelastic material is elastic by definition, it is sufficient to write $\mathbf{E}^{\mathrm{el}}$ simply as $\mathbf{E}$. This gives

$$\boldsymbol{\sigma}^{\mathrm{PK2}} = \frac{\partial W}{\partial \mathbf{E}} \tag{17}$$

2nd Piola-Kirchhoff stress

But there is a challenge with this general approach. It is the determination of off-diagonal (shear) terms. As with the shear terms in Hooke's Law, they are not independent of the normal terms, but must be consistent with coordinate transformations that transform normal components into shears and vice-versa. And as with Hooke's Law, the resolution is to define the material behavior for the principal values and rely on coordinate transformations to give the appropriate corresponding behavior of the shear terms.

$$\sigma_i^{\mathrm{PK2}} = \frac{\partial W}{\partial E_i} \tag{18}$$

Using the chain rule,

$$\sigma_i^{\mathrm{PK2}} = \frac{\partial W}{\partial \lambda_i} \frac{\partial \lambda_i}{\partial E_i} \tag{19}$$

where λ is the *stretch ratios*.

$$\lambda = \frac{L_F}{L_o} \tag{20}$$

So the stretch ratio is "one plus engineering strain"

$$\lambda - 1 = \epsilon_{Eng} = \Delta L / L_o \tag{21}$$

Principal Cauchy stress

Recall that a principal Green strain equals

$$E_i = \frac{\Delta L_i}{L_o} + \frac{1}{2}\left(\frac{\Delta L_i}{L_o}\right)^2 \tag{22}$$

So $\partial E_i / \partial \lambda_i = \lambda_i$ and therefore

$$\frac{\partial \lambda_i}{\partial E_i} = \frac{1}{\lambda_i} \tag{23}$$

Substituting this into the Equation (19) gives,

$$\sigma_i^{\mathrm{PK2}} = \frac{1}{\lambda_i}\frac{\partial W}{\partial \lambda_i} \tag{24}$$

Therefore, the principle stress can be calculated by using $\sigma_i = (1+\epsilon_i)^2 \sigma_i^{\mathrm{PK2}}$ ($\sigma_i = \lambda_i^2 \sigma_i^{\mathrm{PK2}}$):

$$\sigma_i = \lambda_i^2 \sigma_i^{\mathrm{PK2}} = \lambda_i \frac{\partial W}{\partial \lambda_i} \tag{25}$$

Invariants

A deformation gradient can be written as

$$\mathbf{F} = \mathbf{V} \cdot \mathbf{R} \tag{26}$$

where $\mathbf{R}$ is the rotation matrix (same as before), and $\mathbf{V}$ is the left stretch tensor. This is also a polar decomposition.

Definition

The invariants are the product of the deformation gradient with its transpose, $\mathbf{F} \cdot \mathbf{F}^T$. Using polar decomposition,

$$\mathbf{F} \cdot \mathbf{F}^T = (\mathbf{V} \cdot \mathbf{R}) \cdot (\mathbf{V} \cdot \mathbf{R})^T = \mathbf{V} \cdot \mathbf{R} \cdot \mathbf{R}^T \cdot \mathbf{V}^T = \mathbf{V} \cdot \mathbf{V}^T \tag{27}$$

And that the principal values of $\mathbf{V}$ are

$$\mathbf{V}_{\text{Pr}} = \begin{bmatrix} (L_F/L_o)_1 & 0 & 0 \\ 0 & (L_F/L_o)_2 & 0 \\ 0 & 0 & (L_F/L_o)_3 \end{bmatrix} \tag{28}$$

Invariants definition

The ratios of $(L_F/L_o)_i$ are replaced by the single symbol, λ_i, called *stretch ratios*. (Note that $\lambda_i = 1 + \epsilon_i$ where ϵ_i is the i^{th} principal engineering strain.) So the above tensor becomes.

$$\mathbf{V}_{\text{Pr}} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \tag{29}$$

The principal values of $\mathbf{F} \cdot \mathbf{F}^T$ are

$$\left(\mathbf{F} \cdot \mathbf{F}^T\right)_{\text{Pr}} = \left(\mathbf{V} \cdot \mathbf{V}^T\right)_{\text{Pr}} = \begin{bmatrix} \lambda_1^2 & 0 & 0 \\ 0 & \lambda_2^2 & 0 \\ 0 & 0 & \lambda_3^2 \end{bmatrix} \tag{30}$$

Invariants definition

$\mathbf{F} \cdot \mathbf{F}^T$ is symmetric, so it does have three invariants.

$$\begin{aligned} I_1 &= \lambda_1^2 + \lambda_2^2 + \lambda_3^2 \\ I_2 &= \lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + \lambda_1^2\lambda_3^2 \\ I_3 &= \det(\dots) = \lambda_1^2\lambda_2^2\lambda_3^2 = \left(\frac{V_F}{V_o}\right)^2 = J^2 \end{aligned} \tag{31}$$

For incompressibile material, $\lambda_1\lambda_2\lambda_3 = 1$. Therefore,

$$\begin{aligned} I_1 &= \lambda_1^2 + \lambda_2^2 + \lambda_3^2 \\ I_2 &= \frac{1}{\lambda_1^2} + \frac{1}{\lambda_2^2} + \frac{1}{\lambda_3^2} \\ I_3 &= (\lambda_1\lambda_2\lambda_3)^2 = 1 \end{aligned} \tag{32}$$

Neo-Hookean Models

The neo-Hookean Model is a hyperelastic material model, similar to Hooke's law, that can be used for predicting the nonlinear stress-strain behavior of materials undergoing large deformations.

The strain energy density function for an incompressible neo-Hookean material in a three-dimensional description is

$$W = C_1 (I_1 - 3) \tag{33}$$

where C_1 is a material constant.
For a compressible neo-Hookean material the strain energy density function is given by

$$W = C_1 (I_1 - 3 - 2 \ln J) + D_1 (J - 1)^2 \tag{34}$$

The neo-Hookean material model does not predict that increase in modulus at large strains and is typically accurate only for strains less than 20%. The model is also inadequate for biaxial states of stress and has been superseded by the Mooney-Rivlin model.

Mooney-Rivlin Models

The Mooney–Rivlin model is a hyperelastic material model where the strain energy density function W is a linear combination of two invariants of the left Cauchy–Green deformation tensor $\mathbf{B}$. The Mooney-Rivlin class of models expresses the mechanical strain energy as a sum of the invariants as follows.

$$W = \sum_i \sum_j C_{ij} \left(I_1 - 3\right)^i \left(I_2 - 3\right)^j + D(J-1)^2 \tag{35}$$

Note that the series is not a function of I_3 because it is a constant value, 1. The constants, C_{ij} and D, will be determined by curve-fitting measured stress-strain curves to the derivative of the equation. The number of terms in the expansion is determined by the application's accuracy requirements.
As an example, the first few terms of the series are

$$W = C_{10}\left(I_1 - 3\right) + C_{01}\left(I_2 - 3\right) + C_{11}\left(I_1 - 3\right)\left(I_2 - 3\right) + C_{20}\left(I_1 - 3\right)^2 + \ldots + D\left(J - 1\right)^2 \tag{36}$$

Principal Cauchy stress

Each principal Cauchy stress is related to the derivative of the above equation with respect to the corresponding λ. For example, the 1st principal Cauchy stress corresponds to derivatives of W with respect to the first stretch ratio, λ_1.

$$\sigma_1 = \lambda_1 \frac{\partial W}{\partial \lambda_1} = \lambda_1 \left(\frac{\partial W}{\partial I_1} \frac{\partial I_1}{\partial \lambda_1} + \frac{\partial W}{\partial I_2} \frac{\partial I_2}{\partial \lambda_1} + \frac{\partial W}{\partial J} \frac{\partial J}{\partial \lambda_1} \right) \tag{37}$$

The derivatives of the strain energy with respect to the invariants, and J, are

$$\frac{\partial W}{\partial I_1} = C_{10} + C_{11}(I_2 - 3) + 2C_{20}(I_1 - 3) + \cdots \qquad \frac{\partial W}{\partial I_2} = C_{01} + C_{11}(I_1 - 3) + \cdots \tag{38}$$

$$\frac{\partial W}{\partial J} = 2D(J - 1) \tag{39}$$

And the derivatives of the invariants, and J, with respect to λ_1 are

$$\frac{\partial I_1}{\partial \lambda_1} = 2\lambda_1 \qquad \frac{\partial I_2}{\partial \lambda_1} = -\frac{2}{\lambda_1^3} \qquad \frac{\partial J}{\partial \lambda_1} = \lambda_2 \lambda_3 \tag{40}$$

All of these terms can be combined to give polynomials relating stretch ratios to principal stresses, with coefficients such as C_{10}, C_{01}, C_{11}, and C_{20} that are determined from curve-fitting these equations to experimental data.

Thank You